**Note: Handouts DO NOT replace the book. In most cases, they only provide a guideline on topics and an intuitive feel.**

# 1 Multiple Continuous Random Variables: Topics

- Conditioning on an event
- Joint and Marginal PDF
- Expectation, Independence, Joint CDF, Bayes rule
- Derived distributions
  - Function of a Single random variable: $Y = g(X)$ for any function $g$
  - Function of a Single random variable: $Y = g(X)$ for linear function $g$
  - Function of a Single random variable: $Y = g(X)$ for strictly monotonic $g$
  - Function of Two random variables: $Z = g(X, Y)$ for any function $g$

# 2 Conditioning on an event

- Read the book Section 3.4

# 3 Joint and Marginal PDF

- Two r.v.s $X$ and $Y$ are **jointly continuous** iff there is a function $f_{X,Y}(x, y)$ with $f_{X,Y}(x, y) \geq 0$, called the **joint PDF**, s.t. $P((X, Y) \in B) = \int_B f_{X,Y}(x, y)dxdy$ for all subsets $B$ of the 2D plane.

- Specifically, for $B = [a, b] \times [c, d] \triangleq \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$,

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_{y=c}^{d} \int_{x=a}^{b} f_{X,Y}(x, y)dxdy$$

- **Interpreting the joint PDF:** For small positive numbers $\delta_1, \delta_2$,

$$P(a \leq X \leq a + \delta_1, c \leq Y \leq c + \delta_2) = \int_{y=c}^{c+\delta_2} \int_{x=a}^{a+\delta_1} f_{X,Y}(x, y)dxdy \approx f_{X,Y}(a, c)\delta_1\delta_2$$

  Thus $f_{X,Y}(a, c)$ is the probability mass per unit area near $(a, c)$.

- **Marginal PDF:** The PDF obtained by integrating the joint PDF over the entire range of one r.v. (in general, integrating over a set of r.v.'s)

$$\begin{aligned} P(a \leq X \leq b) &= P(a \leq X \leq b, -\infty \leq Y \leq \infty) = \int_{x=a}^{b} \int_{y=-\infty}^{\infty} f_{X,Y}(x, y)dydx \\ \Longrightarrow f_X(x) &= \int_{y=-\infty}^{\infty} f_{X,Y}(x, y)dy \end{aligned}$$

- Example 3.12, 3.13

## 4 Conditional PDF

- Conditional PDF of $X$ given that $Y = y$ is defined as

$$f_{X|Y}(x|y) \triangleq \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- For any $y$, $f_{X|Y}(x|y)$ is a legitimate PDF: integrates to 1.
- Example 3.15
- **Interpretation:** For small positive numbers $\delta_1, \delta_2$, consider the probability that $X$ belongs to a small interval $[x, x+\delta_1]$ given that $Y$ belongs to a small interval $[y, y+\delta_2]$

$$\begin{aligned} P(x \leq X \leq x+\delta_1 | y \leq Y \leq y+\delta_2) &= \frac{P(x \leq X \leq x+\delta_1, y \leq Y \leq y+\delta_2)}{P(y \leq Y \leq y+\delta_2)} \\ &\approx \frac{f_{X,Y}(x,y)\delta_1\delta_2}{f_Y(y)\delta_2} \\ &= f_{X|Y}(x|y)\delta_1 \end{aligned}$$

- **Since $f_{X|Y}(x|y)\delta_1$ does not depend on $\delta_2$, we can think of the limiting case when $\delta_2 \to 0$ and so we get**

$$P(x \leq X \leq x+\delta_1 | Y = y) = \lim_{\delta_2 \to 0} P(x \leq X \leq x+\delta_1 | y \leq Y \leq y+\delta_2) \approx f_{X|Y}(x|y)\delta_1 \quad \delta_1 \text{ small}$$

- In general, for any region $A$, we have that

$$P(X \in A | Y = y) = \lim_{\delta \to 0} P(X \in A | y \leq Y \leq y+\delta) = \int_{x \in A} f_{X|Y}(x|y)dx$$

## 5 Expectation, Independence, Joint & Conditional CDF, Bayes rule

- **Expectation:** See page 172 for $E[g(X)|Y = y]$, $E[g(X,Y)|Y = y]$ and total expectation theorem for $E[g(X)]$ and for $E[g(X,Y)]$.
- **Independence:** $X$ and $Y$ are independent iff $f_{X|Y} = f_X$ (or iff $f_{X,Y} = f_X f_Y$, or iff $f_{Y|X} = f_Y$)
- If X and Y independent, any two events $\{X \in A\}$ and $\{Y \in B\}$ are independent.
- If X and Y independent, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ and $Var[X+Y] = Var[X]+Var[Y]$ Exercise: show this.
- **Joint CDF:**

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f_{X,Y}(s,t)dsdt$$

- Obtain joint PDF from joint CDF:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$$

- **Conditional CDF:**

$$F_{X|Y}(x|y) = P(X \leq x | Y = y) = \lim_{\delta \to 0} P(X \leq x | y \leq Y \leq y + \delta) = \int_{t=-\infty}^{x} f_{X|Y}(t|y) dt$$

- **Bayes rule when unobserved phenomenon is continuous.** Pg 175 and Example 3.18

- **Bayes rule when unobserved phenomenon is discrete.** Pg 176 and Example 3.19. For e.g., say discrete r.v. $N$ is the unobserved phenomenon. Then for $\delta$ small,

$$\begin{aligned} P(N = i | X \in [x, x+\delta]) &= P(N = i | X \in [x, x+\delta]) \\ &= \frac{P(n = i) P(X \in [x, x+\delta] | N = i)}{P(X \in [x, x+\delta])} \\ &\approx \frac{p_N(i) f_{X|N=i}(x) \delta}{\sum_j p_N(j) f_{X|N=j}(x) \delta} \\ &= \frac{p_N(i) f_{X|N=i}(x)}{\sum_j p_N(j) f_{X|N=j}(x)} \end{aligned}$$

  Notice that the right hand side is independent of $\delta$. Thus we can take $\lim_{\delta \to 0}$ on both sides and the right side will not change. Thus we get

$$P(N = i | X = x) = \lim_{\delta \to 0} P(N = i | X \in [x, x+\delta]) = \frac{p_N(i) f_{X|N=i}(x)}{\sum_j p_N(j) f_{X|N=j}(x)}$$

- More than 2 random variables (Pg 178, 179) **

# 6 Derived distributions: PDF of $g(X)$ and of $g(X,Y)$

- **Obtaining PDF of $Y = g(X)$.** ALWAYS use the following 2 step procedure:
  - Compute CDF first. $F_Y(y) = P(g(X) \leq y) = \int_{x|g(x) \leq y} f_X(x) dx$
  - Obtain PDF by differentiating $F_Y$, i.e. $f_Y(y) = \frac{\partial F_Y}{\partial y}(y)$

- Example 3.20, 3.21, 3.22

- **Special Case 1: Linear Case: $Y = aX + b$.** Can show that

$$f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$$

  Proof: see Pg 183.

- Example 3.23, 3.24

- **Special Case 2: Strictly Monotonic Case.**

- Consider $Y = g(X)$ with $g$ being a **strictly monotonic** function of $X$.
- Thus $g$ is a one to one function.
- Thus there exists a function $h$ s.t. $y = g(x)$ iff $x = h(y)$ (i.e. $h$ is the inverse function of $g$, often denotes as $h \triangleq g^{-1}$).
- Then can show that

$$f_Y(y) = f_X(h(y))|\frac{dh}{dy}(y)|$$

- Proof for strictly monotonically increasing $g$:
  $F_Y(y) = P(g(X) \leq Y) = P(X \leq h(Y)) = F_X(h(y))$.
  Differentiate both sides w.r.t $y$ (apply chain rule on the right side) to get:

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{dF_X(h(y))}{dy} = f_X(h(y))\frac{dh}{dy}(y)$$

  For strictly monotonically decreasing $g$, using a similar procedure, we get $f_Y(y) = -f_X(h(y))\frac{dh}{dy}(y)$
- See Figure 3.22, 3.23 for intuition

* Example 3.21 (page 186)
* **Functions of two random variables.** Again use the 2 step procedure, first compute CDF of $Z = g(X, Y)$ and then differentiate to get the PDF.
* CDF of $Z$ is computed as: $F_Z(z) = P(g(X, Y) \leq z) = \int_{x,y:g(x,y)\leq z} f_{X,Y}(x, y)dydx$.
* Example 3.26, 3.27
* Example 3.28
* Special case 1: PDF of $Z = e^{sX}$ (moment generating function): Chapter 4, 4.1
* Special case 2: PDF of $Z = X + Y$ when $X$, $Y$ are independent: convolution of PDFs of X and Y: Chapter 4, 4.2