High-resolution Parametric Subspace Methods

The first parametric subspace-based method was the *Pisarenko method*,, which was further modified, leading to the MUltiple SIgnal Classification (MUSIC) method.

MUSIC Method: Recall the model

$$\boldsymbol{x}(t) = A\boldsymbol{s}(t) + \boldsymbol{n}(t),$$

where A is a Vandermonde matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_L} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-j\omega_1(N-1)} & e^{-j\omega_2(N-1)} & \cdots & e^{-j\omega_L(N-1)} \end{bmatrix}.$$

$$\begin{aligned} R &= \mathrm{E}\left\{\boldsymbol{x}(t)\boldsymbol{x}^H(t)\right\} \\ &= A\mathrm{E}\left\{\boldsymbol{s}(t)\boldsymbol{s}^H(t)\right\}A^H + \sigma^2 I \\ &= ASA^H + \sigma^2 I, \end{aligned}$$

and $S = \mathrm{E}\left\{\boldsymbol{s}(t)\boldsymbol{s}^H(t)\right\}$ is a full-rank diagonal matrix.

From the eigenanalysis of R, we see that

- the eigenvalues of R are:

$$\begin{aligned} \lambda_k &> \sigma^2, \quad k = 1, \ldots, L \quad \text{signal subspace} \\ \lambda_k &= \sigma^2, \quad k = L+1, \ldots, N \quad \text{noise subspace,} \end{aligned}$$

- the noise subspace eigenvectors are orthogonal to the column space of A.

Let the signal and noise subspace eigenvectors be given by

$$\begin{aligned} E &= [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_L] \quad \text{signal subspace,} \\ G &= [\boldsymbol{u}_{L+1}, \boldsymbol{u}_{L+2}, \ldots, \boldsymbol{u}_N] \quad \text{noise subspace.} \end{aligned}$$

Then

$$RG = \sigma^2 G.$$

On the other hand

$$RG = (ASA^H + \sigma^2 I)G = ASA^H G + \sigma^2 G.$$

Comparing the above two equations, we conclude

$$A^H G = \mathbf{0}.$$

The *noise subspace* eigenvectors of R are *orthogonal* to the columns of A. In turn, the *signal-subspace* eigenvectors span the same subspace as the column space of A.

The true frequencies $\{\omega\}_{l=1}^{L}$ are the solutions to

$$\boldsymbol{a}^H(\omega)GG^H\boldsymbol{a}(\omega) = N - \boldsymbol{a}^H(\omega)EE^H\boldsymbol{a}(\omega) = 0$$

where

$$GG^H = P_A^{\perp}, \quad EE^H = I - GG^H = P_A.$$

MUSIC Spectral Estimate:

$$P_{\text{MUSIC}}(\omega) = \frac{1}{\boldsymbol{a}^H(\omega)GG^H\boldsymbol{a}(\omega)} = \frac{1}{N - \boldsymbol{a}^H(\omega)EE^H\boldsymbol{a}(\omega)}.$$

In practice, we do not know R, so:

$$\widehat{P}_{\text{MUSIC}}(\omega) = \frac{1}{\boldsymbol{a}^H(\omega)\widehat{G}\widehat{G}^H\boldsymbol{a}(\omega)} = \frac{1}{N - \boldsymbol{a}^H(\omega)\widehat{E}\widehat{E}^H\boldsymbol{a}(\omega)}.$$

where

$$\widehat{R} = \frac{1}{K}\sum_{k=1}^{K}\boldsymbol{x}(k)\boldsymbol{x}^H(k),$$

$$\widehat{G}\widehat{G}^H = \widehat{P}_A^{\perp}, \qquad \widehat{E}\widehat{E}^H = I - \widehat{G}\widehat{G}^H = \widehat{P}_A.$$

Remarks:

- the number of signals L must be *known* or *estimated*,
- MUSIC involves *eigendecomposition*,
- computational cost can be quite intensive if the MUSIC estimate is evaluated with a *fine grid*.

Root-MUSIC Algorithm

Instead of computing the spectral MUSIC estimate, root the polynomial

$$\boldsymbol{a}^T(1/z)\widehat{G}\widehat{G}^H\boldsymbol{a}(z) = 0,$$

where

$$a(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-N+1} \end{bmatrix}.$$

Remarks:

- MUSIC polynomial has the order $2N-2$, and, therefore, $2N-2$ roots,
- the roots form $N-1$ pairs, where one root is the conjugate reciprocal of another, i.e. if z is a root, then $1/z^*$ will be a root as well.

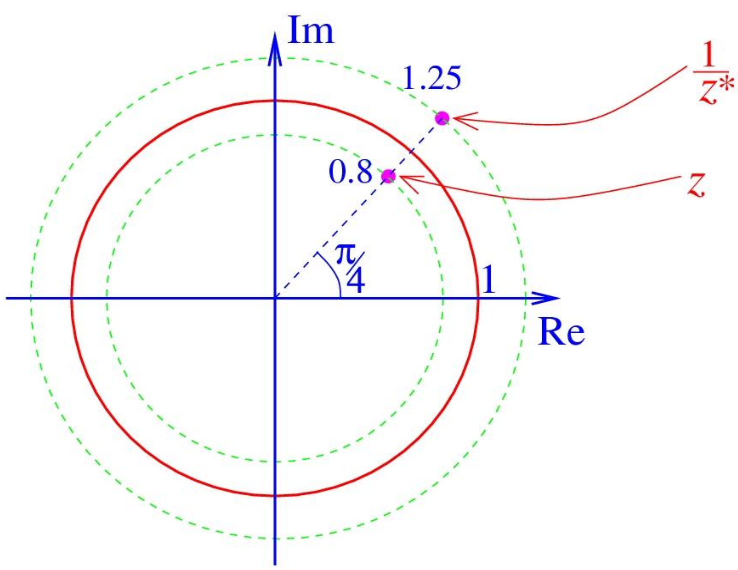
$$\begin{aligned} 0 &= \boldsymbol{a}^T(1/z)\widehat{G}\widehat{G}^H\boldsymbol{a}(z) = [\boldsymbol{a}^T(1/z)\widehat{G}\widehat{G}^H\boldsymbol{a}(z)]^H \\ &= \boldsymbol{a}^T(z^*)\widehat{G}\widehat{G}^H\boldsymbol{a}(1/z^*) \\ &= \boldsymbol{a}^T(1/\widetilde{z})\widehat{G}\widehat{G}^H\boldsymbol{a}(\widetilde{z}) \quad \text{with } \widetilde{z} = \frac{1}{z^*}. \end{aligned}$$

If $z \equiv$ root, then $1/z^* \equiv$ root too!

Example: Let $z = 0.8\, e^{j\pi/4}$ be a root of the MUSIC polynomial. Then, the conjugate-reciprocal root is

$$\frac{1}{z^*} = \frac{1}{0.8 e^{-j\pi/4}} = 1.25\, e^{j\pi/4}.$$

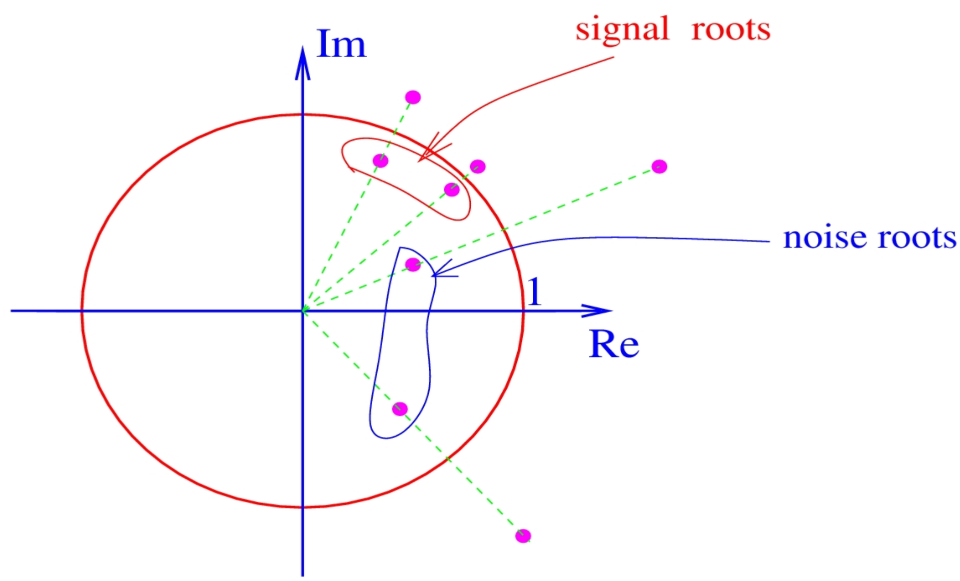
Thus, the *angle* of the root does not *change* but it lies at the opposite side of the unit circle.



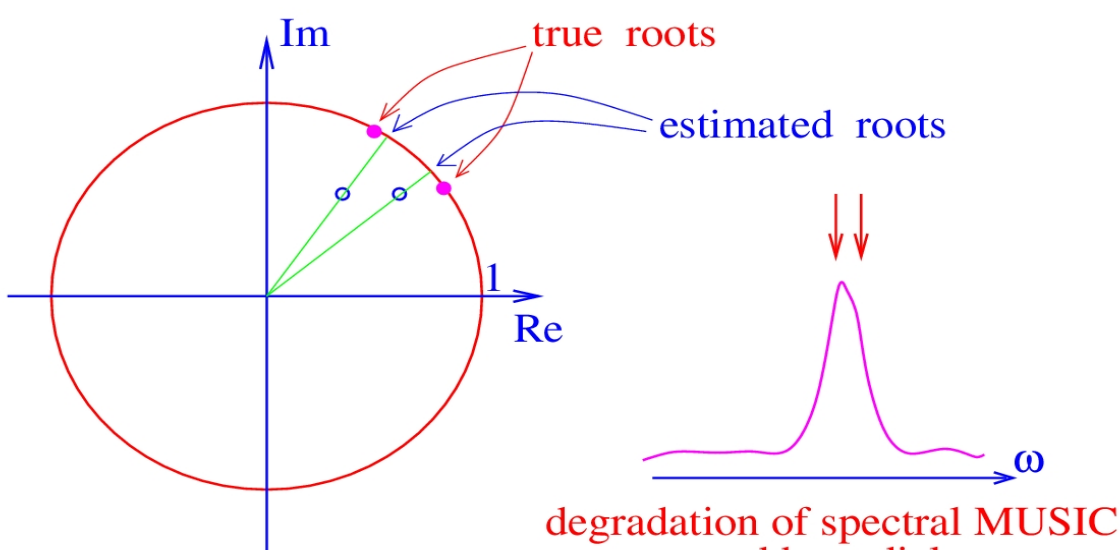
Select only the roots that lie *inside* the unit circle. Estimates of signal frequencies:

$$\widehat{\omega} = \angle\{z_l\}, \quad l = 1, 2, \ldots, L$$

using L roots closest to the unit circle (so-called *signal roots*) with $|z| \leq 1$.



Root-MUSIC has *better performance* at low SNR or small number of samples than spectral MUSIC because it is *insensitive to radial errors* (more precisely, sensitive only to errors that cause subspace swapping).



Root-MUSIC has *much simpler implementation* than spectral MUSIC because it does not require any search over frequency.

Minimum-norm Method

$$\widehat{P}_{\mathrm{MN}}(\omega) = \frac{1}{|\boldsymbol{a}^H(\omega)\boldsymbol{w}|^2},$$

where the vector $\boldsymbol{w}$

- has the first element equal to 1 and minimum norm,
- $\boldsymbol{w}$ belongs to the sample noise subspace.

Optimization problem:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H\boldsymbol{w} \quad \text{subject to} \quad \boldsymbol{w}^H\boldsymbol{e}_1 = 1, \quad \widehat{G}\widehat{G}^H\boldsymbol{w} = \boldsymbol{w}.$$

Substituting the second constraint into the objective function and first constraint yields:

$$\begin{aligned}
\boldsymbol{w}^H\boldsymbol{w} &= \boldsymbol{w}^H\widehat{G}\underbrace{\widehat{G}^H\widehat{G}}_{I}\widehat{G}^H\boldsymbol{w} = \boldsymbol{w}^H\widehat{G}\widehat{G}^H\boldsymbol{w}, \\
\boldsymbol{w}^H\boldsymbol{e}_1 &= \boldsymbol{w}^H\widehat{G}\widehat{G}^H\boldsymbol{e}_1 = 1.
\end{aligned}$$

With these expressions, our optimization problem transforms to

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H\widehat{G}\widehat{G}^H\boldsymbol{w} \quad \text{subject to} \quad \boldsymbol{w}^H\widehat{G}\widehat{G}^H\boldsymbol{e}_1 = 1.$$

Hence,

$$Q(\boldsymbol{w}) = \boldsymbol{w}^H \widehat{G}\widehat{G}^H \boldsymbol{w} + \lambda(1 - \boldsymbol{w}^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1) + \lambda^*(1 - \boldsymbol{e}_1^H \widehat{G}\widehat{G}^H \boldsymbol{w})$$

$$(\nabla Q)^* = \widehat{G}\widehat{G}^H \boldsymbol{w} - \lambda \widehat{G}\widehat{G}^H \boldsymbol{e}_1 \quad \Longrightarrow \quad \widehat{G}\widehat{G}^H \boldsymbol{w} = \lambda \widehat{G}\widehat{G}^H \boldsymbol{e}_1.$$

Substituting the solution $\widehat{G}\widehat{G}^H \boldsymbol{w} = \lambda \widehat{G}\widehat{G}^H \boldsymbol{e}_1$ to the constraint equation $\boldsymbol{w}^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1 = 1$, we get

$$\lambda^* \boldsymbol{e}_1^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1 = 1 \quad \Longrightarrow \lambda = \frac{1}{\boldsymbol{e}_1^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1}.$$

Finally,

$$\boldsymbol{w} = \lambda \widehat{G}\widehat{G}^H \boldsymbol{e}_1 = \frac{1}{\boldsymbol{e}_1^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1} \widehat{G}\widehat{G}^H \boldsymbol{e}_1,$$

where we used the second constraint. Substituting this solution into the expression for $\widehat{P}_{\mathrm{MN}}(\omega)$, we get

$$\widehat{P}_{\mathrm{MN}}(\omega) = \frac{1}{|\boldsymbol{a}^H(\omega)\boldsymbol{w}|^2} = \frac{(\boldsymbol{e}_1^H \widehat{G}\widehat{G}^H \boldsymbol{e}_1)^2}{|\boldsymbol{a}^H(\omega)\widehat{G}\widehat{G}^H \boldsymbol{e}_1|^2}.$$

The constant in the numerator does not alter the shape of the spectrum and, as a rule, is omitted:

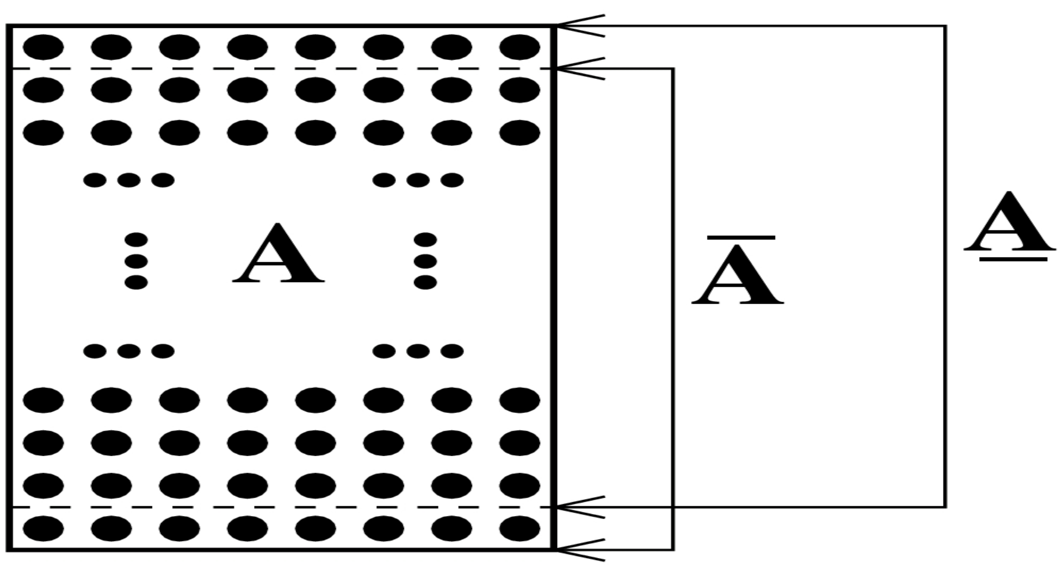
$$\widehat{P}_{\mathrm{MN}}(\omega) = \frac{1}{|\boldsymbol{a}^H(\omega)\widehat{G}\widehat{G}^H \boldsymbol{e}_1|^2} = \frac{1}{|1 - \boldsymbol{a}^H(\omega)\widehat{E}\widehat{E}^H \boldsymbol{e}_1|^2}.$$

ESPRIT Method

Consider

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_L} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-j\omega_1(N-1)} & e^{-j\omega_2(N-1)} & \cdots & e^{-j\omega_L(N-1)} \end{bmatrix}.$$

Let $\overline{A}$ and $\underline{A}$ be the matrices with eliminated first and last row, respectively.



It can be readily shown that

$$\begin{aligned} \overline{A}D &= \underline{A}, \\ D &= \operatorname{diag}\{e^{j\omega_1}, \ldots, e^{j\omega_L}\}. \end{aligned}$$

Let $\overline{E}$ and $\underline{E}$ be formed from the signal eigenvector matrix E in the same way as $\overline{A}$ and $\underline{A}$ from A.

Recall that E and A span the same (signal) subspace. Therefore

$$E = AC \quad \Longrightarrow$$

$$\underline{E} = \underline{A}C = \overline{A}DC, \quad \overline{E} = \overline{A}C \Longrightarrow$$

$$\underline{E}C^{-1}D^{-1}C = \overline{A}\underbrace{DCC^{-1}D^{-1}}_{I}C = \overline{A}C = \overline{E}.$$

Equivalently

$$\underline{E} = \overline{E}C^{-1}DC \quad \Longrightarrow \underline{E} = \overline{E}\Psi$$

where

$$\Psi = C^{-1}DC. \qquad (*)$$

In practice, *both* C and D are *unknown*!

LS solution for Ψ:

$$\Psi = (\overline{E}^H\overline{E})^{-1}\overline{E}^H\underline{E}.$$

From $(*)$ it follows that the diagonal elements of D are the eigenvalues of Ψ!

ESPRIT:

Step 1: Compute the eigendecomposition of the sample covariance matrix $\widehat{R}$ and obtain the sample signal subspace $\widehat{E}$.

Step 2: Form the matrices $\underline{\widehat{E}}$ and $\overline{\widehat{E}}$.

Step 3: Compute

$$\widehat{\Psi} = (\overline{\widehat{E}}^H \overline{\widehat{E}})^{-1} \overline{\widehat{E}}^H \underline{\widehat{E}}.$$

Step 4: Form the eigenvalues $\widehat{\psi}_l, l = 1, 2, \ldots, L$ of $\widehat{\Psi}$ and obtain the frequency estimates as follows:

$$\widehat{\omega}_l = \angle \widehat{\psi}_l.$$

Model-fitting-based Parametric Spectral Analysis

Nonlinear LS: Recall low-rank modeling and obtain the frequencies by minimizing

$$\min_{S,\boldsymbol{\omega}} \left\{ \sum_{t=1}^{K} \|\boldsymbol{x}(t) - A(\boldsymbol{\omega})\boldsymbol{s}(t)\|^2 \right\} = \min_{S,\boldsymbol{\omega}} \|X - A(\boldsymbol{\omega})S\|^2 \quad \Longrightarrow$$

$$\min_{\boldsymbol{\omega}} \mathrm{tr}\{P_A^{\perp}(\boldsymbol{\omega})\widehat{R}\} \quad \Leftrightarrow \quad \max_{\boldsymbol{\omega}} \mathrm{tr}\{P_A(\boldsymbol{\omega})\widehat{R}\}.$$

Nonparametric and Parametric Methods: Relationship

Matrix Inversion Lemma:

$$(H + BCD)^{-1} = H^{-1} - H^{-1}B(C^{-1} + DH^{-1}B)^{-1}DH^{-1}$$

for *arbitrary square nonsingular* H and C.

Consider the familiar expression for the covariance matrix

$$R = ASA^H + \sigma^2 I$$

and apply *matrix inversion lemma* to obtain

$$\begin{aligned} R^{-1} &= (\sigma^2 I + ASA^H)^{-1} = \frac{1}{\sigma^2}(I + \frac{1}{\sigma^2}ASA^H)^{-1} \\ &= \frac{1}{\sigma^2}[I - A(\sigma^2 S^{-1} + A^H A)^{-1}A^H], \end{aligned}$$

which implies

$$\begin{aligned} \lim_{\sigma^2 \to 0}\{\sigma^2 R^{-1}\} &= \lim_{\sigma^2 \to 0}\{I - A(\sigma^2 S^{-1} + A^H A)^{-1}A^H\} \\ &= I - A(A^H A)^{-1}A^H = P_A^{\perp}. \end{aligned}$$

Compare Capon and MUSIC spectra:

$$P_{\text{CAPON}}(\omega) = \frac{1}{\boldsymbol{a}^H(\omega)R^{-1}\boldsymbol{a}(\omega)}, \ P_{\text{MUSIC}}(\omega) = \frac{1}{\boldsymbol{a}^H(\omega)P_A^{\perp}\boldsymbol{a}(\omega)}$$

as well as AR (max entropy) and min-norm spectra:

$$P_{\mathrm{AR}}(\omega) = \frac{1}{|\boldsymbol{a}^H(\omega) R^{-1} \boldsymbol{e}_1|^2}, \quad P_{\mathrm{MN}}(\omega) = \frac{1}{|\boldsymbol{a}^H(\omega) P_A^{\perp} \boldsymbol{e}_1|^2}.$$

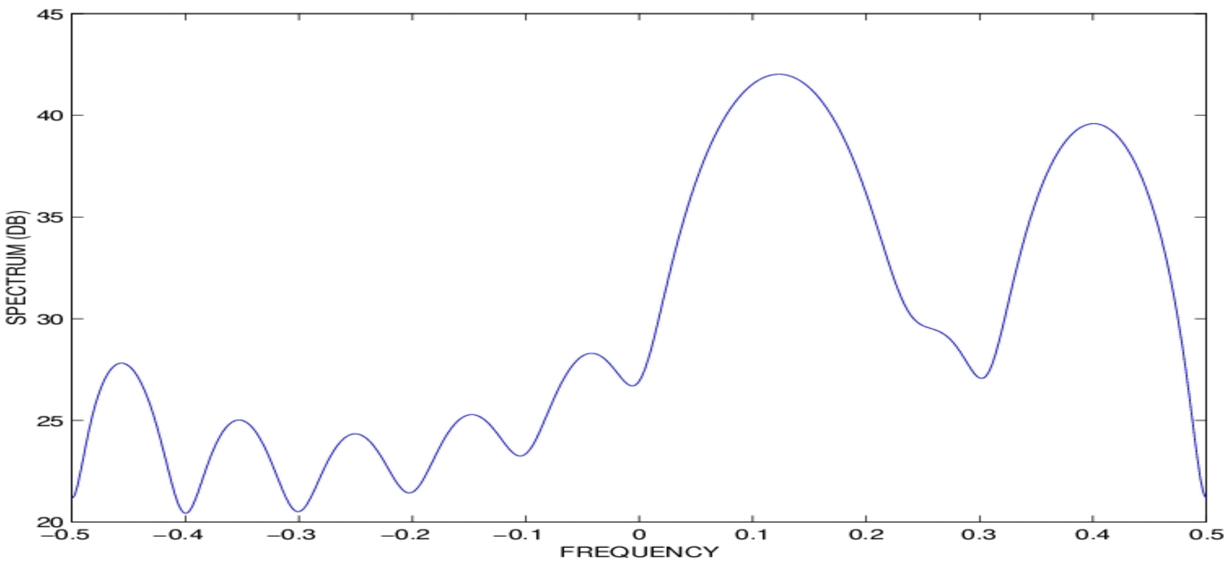
Clearly, for high SNR ($\sigma \to 0$), we obtain

$$\begin{aligned} P_{\mathrm{CAPON}}(\omega) &\sim P_{\mathrm{MUSIC}}(\omega), \\ P_{\mathrm{AR}}(\omega) &\sim P_{\mathrm{MN}}(\omega). \end{aligned}$$

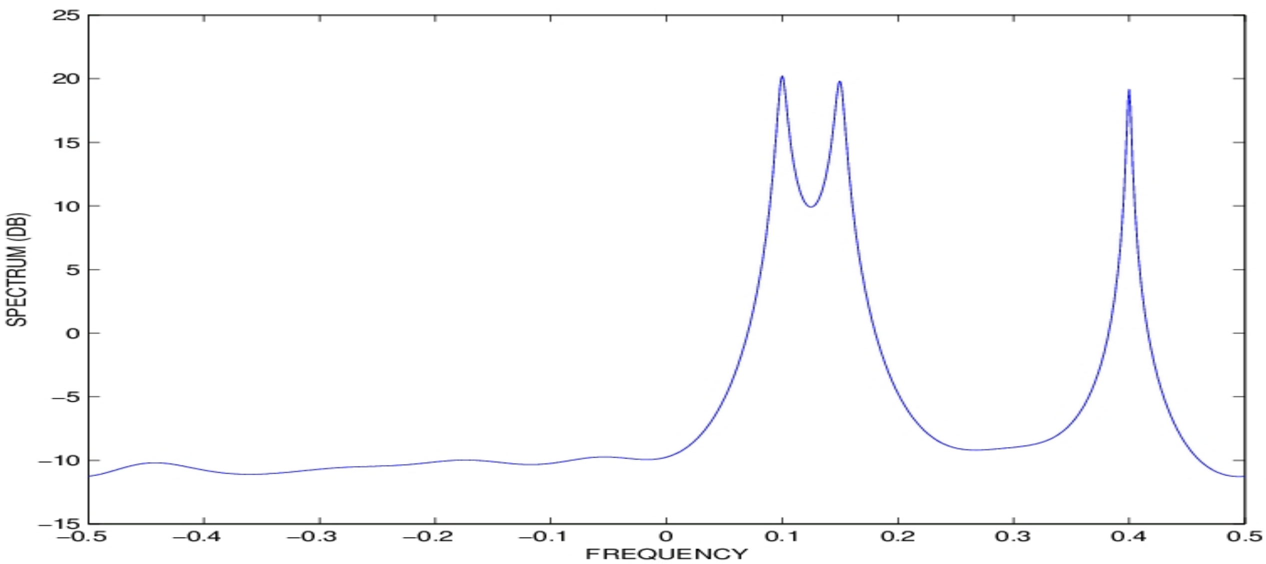
Matlab Example

- 3 equi-power $(A_1 = A_2 = A_3)$ complex exponentials with frequencies $f_1 = 0.1$, $f_2 = 0.15$, and $f_3 = 0.4$,
- zero-mean unit-variance complex Gaussian noise
- SNR and number of samples used to estimate the covariance matrix varied.

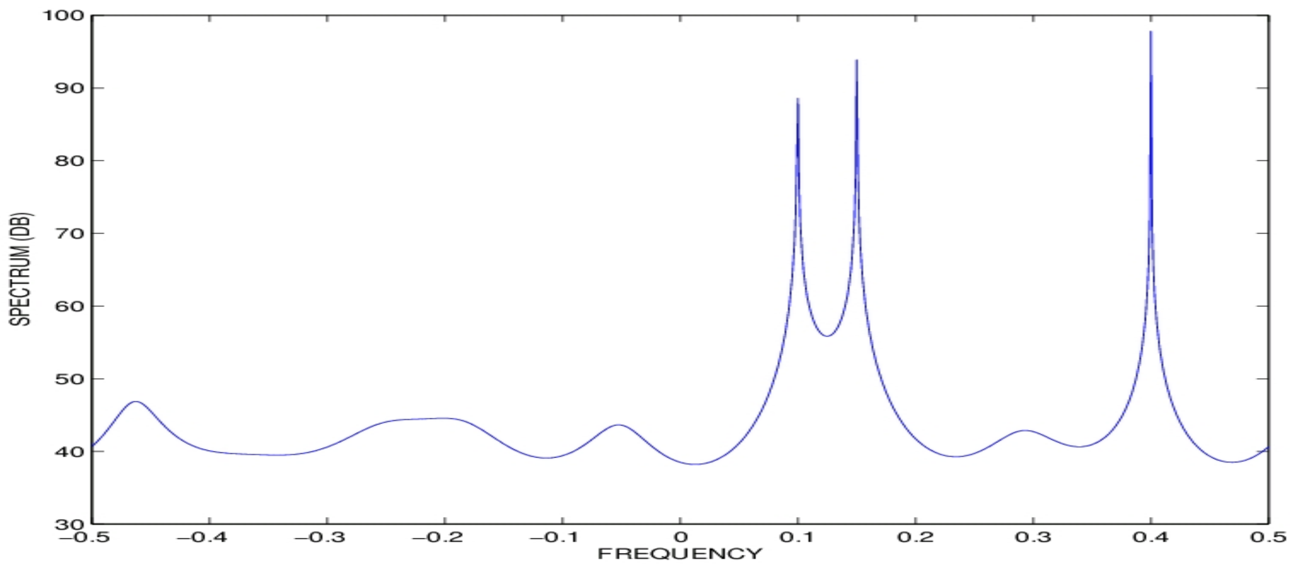
Blackman-Tukey Spectral Estimate, $\mathrm{SNR} = 100$ dB, 100 Samples



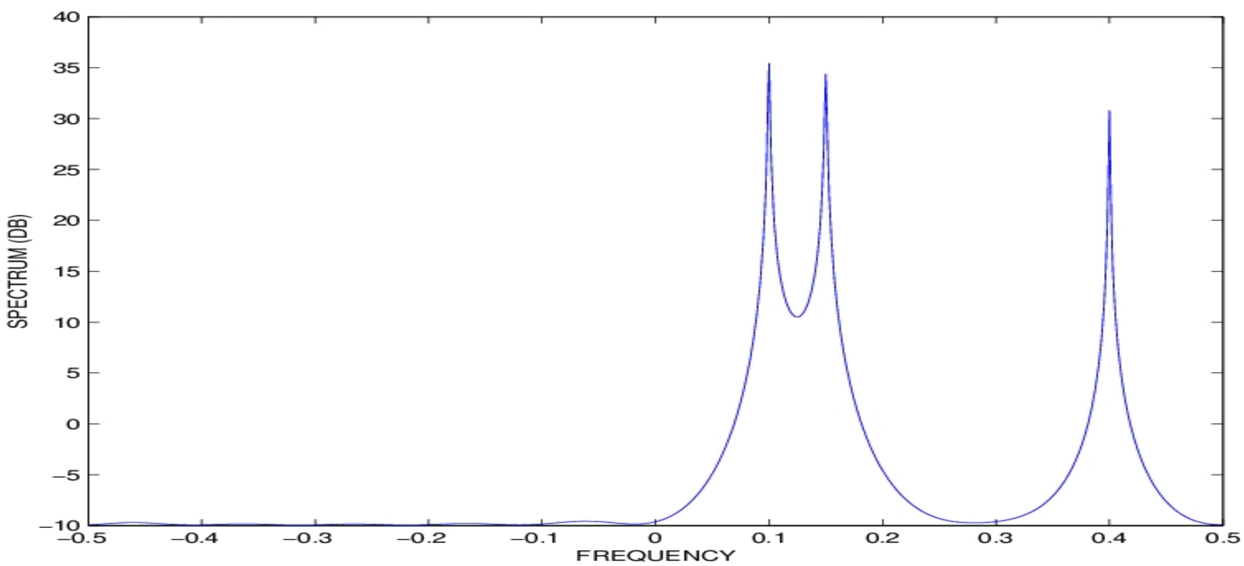
Capon Spectral Estimate, $\mathrm{SNR} = 100$ dB, 100 Samples



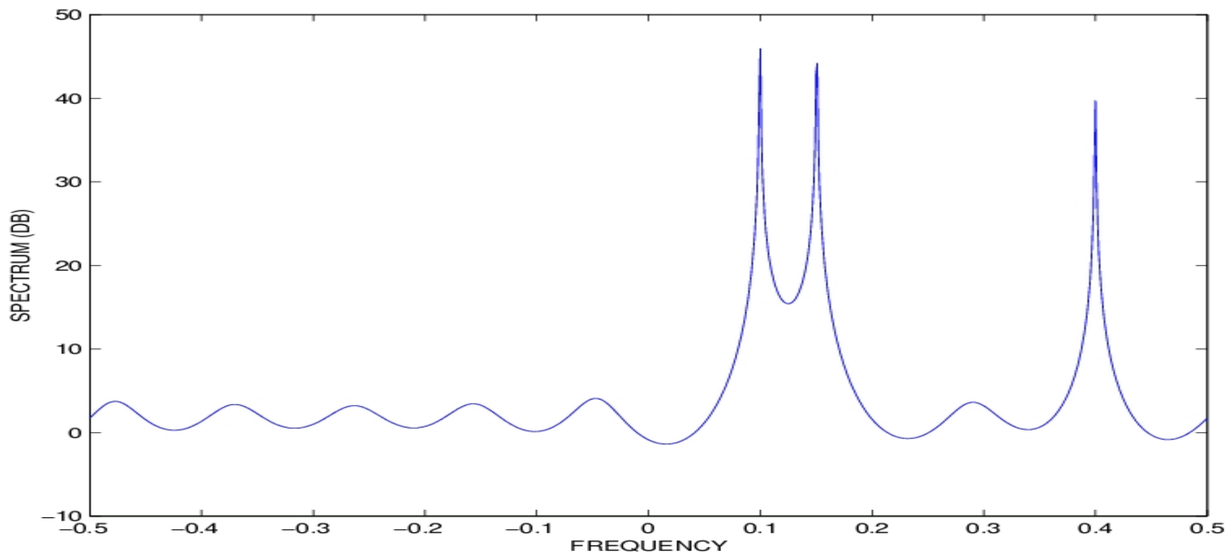
Maximum-entropy (AR) Spectral Estimate, $\mathrm{SNR} = 100$ dB, 100 Samples



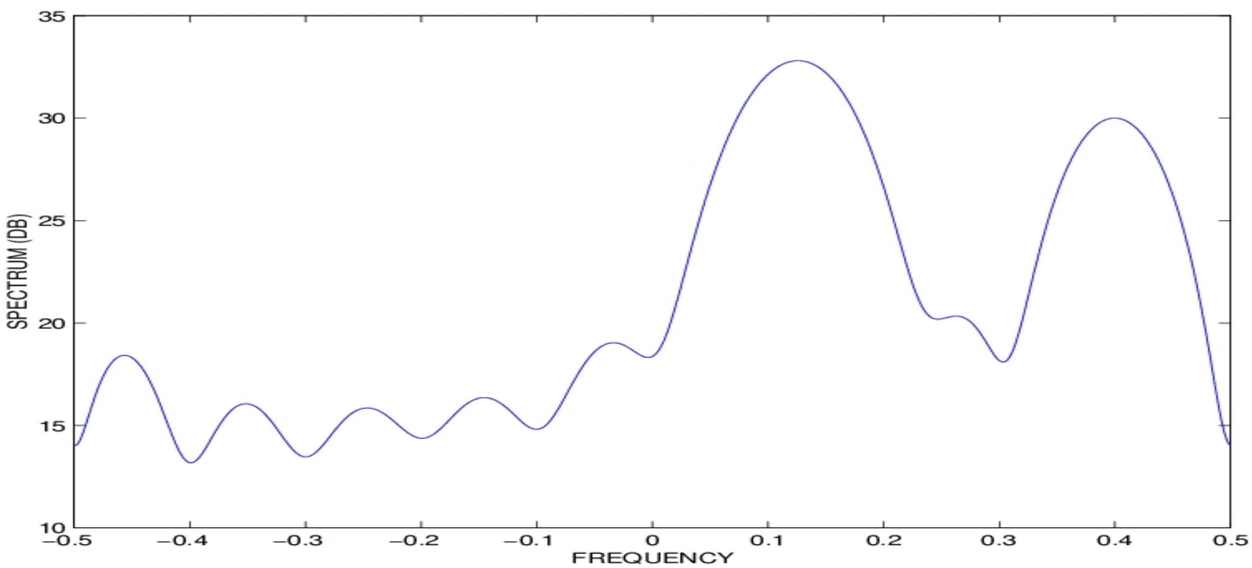
MUSIC Spectral Estimate, $\mathrm{SNR} = 100$ dB, 100 Samples



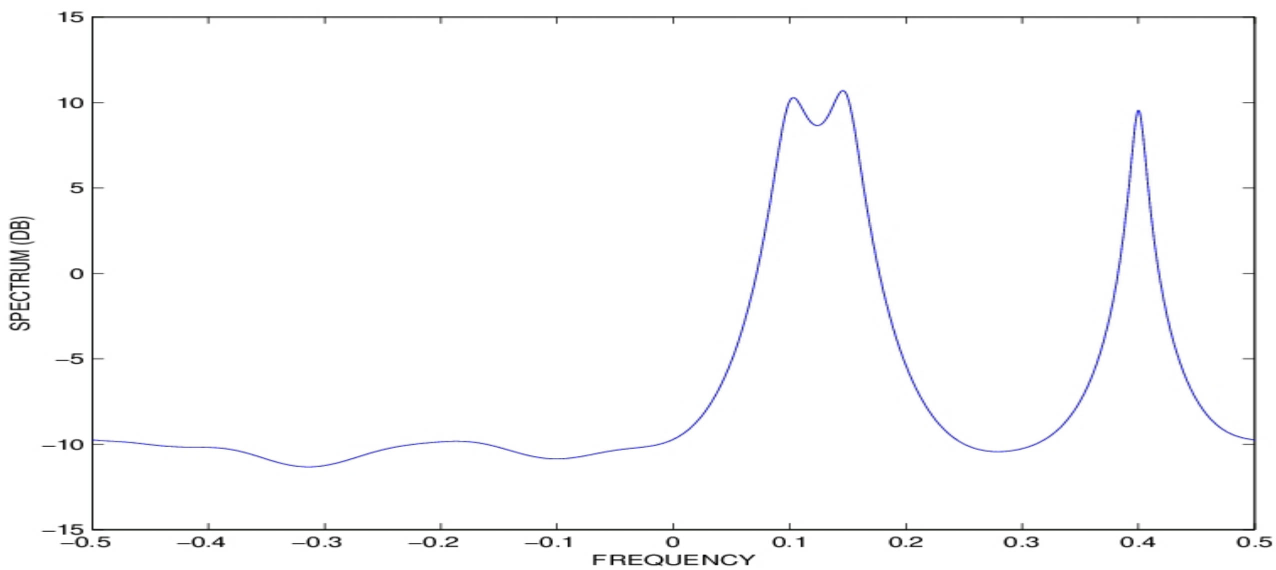
Minimum-norm Spectral Estimate, $\mathrm{SNR} = 100$ dB, 100 Samples



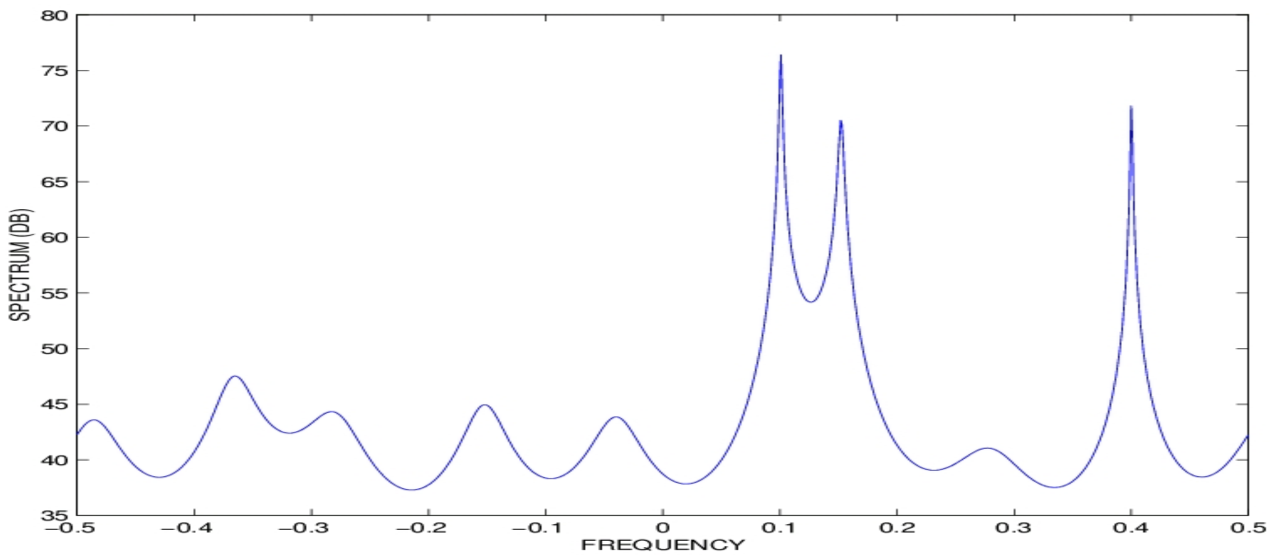
Blackman-Tukey Spectral Estimate, $\mathrm{SNR} = 10$ dB, 100 Samples



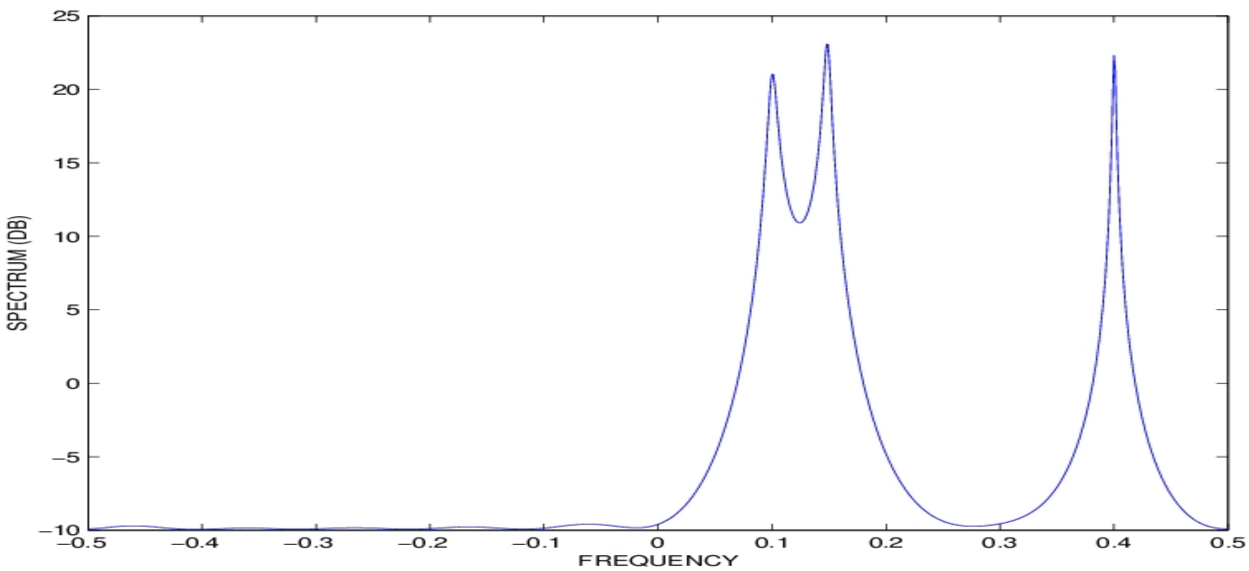
Capon Spectral Estimate, $\mathrm{SNR} = 10$ dB, 100 Samples



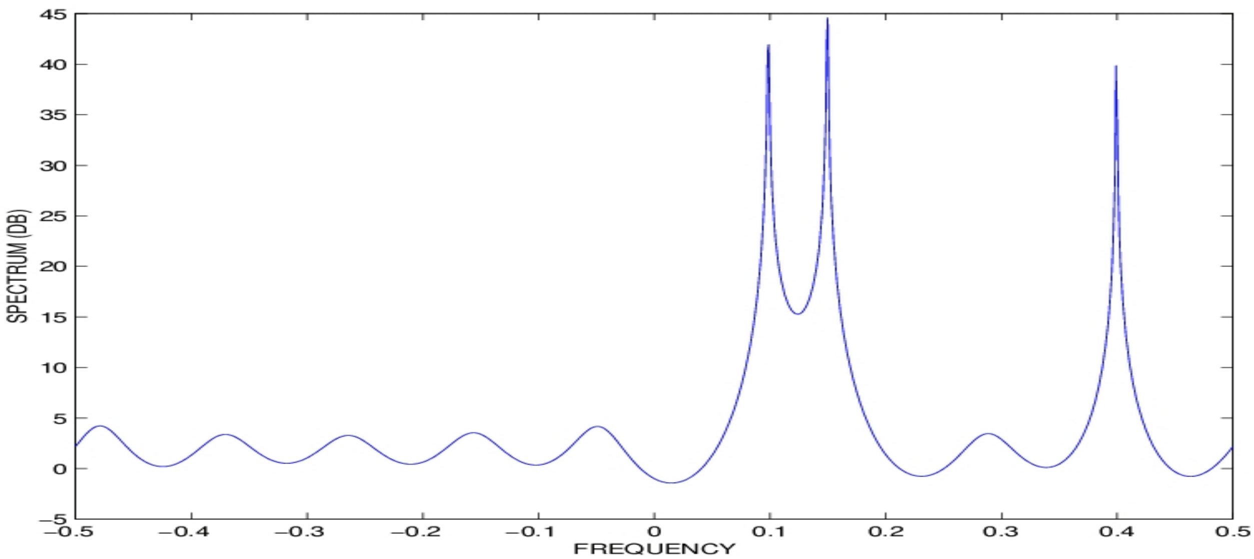
Maximum-entropy (AR) Spectral Estimate, $\mathrm{SNR} = 10$ dB, 100 Samples



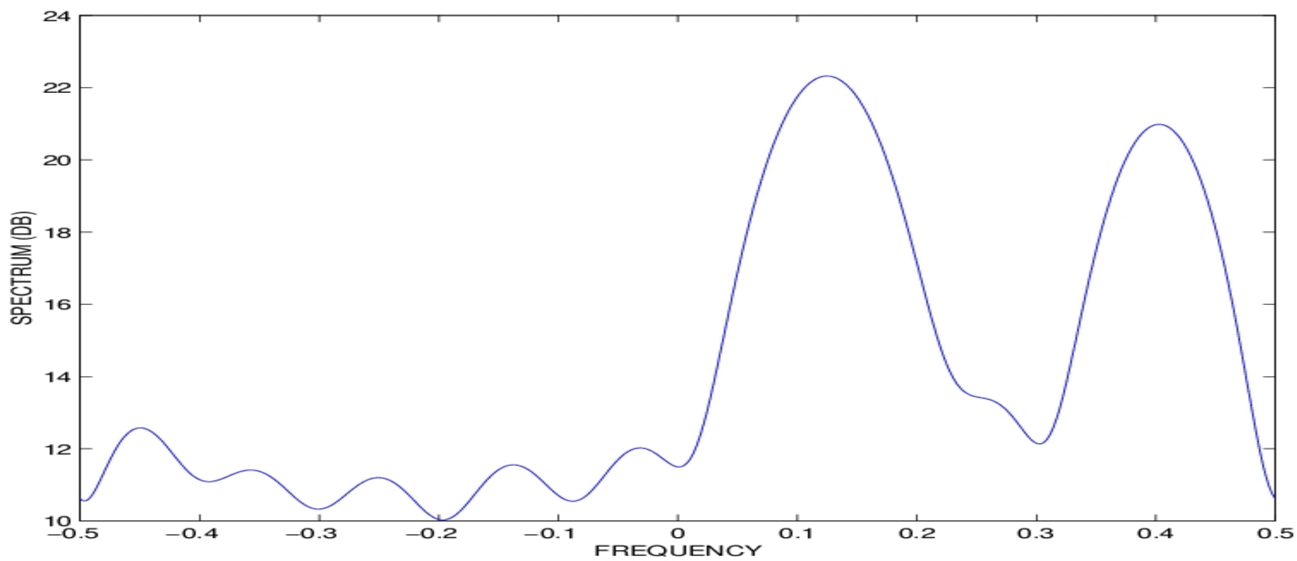
MUSIC Spectral Estimate, $\mathrm{SNR} = 10$ dB, 100 Samples



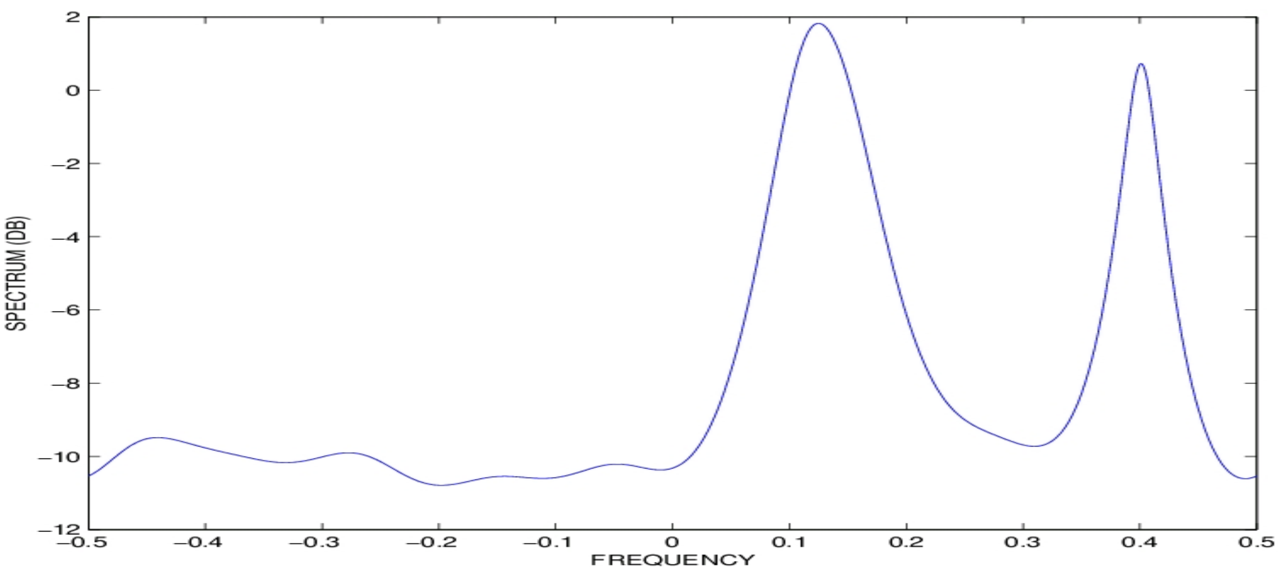
Minimum-norm Spectral Estimate, $\mathrm{SNR} = 10$ dB, 100 Samples



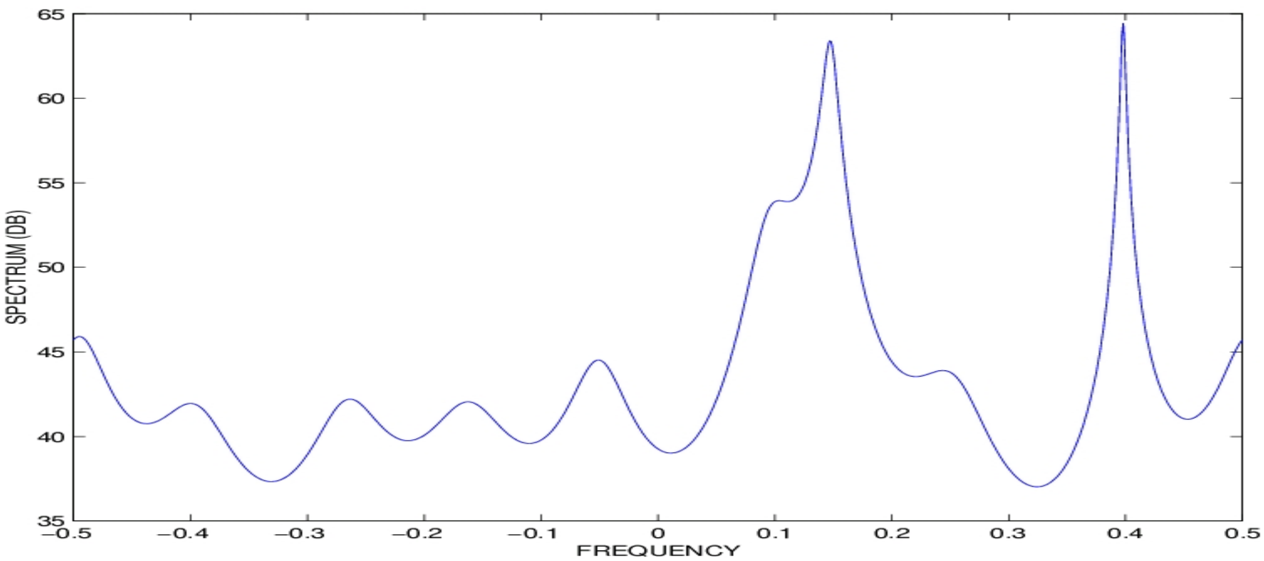
Blackman-Tukey Spectral Estimate, $\mathrm{SNR} = 0$ dB, 100 Samples



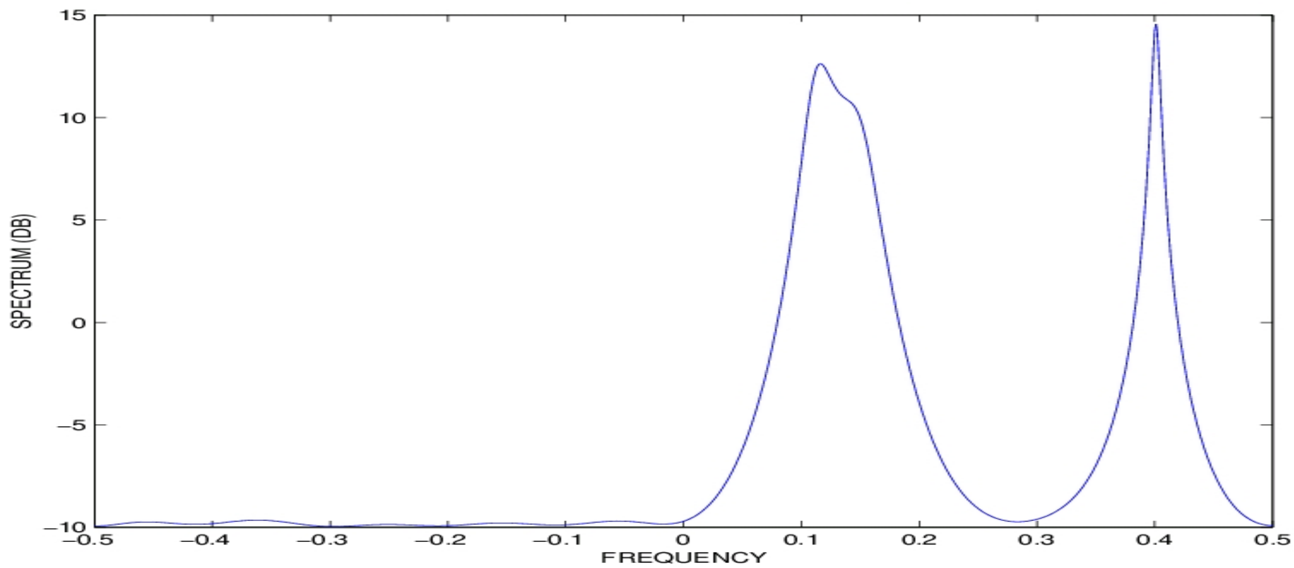
Capon Spectral Estimate, $\mathrm{SNR} = 0$ dB, 100 Samples



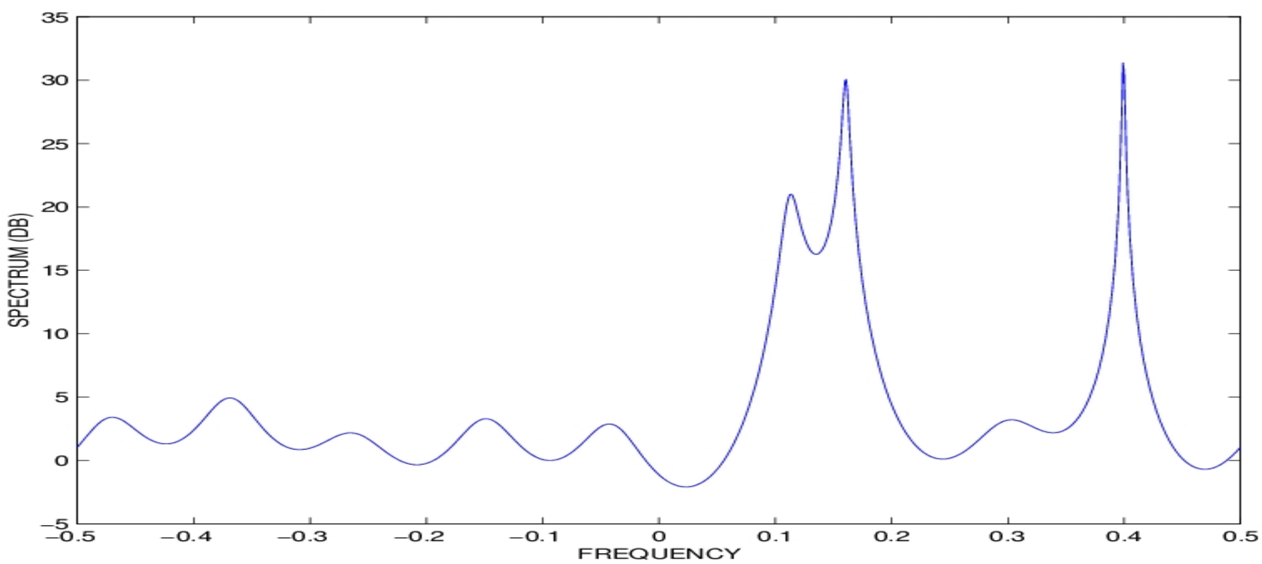
Maximum-entropy (AR) Spectral Estimate, $\mathrm{SNR} = 0$ dB, 100 Samples



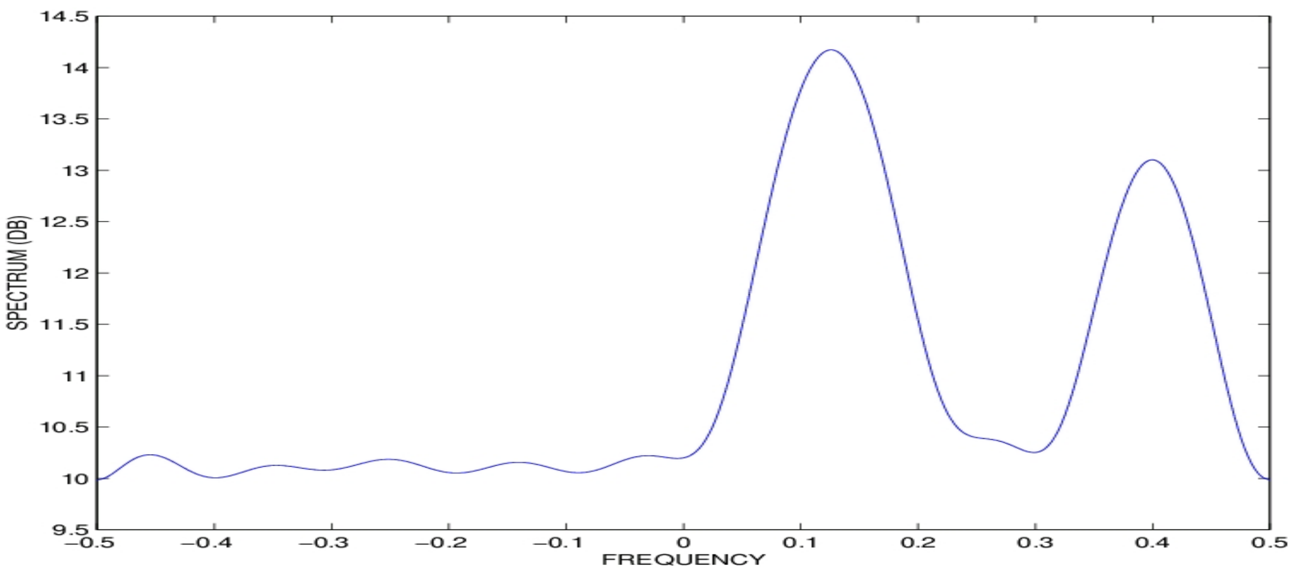
MUSIC Spectral Estimate, $\mathrm{SNR} = 0$ dB, 100 Samples



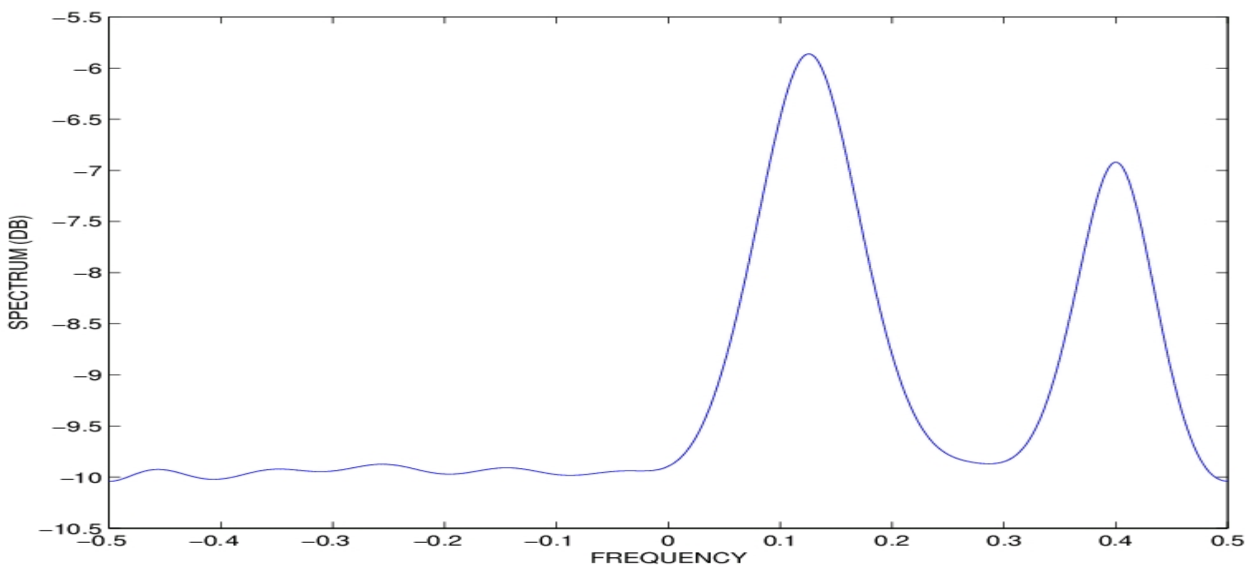
Minimum-norm Spectral Estimate, $\mathrm{SNR} = 0$ dB, 100 Samples



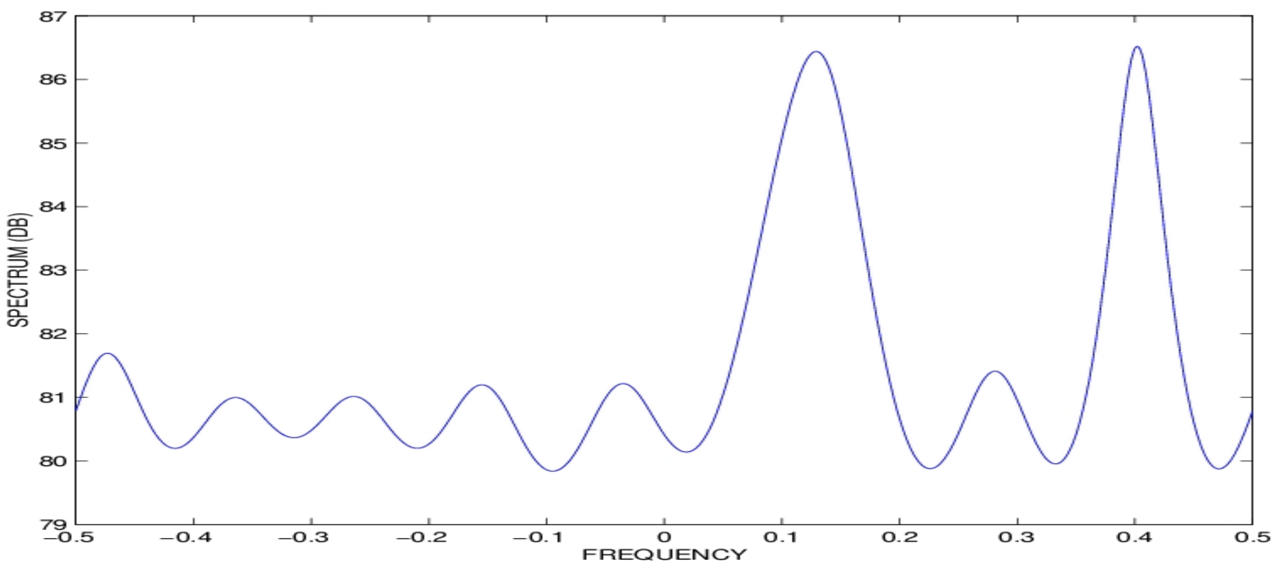
Blackman-Tukey Spectral Estimate, $\mathrm{SNR} = -10$ dB, 10000 Samples



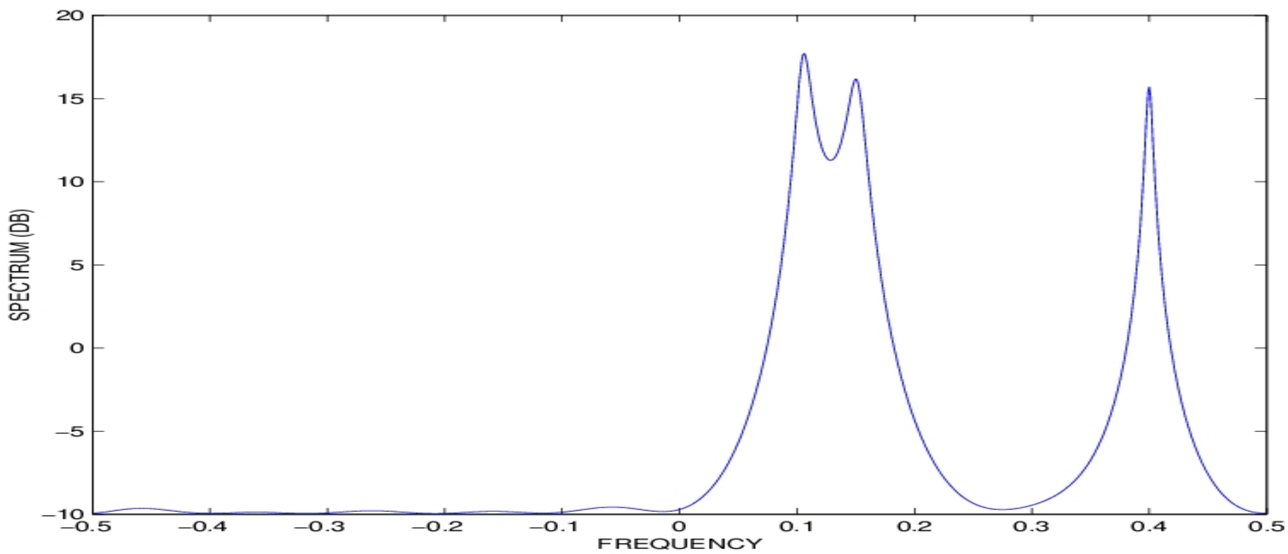
Capon Spectral Estimate, $\mathrm{SNR} = -10$ dB, 10000 Samples



Maximum-entropy (AR) Spectral Estimate, $\mathrm{SNR} = -10$ dB, 10000 Samples



MUSIC Spectral Estimate, $\mathrm{SNR} = -10$ dB, 10000 Samples



Minimum-norm Spectral Estimate, $\mathrm{SNR} = -10$ dB, 10000 Samples