Kundur writes (pg 110) that

“A rigorous treatment of synchronous machine performance including saturation effects is a futile exercise. Any practical method of accounting for saturation effects must be based on semi-heuristic reasoning and judiciously chosen approximations, with due consideration to simplicity of model structure, data availability, and accuracy of results.”

Some assumptions (see Kundur pg. 112-113):

1. Leakage inductances are independent of saturation since the path of the leakage flux is mainly in the air. Therefore we may confine our analysis of saturation to the mutual inductances, represented by $L_{AD}$ and $L_{AQ}$.

2. The leakage fluxes do not contribute to the iron saturation. This is reasonable because these fluxes are small (since their paths are mainly in air, and air has high permeability), and their paths coincide with that of the main flux for only a small part of its path. So we may determine saturation of the inductances as a function of $\lambda_{AD}$ and $\lambda_{AQ}$.

3. The saturation relationship between the resultant air-gap flux and the mmf under loaded conditions is the same as under no-load conditions. This allows the saturation characteristics to be represented by the open-circuit saturation curve, which is usually the only saturation data readily available.

An additional assumption that is sometimes made is that $L_{AQ}$ does not saturate, simply because the quadrature axis flux is usually quite small in comparison to the direct axis flux due to the effect of the main field winding. This assumption is quite good for salient pole machines but not so good for round-rotor machines.
Recall, from our equivalent circuit (shown below), that \( \lambda_{AD} = (i_d + i_F + i_D)L_{AD} \).

\[
\begin{align*}
v_d &= -r_d i_d - l_d i_d - L_{AD} [i_d + i_F + i_D] - \omega \lambda_q \\
-v_F &= -r_F i_F - l_F i_F - L_{AD} [i_d + i_F + i_D] \\
v_D &= 0 = -r_D i_D - l_D i_D - L_{AD} [i_d + i_F + i_D]
\end{align*}
\]

Direct-axis equivalent circuit:
The above is the same as Fig. 4.5 in your text

Define the following terms:
- Magnetization current: \( i_M = (i_d + i_F + i_D) \Rightarrow \lambda_{AD} = L_{AD} i_M \)
- Maximum per-unit flux linkage without saturation: \( \lambda_{ADT} \)
- \( i_{M0} \): current that would produce \( \lambda_{AD} \) if no saturation effects
- \( i_{MS} \): current that produces \( \lambda_{AD} \) with saturation effects
- \( \lambda' \): Flux linkage resulting from \( i_{MS} \) if no saturation effects

Define \( L_{AD0} \) as the inductance corresponding to the air-gap line. It is the inductance when \( i_M \) is small, i.e., it is the non-saturated inductance. Thus, \( \lambda' = L_{AD0} i_{MS} \).

The magnetization curve appears as in the following figure:
From the figure, we can write that: \[ \frac{i_{M0}}{\lambda_{AD}} = \frac{i_{MS}}{\lambda'} \Rightarrow \lambda_{AD} = \frac{i_{M0}}{i_{MS}} \lambda' \]

But from the air-gap line equation, \[ \lambda' = L_{AD0} i_{MS} \], and substitution of this relation into the previous one yields:

\[ \hat{\lambda}_{AD} = \frac{i_{M0}}{i_{MS}} \lambda' = \frac{i_{M0}}{i_{MS}} L_{AD0} i_{MS} \]

\[ (*) \]

Define \( K_S = \frac{i_{M0}}{i_{MS}} \). \( K_S \) is the fraction of the saturated current necessary to achieve the same flux linkage with no saturation (i.e., on the airgap line). Clearly, \( 0 < K_S \leq 1 \), where \( K_S \) close to 0 indicates a highly saturated \( i_{MS} \); \( K_S \) close to 1 indicates a non-saturated \( i_{MS} \). So here we need to recognize a very important feature: \( K_S \) depends on the saturation level which depends on \( \lambda_{AD} \). Substitution of \( K_S \) into \( (*) \) results in:

\[ \lambda_{AD} = \frac{i_{M0}}{i_{MS}} \lambda' = K_S L_{AD0} i_{MS} \]

So \( K_S \) is a factor that we use to account for the difference between the magnetization curve and the air-gap line. Observation: We are trying to use \( K_S \) to compute \( \lambda_{AD} \), yet \( K_S \) depends on \( \lambda_{AD} \), that is, \( \lambda_{AD} = K_S(\lambda_{AD}) L_{AD0} i_{MS} \). This is why Kundur wrote, “A rigorous treatment of…saturation effects is a futile exercise” (see p. 1 of these notes).
So how might we determine $K_S$?

Observe:

$$K_S = \frac{i_{M0}}{i_{MS}} = \frac{i_{M0}}{i_{M0} + \Delta i_M}$$

where $\Delta i_M = i_{MS} - i_{M0}$.

So evaluation of $K_S$ requires evaluation of $\Delta i_M$, and our problem is now to get $\Delta i_M$.

Note from Fig. 1 below that $\Delta i_M$ grows exponentially larger with $\lambda_{AD} - \lambda_{ADT}$.

![Diagram showing exponential growth of $\Delta i_M$](image)

**Fig. 1**

So we reason that a good approximation to $\Delta i_M$ is given by

$$\Delta i_M = A_S e^{B_S (\lambda_{AD} - \lambda_{ADT})}$$

So that

$$K_S = \frac{i_{M0}}{i_{M0} + A_S e^{B_S (\lambda_{AD} - \lambda_{ADT})}}$$
Now it is clear from the above that $K_S$ is a function of $\lambda_{AD}$, i.e.,

$$K_S = K_S(\lambda_{AD})$$

So that the mutual flux is given by

$$\lambda_{AD} = K_S(\lambda_{AD}) L_{AD0} i_{MS}$$

So how do we use it?

Assume that we have values for $\lambda_d$, $\lambda_F$, $\lambda_D$, $\lambda_q$, $\lambda_Q$, and $\lambda_G$. Then the steps for including saturation are:

1. Usually, two values of saturation are given that allow computation of $A_S$ and $B_S$ (these are usually called S1.0 and S1.2 – more on that on p. 8 below and in Section 5.9 of VMAF).
2. Use the auxiliary equations to obtain the unsaturated values of $\lambda_{AD}$ and $\lambda_{AQ}$:

$$\lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D$$

$$\lambda_{AQ} = \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G$$

where

$$\frac{1}{L_{MD}} = \left[ \frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D} \right]$$

$$\frac{1}{L_{MQ}} = \left[ \frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G} \right]$$

3. For a salient pole machine, let $\lambda = \lambda_{AD}$.
   
   For a round-rotor machine, let $\lambda = \sqrt{\lambda_{AD}^2 + \lambda_{AQ}^2}$

4. Check if $\lambda > \lambda_{ADT}$. If not, use the above unsaturated values. If so, proceed to step 5.

5. Obtain currents from 4.124, shown below:
6. Compute the magnetizing current as
   \[ i_{M0} = i_d + i_F + i_D \]

7. Compute \( K_S \) according to:
   \[ \Delta i_M = A_S e^{B_S (\lambda - \lambda_{ADT})} \]
   \[ i_{MS} = i_{M0} + \Delta i_M \]
   \[ K_S = \frac{i_{M0}}{i_{MS}} \]

8. Update \( \lambda_{AD} \) and \( \lambda_{AQ} \) according to:
   a. Replace \( L_{AD} \) with \( L_{AD} \leftarrow K_S L_{AD} \), and then compute:
      \[ \frac{1}{L_{MD}} = \left[ \frac{1}{L_{AD}} + \frac{1}{l_d} + \frac{1}{l_F} + \frac{1}{l_D} \right] \rightarrow \lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D \]
   b. If salient pole, then \( \lambda_{AQ} = \lambda_{AQ} \) (i.e., no change), but if round-rotor, then replace \( L_{AQ} \) with \( L_{AQ} \leftarrow K_S L_{AQ} \), and then compute:
      \[ \frac{1}{L_{MQ}} = \left[ \frac{1}{L_{AQ}} + \frac{1}{l_q} + \frac{1}{l_Q} + \frac{1}{l_G} \right] \rightarrow \lambda_{AQ} = \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G \]
And then you can use the updated values of $\lambda_{AD}$ and $\lambda_{AQ}$ in the following to perform a numerical integration and get the next time step...

\[
\dot{\lambda}_d = -\frac{r}{l_d} \lambda_d + \frac{r}{l_d} \lambda_{AD} - \omega \lambda_q - v_d \tag{4.126}
\]

\[
\dot{\lambda}_F = -\frac{r_F}{l_F} \lambda_F + \frac{r_F}{l_F} \lambda_{AD} + v_F \tag{4.128}
\]

\[
\dot{\lambda}_D = -\frac{r_D}{l_D} \lambda_D + \frac{r_D}{l_D} \lambda_{AD} \tag{4.129}
\]

\[
\dot{\lambda}_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \lambda_{AQ} + \omega \lambda_d - v_q \tag{4.130}
\]

\[
\dot{\lambda}_Q = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \lambda_{AQ} \tag{4.131a}
\]

\[
\dot{\lambda}_G = -\frac{r_G}{l_G} \lambda_G + \frac{r_G}{l_G} \lambda_{AQ} \tag{4.131b}
\]

\[
\dot{\omega} = \frac{T_m}{\tau_j} + \left[ \frac{\lambda_{AQ}}{l_q 3\tau_j} \lambda_d - \frac{\lambda_{AD} \lambda_q}{l_d 3\tau_j} \right] + \left[ \frac{-D}{\tau_j} \right] \omega \tag{4.133}
\]

\[
\dot{\delta} = \omega - 1 \tag{4.102}
\]

Relation to input data to most commercial stability programs:

The input requirements for characterizing generator saturation for most commercial-grade stability programs are in terms of a parameter called $S$, defined by

\[
S = \frac{\Delta i_M}{i_M 0}
\]

where $\Delta i_M = i_{MS} - i_{M0}$ as before.

Recall that

\[
K_S = \frac{i_{M0}}{i_{MS}} = \frac{i_{M0}}{i_{M0} + \Delta i_M}
\]
The relation between $S$ and $K_S$ is derived from the below:

$$S = \frac{\Delta i_M}{i_{M_0}} \Rightarrow S + 1 = \frac{\Delta i_M}{i_{M_0} + \Delta i_M} = \frac{i_{M_0} + \Delta i_M}{i_{M_0}}$$

$$\Rightarrow \frac{l}{S + 1} = \frac{i_{M_0}}{i_{M_0} + \Delta i_M} = K_S \Rightarrow S = \frac{l - K_S}{K_S}$$

The specific data entry into most programs (including PSS/E) is

- $S(1.0)$: value of $S$ when open circuit terminal voltage is 1.0pu
- $S(1.2)$: value of $S$ when open circuit terminal voltage is 1.2pu

Note that $S(1.2)$ should always be larger than $S(1.0)$. In the Diablo Canyon data, $S(1.0)$ is 0.0769 and $S(1.2)$ is 0.41. The corresponding values of $K_S$ are 0.9286 and 0.7092, respectively. Use of $S(1.0)$ and $S(1.2)$ to compute $A_S$ and $B_S$ is provided in Section 5.9.1.

Final note on saturation. Section 4.12.3 develops a model where saturation is neglected. Such a model is useful for linearized analysis (although so is the current state-space model).

The approach for developing the flux-linkage model without saturation, as in Section 4.12.3) is simple – just substitute the auxiliary equations, i.e., the expressions for $\lambda_{AD}$ and $\lambda_{AQ}$, i.e.,

$$\lambda_{AD} = \frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MD}}{l_F} \lambda_F + \frac{L_{MD}}{l_D} \lambda_D$$

$$\lambda_{AQ} = \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G$$

into the state equations:

$$\dot{\lambda}_d = -\frac{r}{l_d} \lambda_d + \frac{r}{l_d} \lambda_{AD} - \omega \lambda_q - v_d \quad (4.126)$$

$$\dot{\lambda}_F = -\frac{r}{l_F} \lambda_F + \frac{r}{l_F} \lambda_{AD} + v_F \quad (4.128)$$
\[
\dot{\lambda}_D = -\frac{r_D}{l_D} \lambda_D + \frac{r_D}{l_D} \dot{\lambda}_{AD} \quad (4.129)
\]
\[
\dot{\lambda}_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \dot{\lambda}_{AQ} + \omega \lambda_d - v_q \quad (4.130)
\]
\[
\dot{\lambda}_G = -\frac{r_G}{l_G} \lambda_G + \frac{r_G}{l_G} \dot{\lambda}_{AQ} \quad (4.131a)
\]
\[
\dot{\lambda}_Q = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \dot{\lambda}_{AQ} \quad (4.131b)
\]
\[
\dot{\omega} = \frac{T_m}{\tau_j} + \left[ \frac{\lambda_{AQ}}{l_q 3 \tau_j} \lambda_d - \frac{\lambda_{AD}}{l_d 3 \tau_j} \lambda_q \right] + \left[ \frac{-D}{\tau_j} \right] \omega \quad (4.133)
\]
\[
\dot{\delta} = \omega - 1 \quad (4.102)
\]

VMAF does this for the d-axis equations (\(\lambda_d, \lambda_F\), and \(\lambda_D\)) on p. 121 but just gives the results for the q-axis equations (\(\lambda_q, \lambda_G\), and \(\lambda_Q\)). And so we will do it for the q-axis equations, starting with the G-equation (4.131a); we also do it for the torque equation.

First, we handle the state equation for \(\lambda_G\).
\[
\dot{\lambda}_G = -\frac{r_G}{l_G} \lambda_G + \frac{r_G}{l_G} \left( \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G \right)
\]
\[
= -\frac{r_G}{l_G} \lambda_G + \frac{r_G L_{MQ}}{l_G l_Q} \lambda_q + \frac{r_G L_{MQ}}{l_G l_Q} \lambda_Q + \frac{r_G L_{MQ}}{l_G^2} \lambda_G
\]
\[
= -\frac{r_G}{l_G} \left( 1 - \frac{L_{MQ}}{l_G} \right) \lambda_G + \frac{r_G L_{MQ}}{l_G l_Q} \lambda_q + \frac{r_G L_{MQ}}{l_G l_Q} \lambda_Q
\]

Now for the \(\lambda_q\) equation (4.130):
\[
\dot{\lambda}_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \lambda_{AQ} + \omega \lambda_d - v_q = -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \left( \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G \right) + \omega \lambda_d - v_q
\]

\[
= -\frac{r}{l_q} \lambda_q + \frac{r}{l_q} \frac{L_{MQ}}{l_q} \lambda_q + \frac{r}{l_q} \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{r}{l_q} \frac{L_{MQ}}{l_G} \lambda_G + \omega \lambda_d - v_q
\]

\[
= -\frac{r}{l_q} \left( 1 + \frac{L_{MQ}}{l_q} \right) \lambda_q + \frac{r}{l_q} \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{r}{l_q} \frac{L_{MQ}}{l_G} \lambda_G + \omega \lambda_d - v_q
\]

And now for the \( \lambda_Q \) equation (4.131b):

\[
\dot{\lambda}_Q = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \lambda_{AQ} = -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \left( \frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G \right)
\]

\[
= -\frac{r_Q}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \frac{L_{MQ}}{l_q} \lambda_q + \frac{r_Q}{l_Q} \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{r_Q}{l_Q} \frac{L_{MQ}}{l_G} \lambda_G
\]

\[
= -\frac{r_Q}{l_Q} \left( 1 - \frac{L_{MQ}}{l_Q} \right) \lambda_Q + \frac{r_Q}{l_Q} \frac{L_{MQ}}{l_q} \lambda_q + \frac{r_Q}{l_Q} \frac{L_{MQ}}{l_G} \lambda_G
\]

And finally for the torque equation (4.133):

\[
\dot{\omega} = \frac{T_m}{\tau_j} + \left[ \frac{\lambda_{AQ}}{l_d 3 \tau_j} - \frac{\lambda_{AD}}{l_d 3 \tau_j} - \lambda_q \right] \omega
\]

\[
= \frac{T_m}{\tau_j} + \left[ \frac{\frac{L_{MQ}}{l_q} \lambda_q + \frac{L_{MQ}}{l_Q} \lambda_Q + \frac{L_{MQ}}{l_G} \lambda_G}{l_d 3 \tau_j} - \frac{\frac{L_{MD}}{l_d} \lambda_d + \frac{L_{MF}}{l_f} \lambda_f + \frac{L_{MD}}{l_D} \lambda_D}{l_d 3 \tau_j} \right] \left[ \frac{\lambda_q}{l_d 3 \tau_j} \right] \omega
\]

\[
= \frac{T_m}{\tau_j} + \left[ \frac{\frac{L_{MQ}}{l_q}}{l_d 3 \tau_j} \lambda_q + \frac{\frac{L_{MQ}}{l_Q}}{l_d 3 \tau_j} \lambda_Q + \frac{\frac{L_{MQ}}{l_G}}{l_d 3 \tau_j} \lambda_G - \frac{\frac{L_{MD}}{l_d}}{l_d 3 \tau_j} \lambda_d - \frac{\frac{L_{MF}}{l_f}}{l_d 3 \tau_j} \lambda_f - \frac{\frac{L_{MD}}{l_D}}{l_d 3 \tau_j} \lambda_D \right] \omega
\]

The above state equations are included in the state equations as follows:
\[
\begin{bmatrix}
\dot{\lambda}_d \\
\dot{\lambda}_f \\
\dot{\lambda}_d \\
\dot{\lambda}_q \\
\dot{\lambda}_q \\
\dot{\omega} \\
\dot{\delta}
\end{bmatrix} = 
\begin{bmatrix}
-\frac{r}{\ell_d} \left(1 - \frac{L_{MD}}{\ell_d}\right) & \frac{r}{\ell_F} \frac{L_{MD}}{\ell_F} & \frac{r}{\ell_D} \frac{L_{MD}}{\ell_D} & -\omega & 0 & 0 & 0 \\
-\frac{r_F}{\ell_F} \left(1 - \frac{L_{MD}}{\ell_F}\right) & \frac{r}{\ell_F} \frac{L_{MD}}{\ell_F} & \frac{r}{\ell_D} \frac{L_{MD}}{\ell_D} & 0 & 0 & 0 & 0 \\
\frac{r_D}{\ell_D} \frac{L_{MD}}{\ell_D} & \frac{r_D}{\ell_D} \frac{L_{MD}}{\ell_D} & \frac{r}{\ell_D} \left(1 - \frac{L_{MD}}{\ell_D}\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{r}{\ell_q} \left(1 - \frac{L_{MQ}}{\ell_q}\right) & \frac{r}{\ell_G} \frac{L_{MQ}}{\ell_G} & \frac{r}{\ell_q} \frac{L_{MQ}}{\ell_q} & 0 \\
0 & 0 & 0 & \frac{r_G}{\ell_G} \frac{L_{MQ}}{\ell_G} & \frac{r}{\ell_G} \left(1 - \frac{L_{MQ}}{\ell_G}\right) & \frac{r_G}{\ell_G} \frac{L_{MQ}}{\ell_G} & 0 \\
0 & 0 & 0 & \frac{r_Q}{\ell_Q} \frac{L_{MQ}}{\ell_Q} & \frac{r}{\ell_Q} \left(1 - \frac{L_{MQ}}{\ell_Q}\right) & \frac{r_Q}{\ell_Q} \frac{L_{MQ}}{\ell_Q} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{D}{\tau_j} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
-\nu_d \\
-\nu_f \\
-\nu_d \\
-\nu_q \\
-\nu_q \\
T_m \\
\tau_j \\
-1
\end{bmatrix}
\]

(4.138)
Comments:

1. There is a very good treatment of saturation in IEEE Std 1110-2002, “IEEE Guide for Synchronous Generator Modeling Practices and Applications in Power System Stability Analyses,” see chapter 6. Indeed, this is a very good standard to complement our entire course, and I encourage you to download, print, and include in your binder/folder.

2. Note the presence of $v_d$ and $v_q$ on the right-hand-side. Recall that $v_{0dq}=Pv_{abc}$ and so $v_d$ and $v_q$ come from the phase voltages $v_a$, $v_b$, and $v_c$. Since the phase voltages are affected by the load currents, so are $v_d$ and $v_q$. So, we need to represent the load in order to complete the model. This is the subject of section 4.13. It is done in Section 4.13.2 for the current-state space model and in Section 4.13.3 for the flux linkage state space model. I have already addressed it in class, in the notes called “LoadEquations,” for the current state space model only. I will leave you to read Section 4.13.3 for the flux linkage state space model. There is also a nice example (Ex 4.4, p. 128) which illustrates integration of the load equations.

3. Section 4.12.4 is the first instance in VMAF where saturation is addressed, but not the last. It is also addressed in Section 5.9.1 and in Appendix D.1.2.