Homework #6: Use the Generalized Reduced Gradient procedure to solve the below problem.

<u>Problem statement</u>:

The problem statement then becomes the following:

$$\min f(\underline{x},u) = 1.5 + x_2 + 3x_2^2 + 0.5u + 0.5u^2$$

subject to

$$g_1(\underline{x},\underline{u}) = 4 - \cos x_1 - 10\sin x_1 - x_2 = 0$$

$$g_2(\underline{x},\underline{u}) = 2 - \cos x_1 + 10\sin x_1 - u = 0$$

<u>Lagrangian</u>:

The Lagrangian function becomes:

$$\mathcal{L}(\underline{x},u,\underline{\lambda}) = f(\underline{x},u) + \underline{\lambda}^T \underline{g}(\underline{x},u,)$$

$$= f(\underline{x},u) + \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}^T \begin{bmatrix} g_1(\underline{x},\underline{u})\\ g_2(\underline{x},\underline{u})\end{bmatrix} = f(\underline{x},u) + \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix} g_1(\underline{x},\underline{u})\\ g_2(\underline{x},\underline{u})\end{bmatrix} = f(\underline{x},u) + \lambda_1 g_1(\underline{x},\underline{u}) + \lambda_2 g_2(\underline{x},\underline{u})$$

$$= 1.5 + x_2 + 3x_2^2 + 0.5u + 0.5u^2 + \lambda_1(4 - \cos x_1 - 10\sin x_1 - x_2) + \lambda_2(2 - \cos x_1 + 10\sin x_1 - u)$$

<u>Optimality conditions</u>:

The appropriate optimality conditions are given by

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial \underline{x}} = \frac{\partial f(\underline{x},u)}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}}[\underline{\lambda}^T \underline{g}(\underline{x},\underline{u})] = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial u} = \frac{\partial f(\underline{x},u)}{\partial u} + \frac{\partial}{\partial u}[\underline{\lambda}^T \underline{g}(\underline{x},\underline{u})] = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial \underline{\lambda}} = \frac{\partial}{\partial \underline{\lambda}}[\underline{\lambda}^T \underline{g}(\underline{x},\underline{u})] = 0 \qquad (3)$$

which are

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial x_1} = \lambda_1(\sin x_1 - 10\cos x_1) + \lambda_2(\sin x_1 + 10\cos x_1) = 0 \qquad (1a)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial x_2} = 1 + 6x_2 - \lambda_1 = 0 \qquad (1b)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial u} = 0.5 + u - \lambda_2 = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial \lambda_1} = 4 - \cos x_1 - 10\sin x_1 - x_2 = 0 \qquad (3a)$$

$$\frac{\partial \mathcal{L}(\underline{x},u,\underline{\lambda})}{\partial \lambda_2} = 2 - \cos x_1 + 10\sin x_1 - u = 0 \qquad (3b)$$

$$\begin{bmatrix} P_{G1} \\ P_{G2} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0.5 \\ 3.5 \\ 4.0 \end{bmatrix}$$

1. Let k = 1. Guess an initial control vector $\underline{u}^{(k)}$. (You can use economic dispatch with losses or without losses to make the initial guess).

   ➔From economic dispatch solution (see class notes), $u^{(1)}=3.5$.

2. Given $\underline{u}^{(k)}$, solve for $\underline{x}^{(k)}$ from (8c), repeated here for convenience:

$$\frac{\partial \mathcal{L}(\underline{x},\underline{u},\underline{\lambda},\underline{\mu})}{\partial \underline{\lambda}} = \frac{\partial}{\partial \underline{\lambda}}[\underline{\lambda}^T \underline{g}(\underline{x},\underline{u})] = \underline{0} \tag{8c}$$

   This is just a power flow solution!

From above equation (3b),

$g_1 = 4 - \cos x_1 - 10\sin x_1 - x_2 = 0$, ➔ $\frac{\partial g_1}{\partial x_1} = \sin x_1 - 10\cos x_1$; $\frac{\partial g_1}{\partial x_2} = -1$

$g_2 = 2 - \cos x_1 + 10\sin x_1 - u = 0$ ➔ $\frac{\partial g_2}{\partial x_1} = \sin x_1 + 10\cos x_1$; $\frac{\partial g_2}{\partial x_2} = 0$

Jacobian is: $\begin{bmatrix} \sin x_1 - 10\cos x_1 & -1 \\ \sin x_1 + 10\cos x_1 & 0 \end{bmatrix}$

Update equation:

$\underline{J}\Delta\underline{x}^{(0)} = -\underline{f}(\underline{x}^{(0)})$ ➔ $\begin{bmatrix} \sin x_1 - 10\cos x_1 & -1 \\ \sin x_1 + 10\cos x_1 & 0 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = -\begin{bmatrix} 4 - \cos x_1 - 10\sin x_1 - x_2 \\ 2 - \cos x_1 + 10\sin x_1 - u \end{bmatrix}$

For $u^{(1)}=3.5$, I found x1=0.2495, x2=0.5619.

3. Compute:

$$\underline{\lambda}^{(k)} = -\left\{\left[\frac{\partial \underline{g}(\underline{x},\underline{u})}{\partial \underline{x}}\right]^T\right\}^{-1} \frac{\partial f(\underline{x},\underline{u})}{\partial \underline{x}}\Bigg|_{\underline{x}^{(k)},\underline{u}^{(k)}} \tag{20}$$

Note that the matrix $\left[\frac{\partial \underline{g}(\underline{x},u)}{\partial \underline{x}}\right]$ is Jacobian developed above. The second term is $\frac{\partial f(\underline{x},u)}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial f(\underline{x},u)}{\partial x_1} \\ \frac{\partial f(\underline{x},u)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1+6x_2 \end{bmatrix}$, and the whole expression becomes

$$\underline{\lambda}^{(1)} = -\left\{\begin{bmatrix} \sin x_1 - 10\cos x_1 & -1 \\ \sin x_1 + 10\cos x_1 & 0 \end{bmatrix}^T\right\}^{-1} \begin{bmatrix} 0 \\ 1+6x_2 \end{bmatrix}\Bigg|_{\underline{x}^{(1)}=\begin{bmatrix} 0.2495 \\ 0.5619 \end{bmatrix}, \underline{u}^{(1)}=3.5}$$

$$-\left\{\begin{bmatrix} \sin x_1 - 10\cos x_1 & \sin x_1 + 10\cos x_1 \\ -1 & 0 \end{bmatrix}\right\}^{-1} \begin{bmatrix} 0 \\ 1+6x_2 \end{bmatrix}\Bigg|_{\underline{x}^{(1)}=\begin{bmatrix} 0.2495 \\ 0.5619 \end{bmatrix}, \underline{u}^{(1)}=3.5}$$

This results in λ_1=4.3715, λ_2=4.1543.

4. Compute the "steepest ascent" direction, i.e., the gradient of f, according to (19)

$$\nabla_{\underline{u}} f(\underline{x},\underline{u}) = \frac{df(\underline{x},\underline{u})}{d\underline{u}} = \left[\frac{\partial f(\underline{x},\underline{u})}{\partial \underline{u}} - \left[\frac{\partial \underline{g}(\underline{x},\underline{u})}{\partial \underline{u}}\right]^T \underline{\lambda}\right]_{\underline{x}^{(k)},\underline{u}^{(k)}} \tag{19}$$

(the reduced gradient).

The first derivative term is $\frac{\partial f(\underline{x},\underline{u})}{\partial \underline{u}} = 0.5 + u$

The second derivative term is $\left[\frac{\partial \underline{g}(\underline{x},\underline{u})}{\partial \underline{u}}\right] = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Lambda is (from above) $\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 4.3715 \\ 4.1543 \end{bmatrix}$

And the entire expression becomes:

$$\nabla_{\underline{u}} f(\underline{x},\underline{u}) = \frac{df(\underline{x},\underline{u})}{d\underline{u}} = \left[0.5 + u - \begin{bmatrix} 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} 4.3715 \\ 4.1543 \end{bmatrix}\right]_{\underline{x}^{(1)}=\begin{bmatrix} 0.2495 \\ 0.5619 \end{bmatrix}, \underline{u}^{(1)}=3.5}$$

This results in $\nabla_{\underline{u}} f(\underline{x},\underline{u}) = -0.1543$.

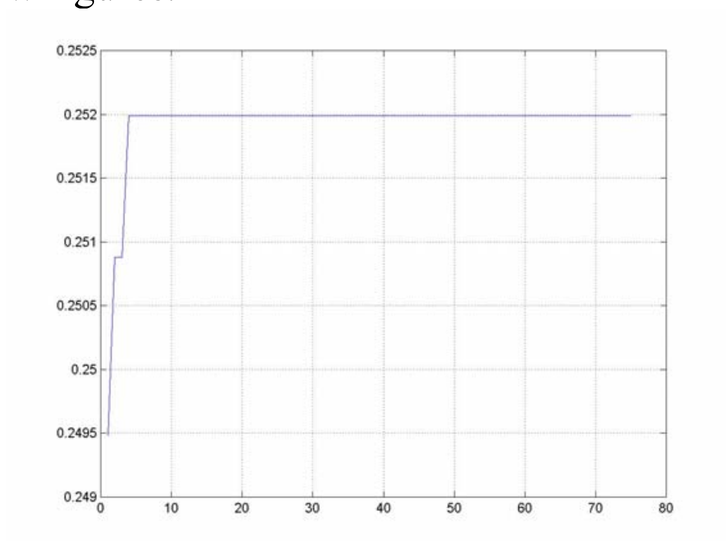
5. Update the control vector by moving it in the direction of steepest descent.

$$\underline{u}^{(k+1)} = \underline{u}^{(k)} - \alpha^{(k)} \nabla_{\underline{u}} f(\underline{x}, \underline{u}) \tag{21}$$

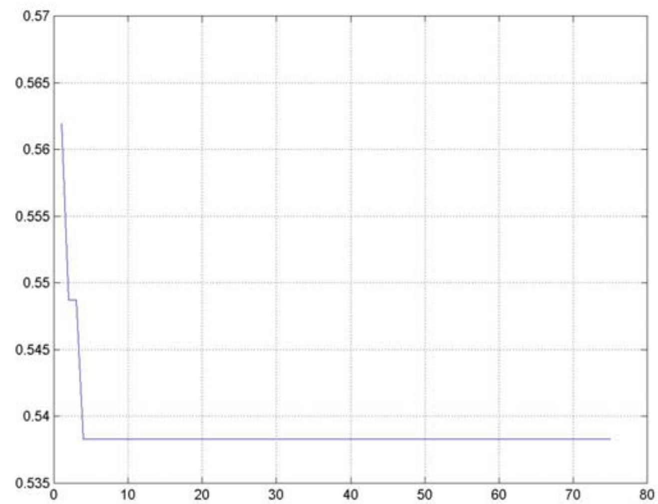
where $\alpha^{(k)}$ is a step size which is reduced for every iteration.
Using $\alpha^{(1)}$ =0.1 and $\alpha^{(k+1)}$ =0.9$\alpha^{(k)}$, we get that
$u^{(2)}$=3.5139.

6. If $\left|\nabla_{\underline{u}} f(\underline{x}, \underline{u})\right| < \varepsilon$, stop. Else, k=k+1, and go to (2).
Using ε=0.00001, the final result was u=3.5174, x1=0.252, x2=0.5383

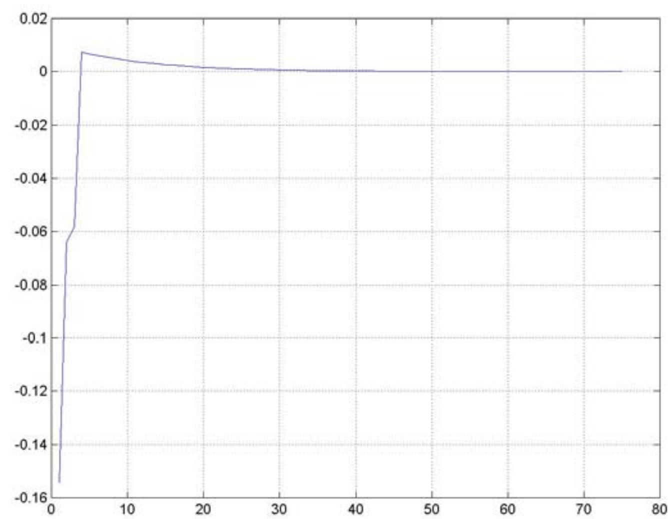
Values of x1, x2, del_u (f), and u are plotted against for each successive step in the below figures.



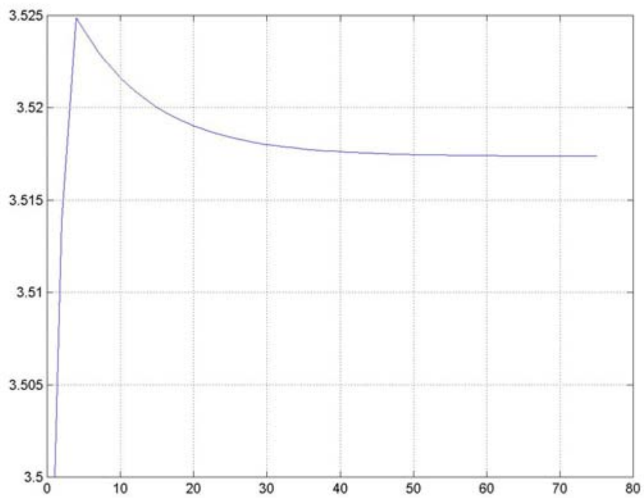
Plot for x1



Plot for x2



Plot for del u (f)



Plot for u

The matlab code for the problem is given below.

```
clear
% Set stopping tolerance
eps=0.00001;
% Set step size reduction per iteration
beta=0.9;
% Set initial step size
alpha=0.1;
% Set initial conditions
x1=0;
x2=0.5;
u=3.5;
unew=100;
k=0;
delf=100;
while abs(delf) > eps
  k=k+1;
  ustore(k)=u;
% Solve the power flow equations to get update on x1 and x2
  b=[4-cos(x1)-10*sin(x1)-x2; 2-cos(x1)+10*sin(x1)-u];
  maxb=max(abs(b(1)), abs(b(2)));
  while maxb > 0.01,
     J=[sin(x1)-10*cos(x1)  -1;sin(x1)+10*cos(x1)  0];
     delx=-1*inv(J)*b;
     new=[x1;x2]+delx;
     x1=new(1);
     x2=new(2);
     b=[4-cos(x1)-10*sin(x1)-x2; 2-cos(x1)+10*sin(x1)-u];
     maxb=max(abs(b(1)), abs(b(2)));
  end
% Write solution to power flow equations
  x1store(k)=x1;
```

```
  x2store(k)=x2;
% Compute Lambda
  J=[sin(x1)-10*cos(x1)  -1;sin(x1)+10*cos(x1)  0];
  lam=-1*inv(J')*[0; 1+6*x2];
  lam1store(k)=lam(1);
  lam2store(k)=lam(2);
% Compute steepest ascent direction
  delf=0.5+u+[0;-1]'*lam;
  delfstore(k)=delf;
% Compute the new value of the control
  unew=u-beta*alpha*delf;
  u=unew;
end
```