APPENDIX D

Review of Control System Analysis Techniques

BODE DIAGRAMS

The frequency response of a linear system is determined experimentally by applying a sinusoidal input signal and then measuring the sinusoidal response of the system. The frequency response data includes the measurement of the amplitude and phase shift of the sinusoidal output compared to the amplitude and phase of the input signal as the input frequency is varied. The relationship between the output and input to the system can be used by the designer to determine the performance of the system. Furthermore, frequency response data can be used to deduce the performance of a system to an arbitrary input that may or may not be periodic.

The magnitude of the amplitude ratio and phase angle can be presented graphically in a number of ways. However, one of the most useful presentations of the data is in the so-called Bode diagram, named after H. W. Bode for his pioneering work in frequency response analysis. In a Bode diagram the logarithm of the magnitude of the system transfer function, $|G(i\omega)|$, and the phase angle, ϕ, are plotted separately versus the frequency.

The frequency response, output-input amplitude ratio, and phase with respect to the input can be determined analytically from the system transfer function written in factored time constant form:

$$G(s) = \frac{k(1 + T_a s)(1 + T_b s) \cdots}{s^r(1 + T_1 s)(1 + T_2 s) \cdots \left(1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega^2_n}\right)} \tag{D.1}$$

This transfer function has simple zeros at $-1/T_a$, $-1/T_b$, . . ., a pole at the origin of order r, simple poles at $-1/T_1$, $-1/T_2$, . . ., and complex poles at $-\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}$. The steady-state response can be shown to be determined by substituting $i\omega$ for the Laplace variable s in the system transfer function. Substituting $i\omega$ for s one can express the transfer function in terms of the magnitude of its amplitude ratio and phase angle as follows:

$$\begin{aligned} 20 \log |G(i\omega)| &= 20 \log k + 20 \log |1 + i\omega T_a| \\ &\quad + 20 \log |1 + i\omega T_b| + \cdots - 20\, r \log |i\omega| \\ &\quad - 20 \log |1 + i\omega T_1| - 20 \log |1 + i\omega T_2| \\ &\quad - 20 \log |1 + 2\zeta(\omega/\omega_n)i - (\omega/\omega_n)^2| \cdots \end{aligned} \tag{D.2}$$

and the phase angle in degrees

$$\angle G(i\omega) = \tan^{-1}\omega T_a + \tan^{-1}\omega T_b + \cdots - r(90°) - \tan^{-1}\omega T_1$$
$$-\tan^{-1}\omega T_2 \cdots - \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega^2_n - \omega^2}\right) \tag{D.3}$$

The magnitude has been expressed in terms of decibels. A magnitude in decibels is defined as follows:

$$\text{Magnitude in dB} = 20 \log \frac{|\text{magnitude of output}|}{|\text{magnitude of input}|} \tag{D.4}$$

where the logarithm is to the base 10.

The Bode diagram now can be constructed using a semilog plot. The magnitude in decibels and phase angle are plotted separately on a linear ordinate versus the frequency on a logarithmic abscissa. Because the Bode diagram is obtained by adding the various factors of $G(i\omega)$ one can construct the Bode diagram quite rapidly.

In the general case the factors that will make up the transfer function are a constant term (system gain), poles at the origin, simple poles and zeros on the real axis, and complex conjugate poles and zeros. The graphical representation of each of these individual factors is described in the following section.

System Gain

The log magnitude of the system gain is as follows:

$$20 \log k = \text{constant dB} \tag{D.5}$$

and the phase angle by

$$\angle k = \begin{matrix} 0° & k > 0 \\ 180° & k < 0 \end{matrix} \tag{D.5}$$

Figure D.1 shows the Bode plot for a positive system gain.

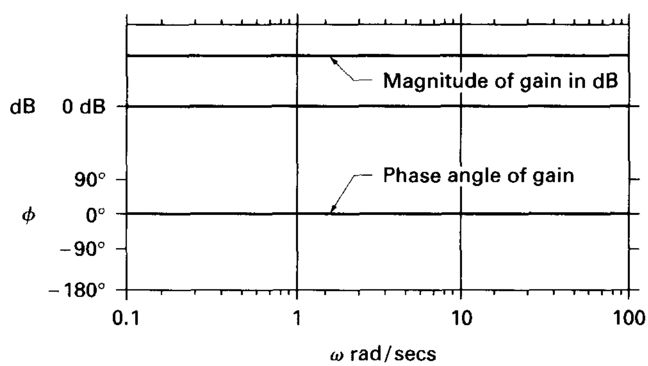

FIGURE D.1
Bode representation of the magnitude and phase of the system gain k.

Poles or Zeros at the Origin $(i\omega)^{\pm r}$

The log magnitude of a pole or zero at the origin of order r can be written as

$$20 \log |(i\omega)^{\pm r}| = \pm 20r \log \omega \text{ dB} \tag{D.6}$$

and the phase angle is given by

$$\angle (i\omega)^{\pm r} = \pm 90° r \tag{D.7}$$

The log-magnitude is 0 dB at $\omega = 1.0$ rad/s and has a slope of 20 dB/decade, where a decade is a factor of 10 change in frequency. Figure D.2 is a sketch of the log magnitude and phase angle for a multiple zero or pole.

Simple Poles or Zeros $(1 + i\omega T)^{\pm 1}$

The log magnitude of a simple pole or zero can be expressed as

$$\pm 20 \log |1 + i\omega T| = \pm 20 \log \sqrt{1 + (\omega T)^2} \tag{D.8}$$

For very low values of ωT, that is, $\omega T \ll 1$, then

$$\pm 20 \log \sqrt{1 + (\omega T)^2} \cong 0 \tag{D.9}$$

and for very large values of ωT, that is, $\omega T \gg 1$, then

$$\pm 20 \log \sqrt{1 + (\omega T)^2} \cong \pm 20 \log \omega T \tag{D.10}$$

From this simple analysis one can approximate the log magnitude plot of a simple pole or zero by two straight line segments as shown in Figure D.3. One of the asymptotic lines is the 0 dB line and the second line segment has a slope of 20 dB/decade that intersects the 0 dB line at the frequency $\omega = 1/T$. The intersection

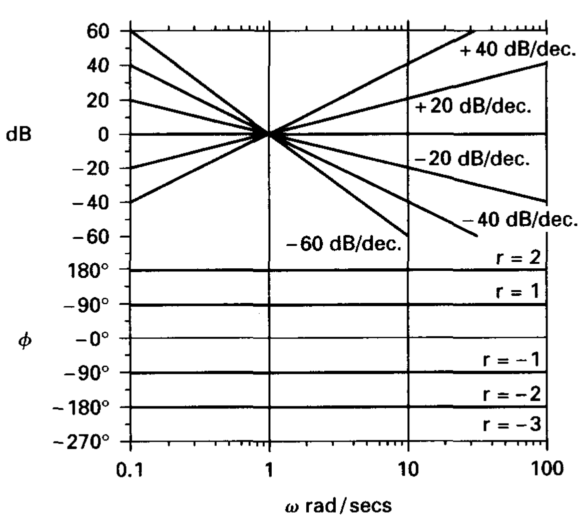

FIGURE D.2
Bode representation of the magnitude and phase of a pole or zero at the origin.

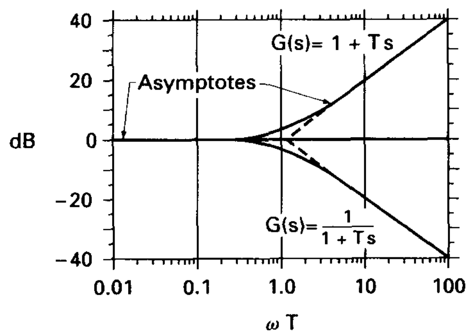

FIGURE D.3
Bode representation of the magnitude of a simple pole or zero.

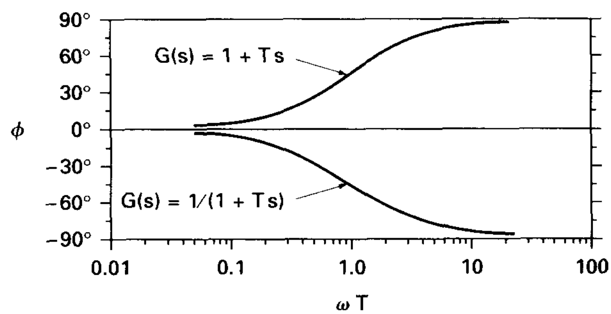

FIGURE D.4
Bode representation of the phase angle of a simple pole or zero.

frequency is called the corner frequency. The actual log magnitude differs from the asymptotic approximation in the vicinity of the corner frequency.

The phase angle for a simple pole or zero is given by

$$\angle(1 + i\omega T)^{\pm 1} = \pm\tan^{-1} \omega T \tag{D.11}$$

Figure D.4 is a sketch of the phase angle.

Complex Conjugate Pole or Zero $[1 + i2\zeta\omega/\omega_n - (\omega/\omega_n)^2]^{\pm 1}$

The log magnitude of the complex pole can be written as

$$\begin{aligned} 20 \log \left| \frac{1}{1 + i2\zeta\omega/\omega_n + (\omega/\omega_n)^2} \right| \\ &= -20 \log[(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2]^{1/2} \qquad \text{(D.12)} \\ &= -10 \log[(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2] \end{aligned}$$

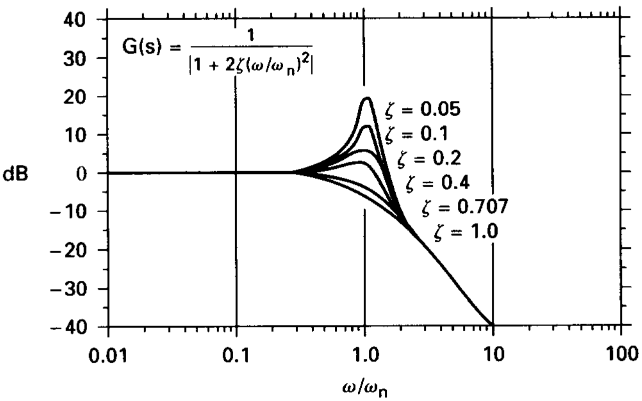

FIGURE D.5
Bode representation of the magnitude of a complex conjugate pole.

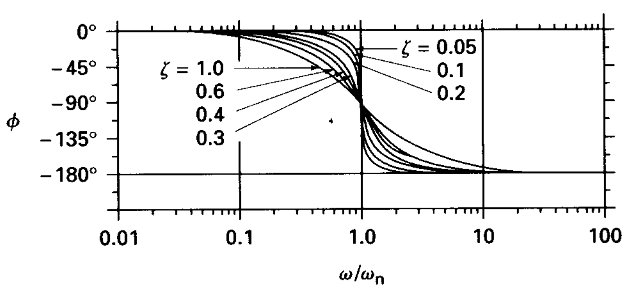

FIGURE D.6
Bode representation of the phase angle of a complex conjugate pole.

The log magnitude can be approximated by two straight line segments. For example, when $\omega/\omega_n \ll 1$

$$20 \log \left| \frac{1}{1 + i2\zeta\omega/\omega_n - (\omega/\omega_n)^2} \right| \cong 0 \tag{D.13}$$

and when $\omega/\omega_n \gg 1$

$$20 \log \left| \frac{1}{1 + i2\zeta\omega/\omega_n - (\omega/\omega_n)^2} \right| \cong -40 \log \omega/\omega_n \tag{D.14}$$

The two straight line asymptotes consist of a straight line along the 0 dB line for $\omega/\omega_n = 1 \ll 1$ and a line having a slope of -40 dB/decade for $\omega/\omega_n \gg 1$. The asymptotes intersect at $\omega/\omega_n = 1$ or $\omega = \omega_n$, where ω_n is the corner frequency. Figure D.5 shows the asymptotes as well as the actual magnitude plot for various damping ratios for a complex pole.

The phase angle for a complex pole is given by

$$\angle[1 + i2\zeta\omega/\omega_n - (\omega/\omega_n)^2]^{-1} = -\tan^{-1}\left[\frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right] \qquad \text{(D.16)}$$

Figure D.6 shows the phase angle for a complex pole. Similar curves can be developed for a complex zero.

If the transfer function is expressed in time constant form, then the Bode diagram easily can be constructed from the simple expressions developed in this section.