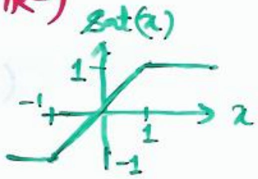
Existence & Uniqueness (Ctnd.)

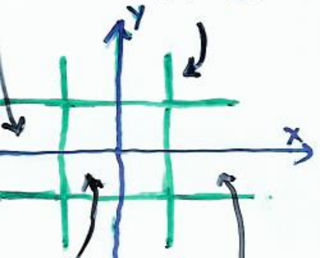
Example: $f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1+x_2) \end{bmatrix}$ is cont. but not (cont.)-diff. on $\mathbb{R}^2$.
(but Lipschitz on $\mathbb{R}^2$)



$$\|f(x)-f(y)\|_2^2 = \left\| \begin{bmatrix} x_2-y_2 \\ \text{sat}(y_1+y_2)-\text{sat}(x_1+x_2) \end{bmatrix} \right\|_2^2$$

$|\text{sat}(x)-\text{sat}(y)| = |-1-y| = |1+y| \leq |-x+y| = |x-y|$

$|\text{sat}(x)-\text{sat}(y)| = 0 \leq |x-y|$



$|\text{sat}(x)-\text{sat}(y)| = |x-y|$

$|\text{sat}(x)-\text{sat}(y)| = |1-y| \leq |x-y|$

$$= (x_2-y_2)^2 + (\text{sat}(x_1+x_2)-\text{sat}(y_1+y_2))^2$$

$$\leq (x_2-y_2)^2 + ((x_1+x_2)-(y_1+y_2))^2 \qquad (|\text{sat}(x)-\text{sat}(y)| \leq |x-y|)$$

$$= (x_1-y_1)^2 + 2(x_1-y_1)(x_2-y_2) + 2(x_2-y_2)^2$$

$$= 2(x_1-y_1)^2 - [(x_1-y_1)^2 - 2(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2] + 3(x_2-y_2)^2$$

$$= 2(x_1-y_1)^2 + 3(x_2-y_2)^2 - ((x_1-y_1)-(x_2-y_2))^2$$

$$\leq 2(x_1-y_1)^2 + 3(x_2-y_2)^2$$

$$\leq 3[(x_1-y_1)^2 + (x_2-y_2)^2] = 3\|x-y\|_2^2$$

$\Rightarrow \|f(x)-f(y)\|_2 \leq \sqrt{3}\,\|x-y\|_2$. Thus globally Lipschitz (weaker than cont. diff.)

Locally Lipschitz at $(t_0, x(t_0))$ guarantees unique solution in nbhd of $(t_0, x(t_0))$, say up to $t_1 = t_0 + \delta$. Further extension would require local Lipschitzness at $(t_1, x(t_1))$, etc. In general exist a max. T s.t. unique solution exists over $[t_0, T)$. As $t \to T$, the solution leaves any compact set over which f is locally Lipschitz.

Example: $\dot{x} = -x^2$, $x(0) = -1$

Here $f = x^2$ is cont. & cont. diff., but differential not uniformly bdd. However diff. bdd on any compact set $\Rightarrow$ Lipschitz over that compact set.

Unique solution $x(t) = \frac{1}{t-1}$ exists over $[0,1)$. As $t \to 1$, x leaves any compact set.

Question: When can the unique solution exist indefinitely?

Thm 2: $\dot{x} = f(t,x)$ has unique solution over $[t_0, t_1]$ if f Lipschitz over $[t_0,t_1] \times \mathbb{R}^n$ and piece-wise cont. in t over $[t_0, t_1]$.

"Lipschitz" requirement of above thm is restrictive: $\dot{x} = -x^3 = f(x)$.
Here f is cont. & cont. diff., but $\frac{\partial f}{\partial x}$ not bounded. Yet unique solution exists:

$$x(t) = \text{sgn}(x_0)\sqrt{x_0^2/1+2x_0^2(t-t_0)} \quad \forall t \geq t_0.$$

Existence & Uniqueness (Ctnd.)

Example: $\dot{x} = A(t)x + g(t)$

$$\Rightarrow \|f(t,x) - f(t,y)\| = \|A(t)(x-y)\| \leq \|A(t)\| \, \|x-y\|$$

So if $A(t)$ is bounded for $t \in [t_0, t_1]$, we have that conditions of Thm 2 hold.

- As we discussed, condition of Thm 2 are quite strong, and so here is another result:

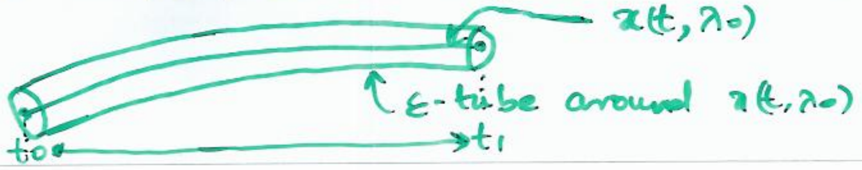
Thm 3: $\dot{x} = f(t,x)$ has unique solution for all $t \geq t_0$ if f locally Lipschitz over $[t_0,\infty) \times W$, $W \subseteq \mathbb{R}^n$ compact set, $x_0 \in W$, and solution of $\dot{x} = f(t,x)$ does not exit W.

Example: $\dot{x} = -x^3 = f(x)$. Then f is locally Lipschitz over $[t_0,\infty) \times \mathbb{R}^n$.

Also if $x_0 = a$, then system never leaves the set $\{x \mid |x| \leq |a|\}$.
(This is because $x > 0 \Rightarrow \dot{x} < 0$, and $x < 0 \Rightarrow \dot{x} > 0$) So Thm 3 applies.

Continuous dependence on Initial Conditions / Parameters

- Does small change in t_0, x_0, or f causes small change in solution?
- Continuous dependence on t_0 since, $x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds$
- cont. dependence on x_0: suppose $\dot{x} = f(t,x)$ uniquely solveable over $[t_0,t_1]$ starting x_0.
  $\forall \varepsilon \, \exists \delta: \|z_0 - x_0\| \leq \delta \Rightarrow \|z(t) - x(t)\| < \varepsilon \;\; \forall t \in [t_0, t_1]$ and $z(t)$ unique.
- cont. dependence on f: $\{f_m\} \xrightarrow[m\to\infty]{} f$ uniformly in $t \overset{?}{\Rightarrow} \{x_m\} \xrightarrow[m\to\infty]{} x$ uniformly?
  Another way to study cont. dependence on f, parametrize f using parameter "λ".
  $\Rightarrow \dot{x} = f(t, x, \lambda)$, which suppose has solution over $[t_0, t_1]$.
  $\forall \varepsilon, \exists \delta: \|\lambda - \lambda_0\| \leq \delta \Rightarrow \|x(t,\lambda) - x(t,\lambda_0)\| \leq \varepsilon \;\; \forall t \in [t_0, t_1]$
- **Thm:** $f(t,x,\lambda)$ cont. in (t,x,λ) & locally Lipschitz over $[t_0,t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$ (D open, connected)
  $\forall \varepsilon \, \exists \delta: \|z_0 - x_0\|, \|\lambda - \lambda_0\| < \delta \Rightarrow \|z(t,\lambda) - x(t,\lambda_0)\| < \varepsilon \;\; \forall t \in [t_0, t_1]$.



Differentiability of solution & Sensitivity Eq.

- Under the additional requirement that $f(t, x, \lambda)$ is cont. differentiable in x & λ (instead of just satisfying some Lipschitz condition) over $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p$, then $x(t, \lambda)$ is differentiable wrt x & λ near x_0, λ_0 where x_0, λ_0 such that $\dot{x} = f(t, x, \lambda_0)$ with $x(t_0) = x_0$ has unique soln. over $[t_0, t_1]$.

- Further $x(t, \lambda) \simeq x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$, where $S(t) = \dfrac{\partial x}{\partial \lambda}$

$$\dot{S}(t) = \left(\frac{\partial f}{\partial x} \Big|_{\substack{x = x(t,\lambda) \\ \lambda = \lambda_0}} \right) S(t) + \left(\frac{\partial f}{\partial \lambda} \Big|_{\substack{x = x(t,\lambda_0) \\ \lambda = \lambda_0}} \right) \text{ with } S(t_0) = 0$$

- Thus if $x(t, \lambda_0)$ is available as solution of $\dot{x} = f(t, x, \lambda_0)$, $x(t_0) = x_0$, then $x(t, \lambda)$ can be obtained by first solving eq. for "sensitivity".

- Another way to approach this is by solving the following together:

$$\dot{x} = f(t, x, \lambda_0) \quad \text{with } x(t_0) = x_0$$

$$\dot{S} = \left(\frac{\partial f}{\partial x} \Big|_{\lambda = \lambda_0} \right) S + \left(\frac{\partial f}{\partial \lambda} \Big|_{\lambda = \lambda_0} \right) \text{ with } S(t_0) = 0.$$

- These are usually solved numerically.

<u>Example</u>: Phase-locked-loop

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -c \sin x_1 - (a + b\cos x_1) x_2$$

$$\lambda = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{with } \lambda_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow f(x, \lambda_0) = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}$$

Also, $\dfrac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -c\cos x_1 + b x_2 \sin x_1 & -(a + b\cos x_1) \end{bmatrix}$, $\dfrac{\partial f}{\partial \lambda} = \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2 \cos x_1 & -\sin x_1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \quad (\text{at } \lambda = \lambda_0)$$

$$\dot{S} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} S + \begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -x_2\cos x_1 & -\sin x_1 \end{bmatrix} \qquad (S = S_{2\times 3})$$

More on differentiability of solution & Sensitivity Equation

$\dot{x} = f(t, x, \lambda)$ with $x(t_0) = x_0$

$\Rightarrow x(t,\lambda) = x_0 + \int_{t_0}^{t} f(s, x(s,\lambda), \lambda)\, ds$

$$\Rightarrow \underbrace{x_\lambda(t,\lambda)}_{\frac{\partial x}{\partial \lambda}} = \int_{t_0}^{t} \underbrace{\left[\frac{\partial f}{\partial x}(s, x(s,\lambda), \lambda)\, x_\lambda(s,\lambda) + \frac{\partial f}{\partial \lambda}(s, x(s,\lambda), \lambda)\right]}_{\text{Assumes } f \text{ is diff. wrt. } x \,\&\, \lambda} ds$$

Since $x_\lambda(t,\lambda)$ is given as an integral over t of a cont. function $\Rightarrow$ $x_\lambda(t,\lambda)$ differentiable wrt t. (assumes f is cont. diff. wrt x & λ)

$$\Rightarrow \frac{\partial}{\partial t} \underbrace{x_\lambda(t,\lambda)}_{S} = \underbrace{\frac{\partial f}{\partial x}(t, x(t,\lambda), \lambda)}_{A(t,\lambda)} x_\lambda(t,\lambda) + \underbrace{\frac{\partial f}{\partial \lambda}(t, x(t,\lambda), \lambda)}_{B(t,\lambda)}$$

$$\Rightarrow \boxed{\begin{aligned} &\dot{S} = A(t,\lambda)\, S + B(t,\lambda) \\ &S(t_0) = \int_{t_0}^{t_0} \cdots\, ds \Rightarrow S(t_0) = 0 \end{aligned}}$$

$S(t)$: sensitivity function