Existence of Periodic Orbits

• Periodic orbits separate 2D plane into two halves (for 2^{nd} order sys)

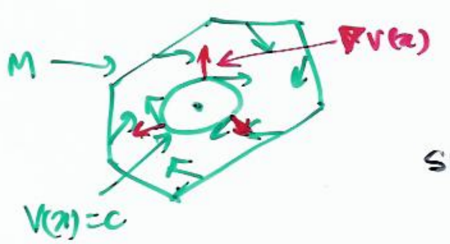
• Poincare-Bendixson Criterion: Consider 2^{nd}-order sys. $\dot{x}=f(x)$ and let M closed bounded subset of $\mathbb{R}^2$ such that

* M contains no eq. pt. or contains single eq. pt. where $\mathrm{Re}\left(\lambda\left(\frac{\partial f}{\partial x}\right)\right)>0$
* M is invariant (trajectory initiating in M remains in M)

Then, M contains a periodic orbit.

Intuition: Bounded trajectory in a plane must approach an orbit or a eq. pt. If M contains no eq. pt., then it must contain a periodic orbit. On the other hand, if M contains a single unstable eq. pt., then in its nbhd trajectories move away ⇒ Exists simple closed curve encircling the eq. pt. such that vector field on the curve points outward. By excluding the region enclosed by the simple curve, we obtain closed bdd M' with no eq.

Let $V(x)=c$ denote the simple closed curve. Then, $\boxed{f(x)^T\nabla V(x)>0}$



$\nabla V(x)$, gradient of $V(x)$ normal to the curve $V(x)=c$

$f(x)^T\nabla V(x)>0 \Leftrightarrow$ angle between normal & vector field $< 90^\circ$

similarly if M is given by, $V(x)=c'$, then, $\boxed{f(x)^T\nabla V(x)\le 0}$

Example: $\left.\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-x_1\end{aligned}\right\}$ Let $M=\{c_1\le V(x)\le c_2\}$, where $V(x)=x_1^2+x_2^2$ and $0<c_1<c_2$.

⇒ eq. pt. $=(0,0)$ Then M has no eq. pt. inside it. Also, M closed, bdd.

Also, $f(x)^T\nabla V(x) = \begin{bmatrix}x_2 & -x_1\end{bmatrix}\begin{bmatrix}2x_1\\ 2x_2\end{bmatrix} = 2x_1x_2-2x_1x_2=0$

⇒ M is invariant. So M has periodic orbit inside.

In fact, M has infinitely many periodic orbits inside.

Example:
$$\dot{x}_1 = x_1+x_2-x_1(x_1^2+x_2^2)$$
$$\dot{x}_2 = -2x_1+x_2-x_2(x_1^2+x_2^2)$$

One possible eq. at $x_1=x_2=0$ (can be shown that it is unique)

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix}1-(x_1^2+x_2^2)-x_1(2x_1) & 1-x_1(2x_2)\\ -2-x_2(2x_1) & 1-(x_1^2+x_2^2)-x_2(2x_2)\end{bmatrix}\right|_{x=0} = \begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

Existence of periodic orbit (ctnd.)

$$A = \begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix} \Rightarrow \lambda I - A = \begin{bmatrix}\lambda-1 & -1\\ 2 & \lambda-1\end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 - 2\lambda + 3$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm j\sqrt{8}}{2} = 1 \pm j\sqrt{2} \Rightarrow \text{unstable focus.}$$

Consider $M = \{x \mid V(x) = x_1^2 + x_2^2 \le c\}$. Then

$$f(x)^T \nabla V(x) = (x_1 + x_2 - x_1(x_1^2+x_2^2))(2x_1) + (-2x_1 + x_2 - x_2(x_1^2+x_2^2))(2x_2)$$

$$= 2[x_1^2 + x_1x_2 - x_1^2(x_1^2+x_2^2) - 2x_1x_2 + x_2^2 - x_2^2(x_1^2+x_2^2)]$$

$$= 2[(x_1^2+x_2^2) - (x_1^2+x_2^2)^2 - x_1x_2]$$

$$\le 2(x_1^2+x_2^2) - 2(x_1^2+x_2^2)^2 - \underbrace{2x_1x_2 + (x_1+x_2)^2}_{(x_1^2+x_2^2)}$$

$$= 3c - 2c^2 = c(3-2c) < 0 \quad \text{for } c > 1.5$$

So exists a periodic orbit inside circle of radius larger than 1.5.

Uniqueness of eq. pt.:

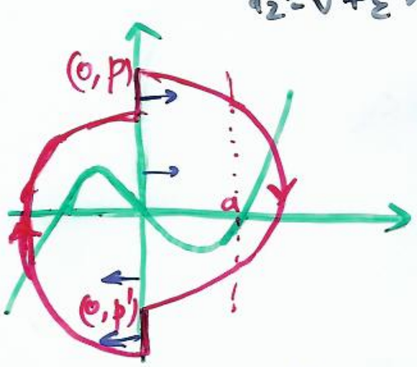
$$[x_1 + x_2 - x_1(x_1^2+x_2^2) = 0] \times x_2$$
$$-\ [-2x_1 + x_2 - x_2(x_1^2+x_2^2) = 0] \times x_1$$
$$x_2^2 + 2x_1^2 = 0 \Rightarrow x_1 = x_2 = 0.$$

Example (-ve resistance oscillator):

$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0$ $\quad h(0) = 0,\ h'(0) < 0,\quad h(\infty) = \infty,\ h(-\infty) = -\infty.$

{Additionally, $h(v) = -h(-v)$, $h(v) < 0$ for $0 < v < a$, $h(v) > 0$ for $v > a$}

Choose $\left.\begin{array}{l} x_1 = v \\ x_2 = \dot{v} + \varepsilon h(v) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dot{x}_1 = x_2 - \varepsilon h(x_1) \\ \dot{x}_2 = -x_1 \end{array}\right\} \Rightarrow$ at eq., $x_1 = x_2 = 0$



Also, $\left.\frac{\partial f}{\partial x}\right|_0 = \begin{bmatrix}-\varepsilon h'(x_1) & 1\\ -1 & 0\end{bmatrix}_0 = \begin{bmatrix}\alpha & 1\\ -1 & 0\end{bmatrix}$

$$\lambda I - \begin{bmatrix}\alpha & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}\lambda-\alpha & -1\\ 1 & \lambda\end{bmatrix} \Rightarrow \det(\ ) = \lambda^2 - \alpha\lambda + 1$$

$$\Rightarrow \lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2} \Rightarrow Re(\lambda) > 0 \Rightarrow \text{unstable}$$

To show existence of a periodic orbit, need to find closed, bdd, inv. M.
M enclosed by "red" curve, where right side is trajectory starting at (0,p)
left side "mirror reflection" of right side

Existence of periodic orbit (contd.)

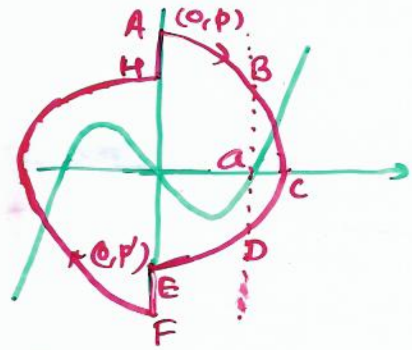
- Consider trajectory starting at $(0, p)$, then it hits x_2-axis again at $(0, p')$, and for large enough p, $|p'| < p$.

- To see this consider $V(x) = x_1^2 + x_2^2$ along the trajectory.

Then $\dot{V} = V_x^T \dot{x} = [x_1 \;\; x_2] \begin{bmatrix} x_2 - \varepsilon h(x_1) \\ -x_1 \end{bmatrix} = x_1 x_2 - \varepsilon x_1 h(x_1) - x_1 x_2$

$$= -\varepsilon x_1 h(x_1)$$

$\Rightarrow \dot{V} > 0$ for $x_1 < a$ and $\dot{V} < 0$ for $x_1 > a$



Thus V increases over AB and DE but decreases over BCD.

By choosing sufficiently large p, we can ensure there is net decrease over ABCDE $\Rightarrow V(E) = (p')^2 < V(A) = p^2 \Rightarrow p' < p$.

- Also if $(x_1(t), x_2(t))$ solution of $\begin{cases} \dot{x}_1 = x_2 - \varepsilon h(x_1) \\ \dot{x}_2 = -x_1 \end{cases}$, then so is $(-x_1(t), -x_2(t))$ since $\begin{cases} -\dot{x}_1 = -x_2 - \varepsilon h(-x_1) \\ -\dot{x}_2 = -(-x_1) \end{cases} \Leftrightarrow \begin{cases} \dot{x}_1 = x_2 - \varepsilon h(x_1) \\ \dot{x}_2 = -x_1 \end{cases}$

This implies "mirror reflection" of ABCDE is also a trajectory.

- Define $M \equiv$ AEFHA. $\Rightarrow$ M closed, bdd, inv. Since trajectory starting at a point on AE or FH $\Rightarrow$ tangential to M trajectory starting at a point on EF or HA $\Rightarrow$ inward to M

- From Poincare-Bendixon criterion, M contains a closed orbit.

- Also, AEFHA is a closed orbit iff $p' = -p$, and so there exist a unique closed orbit. Let p_0 be such p.

- Suppose $p > p_0$ (starting point outside orbit) $\Rightarrow -p' < p$ (p': pt. after half rotation)

Due to symmetry, $\underbrace{p_0 \le}_{\text{no-crossing}} \underbrace{p'' < p}_{\text{symmetry}}$ (p'': pt. after full rotation) $\Rightarrow$ unique closed orbit is stable.

Existence of periodic orbit (ctnd.)

Bendixson Criterion: If, on a simply connected D, $\overbrace{\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}}^{\mathrm{div}(f)} \neq 0$ and does not change sign $\Rightarrow$ no periodic orbit in D.

Since $\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$, on a closed orbit γ, $\int_\gamma (f_1 dx_2 - f_2 dx_1) = 0$

From Green's thm, $\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 = 0$ (S region inside γ)

From hypothesis, D cannot contain a closed orbit γ.

(D simply connected if region enclosed by a simple curve C in D is subset of D. Annular region $0 < c_1 \leq x_1^2 + x_2^2 \leq c_2$ not simply connected)

Example:
$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_1 + bx_2 - x_1^2 x_2 - x_1^3 \end{aligned}\right\} \Rightarrow \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 + (b - x_1^2)$$

So if $b<0$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$ and does not change sign.
$\Rightarrow$ no periodic orbit.

Index Criterion: When a closed curve C is traversed counterclockwise the vector-field orientation changes, and in one traversal of C it rotates by $2\pi k$. k is called index of C. If C encloses an eq. pt., then k also called index of that eq. pt.

- index of node / focus / center / closed orbit $= +1$
- index of hyperbolic saddle $= -1$
- index of closed curve $=$ sum of indices of eq. pts. inside it.

$\Rightarrow$
- periodic orbit contains an eq. pt. (index of orbit $= 1$ $=$ sum of indices of eq. pt. inside)
- All eq. pts inside closed orbit hyperbolic $\Rightarrow$ #(node / focus) $-$ #(saddle) $= 1$.

Example:
$$\left.\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2 \end{aligned}\right\} \Rightarrow \text{eq. pts. at } (0,0) \text{ and } (1,1)$$

Since then index will be -1 for $(0,0)$, or 0 for $\{(0,0),(1,1)\}$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1+x_2 & x_1 \\ 1-2x_2 & 1-2x_1 \end{bmatrix} \Rightarrow \text{at } (0,0) = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}_{\text{saddle}}, \text{ at } (1,1) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{\text{stable focus}}$$

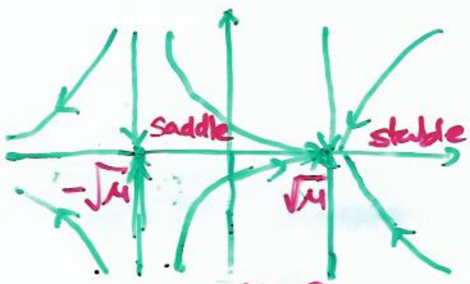
$\Rightarrow$ period orbit can only enclose $(1,1)$; enclosing $(0,0)$ or both $\{(0,0),(1,1)\}$ not possible.

Bifurcation

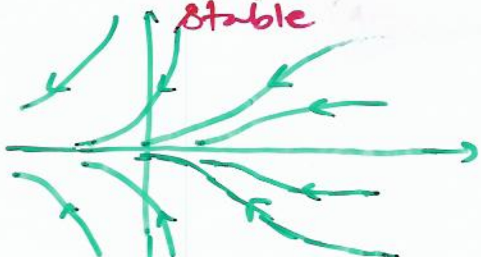
- Bifurcation is "lack of structural stability", where a small change in some parameter causes the nature of equilibrium to change. Parameter known as bifurcation parameter; value where change occurs called bifurcation point.

Saddle-stable: $\begin{rcases} \dot{x}_1 = \mu - x_1^2 \\ \dot{x}_2 = -x_2 \end{rcases} \Rightarrow$ at eq. pt., $\begin{cases} x_1 = \pm\sqrt{\mu},\ x_2 = 0. & \text{if } \mu > 0 \\ \text{no eq. pt.} & \text{if } \mu < 0 \end{cases}$

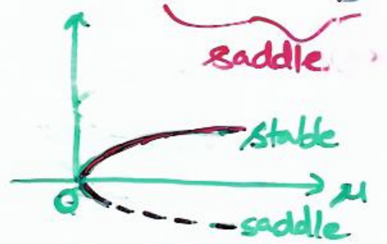
$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ At } (\sqrt{\mu},0) \Rightarrow \underbrace{\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}}_{\text{stable}}, \text{ at } (-\sqrt{\mu},0) \Rightarrow \underbrace{\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}}_{\text{saddle}}$$



$\mu > 0$



$\mu < 0$



saddle-stable bifurcation

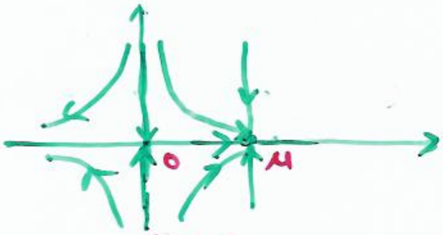
As μ decreases, saddle & stable approach each other, and "collide" when $\mu = 0$, and disappear when $\mu < 0$.

Transcritical: $\begin{rcases} \dot{x}_1 = \mu x_1 - x_1^2 \\ \dot{x}_2 = -x_2 \end{rcases} \Rightarrow$ at eq. pt., $\{ x_1 = 0 \text{ or } \mu,\ x_2 = 0$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \mu - 2x_1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ At } (0,0) \Rightarrow \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}, \text{ at } (\mu,0) \Rightarrow \begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$$

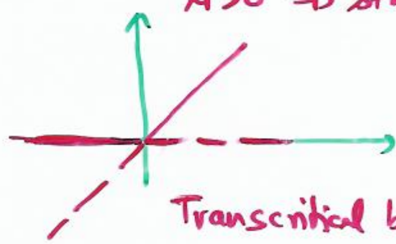
At $(0,0)$: $\mu < 0 \Rightarrow$ stable; $\mu > 0 \Rightarrow$ saddle

At $(\mu,0)$: $\mu < 0 \Rightarrow$ saddle; $\mu > 0 \Rightarrow$ stable



$\mu > 0$

$\mu < 0$



Transcritical bifurcation

In transcritical, behavior does not change so drastically.

$\mu > 0: \{x_1 > 0\} \to (\mu, 0)$; $\mu < 0: \{x_1 > \mu\} \to (0,0)$

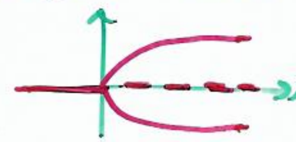
Transcritical: "soft" or "safe"

Saddle-stable: "hard" or "unsafe"

Bifurcation (ctnd.)

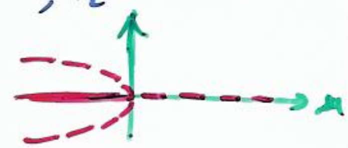
Supercritical pitchfork: $\dot{x}_1 = \mu x_1 - x_1^3$
$\dot{x}_2 = -x_2$ $\Rightarrow$ at eq. $\begin{cases} x_1 = 0 \text{ or } \pm\sqrt{\mu}, & x_2 = 0 & \mu > 0 \\ x_1 = 0 & , x_2 = 0 & \mu < 0 \end{cases}$

$\frac{\partial f}{\partial x} = \begin{bmatrix} \mu - 3x_1^2 & 0 \\ 0 & -1 \end{bmatrix}$. $\quad (0,0) \Rightarrow \underbrace{\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\mu > 0 \Rightarrow \text{saddle} \\ \mu < 0 \Rightarrow \text{stable}}} \quad (0, \pm\sqrt{\mu}) \Rightarrow \underbrace{\begin{bmatrix} -2\mu & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\mu > 0 \text{ stable} \\ \mu < 0 \text{ nonexistant}}}$



Subcritical pitchfork: $\left.\begin{aligned} \dot{x}_1 &= \mu x_1 + x_1^3 \\ \dot{x}_2 &= -x_2 \end{aligned}\right\} \Rightarrow$ at eq. $\begin{cases} x_1 = 0, \ x_2 = 0 & \mu > 0 \\ x_1 = 0 \text{ or } \pm\sqrt{\mu}, \ x_2 = 0 & \mu < 0 \end{cases}$

$\frac{\partial f}{\partial x} = \begin{bmatrix} \mu + 3x_1^2 & 0 \\ 0 & -1 \end{bmatrix}$. $\quad (0,0) \Rightarrow \underbrace{\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\mu > 0 \Rightarrow \text{saddle} \\ \mu < 0 \Rightarrow \text{stable}}} \quad (0, \pm\sqrt{\mu}) \Rightarrow \underbrace{\begin{bmatrix} 2\mu & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\mu > 0 \Rightarrow \text{nonexistant} \\ \mu < 0 \Rightarrow \text{saddle}}}$



Supercritical "safe" but subcritical "dangerous".

Sup: Stable at $(0,0)$ changes to stable at $(\pm\sqrt{\mu}, 0)$ when μ becomes $-$ve to $+$ve

Sub: stable at $(0,0)$ changes to saddle at $(0,0)$ when μ becomes $-$ve to $+$ve

In all above 4 cases, one eigen value zero at bifurcation point. So these also called, zero eigenvalue bifurcation. (Bifurcation caused by eigen value crossing zero.) Next types caused by complex eigenvalues crossing imaginary axis.

Supercritical Hopf bifurcation:

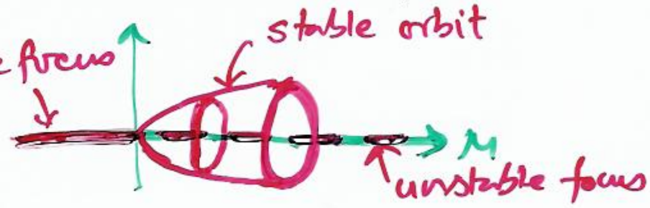
$$\left.\begin{aligned} \dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1 \end{aligned}\right\} \Rightarrow \begin{aligned} & x_1 = r\cos\theta, \ x_2 = r\sin\theta \\ & \dot{r} = \mu r - r^3 \qquad \dot{\theta} = 1. \end{aligned}$$

$$\dot{r} = 0 \Rightarrow \left.\begin{cases} r = 0 & \mu < 0 \\ r = 0 \text{ or } r = \sqrt{\mu} & \mu > 0 \end{cases}\right\} \quad (0,0) \text{ eq. pt.}; \quad (\sqrt{\mu}, t) \text{ orbit}$$

$$\begin{cases} \frac{\partial f}{\partial x} = \begin{bmatrix} x_1(-2x_1) + (\mu - x_1^2 - x_2^2) & x_1(-2x_2) - 1 \\ x_2(-2x_1) + 1 & x_2(-2x_2) + (\mu - x_1^2 - x_2^2) \end{bmatrix}_{(0,0)} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Rightarrow \lambda_{1,2} = \mu \pm j \\ \mu < 0 \Rightarrow (0,0) \text{ stable focus}; \quad \mu > 0 \Rightarrow (0,0) \text{ unstable focus.} \end{cases}$$

$(\sqrt{\mu}, t)$: stable periodic orbit

This is a "safe" bifurcation.



Bifurcation (ctnd.)

Subcritical Hopf bifurcation:

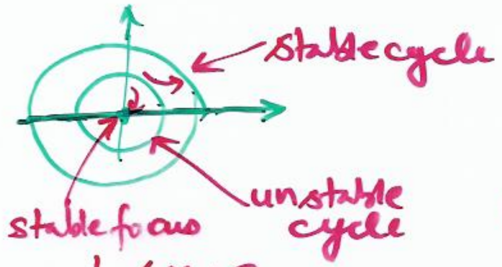
$$\left.\begin{aligned}\dot x_1 &= x_1\left[\mu + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2\right] - x_2\\ \dot x_2 &= x_2\left[\mu + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2\right] + x_1\end{aligned}\right\} \quad x_1 = r\cos\theta,\ x_2 = r\sin\theta.$$

$$\Rightarrow \left.\begin{aligned}\dot r &= \mu r + r^3 - r^5\\ \dot\theta &= 1\end{aligned}\right\} \quad \dot r = 0 \Rightarrow r = 0 \text{ or } \mu + r^2 - r^4 = 0 \Leftrightarrow r^2 = \frac{+1 \pm \sqrt{1+4\mu}}{2}$$

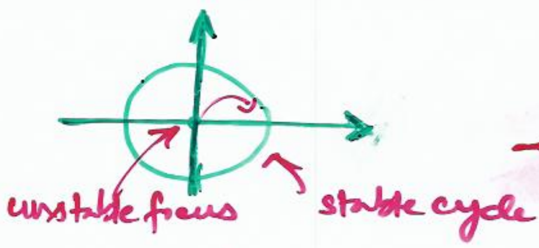
$$\begin{cases} \mu < -\frac{1}{4} \Rightarrow r = 0 \text{ eq. pt.} \\ -\frac{1}{4} < \mu < 0 \Rightarrow r = 0 \text{ eq. pt.} \quad r^2 = (1 \pm \sqrt{1+4\mu})/2 \\ \mu > 0 \Rightarrow r = 0 \text{ eq. pt.},\ r^2 = \dfrac{1 + \sqrt{1+4\mu}}{2} \end{cases}$$

$-\frac{1}{4} < \mu < 0$: (0,0) stable focus, $\sqrt{(1+\sqrt{1+4\mu})/2}$ stable cycle, $\sqrt{(1-\sqrt{1+4\mu})/2}$ unstable cycle.

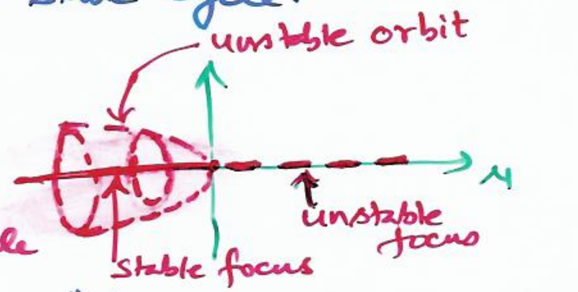
$\mu > 0$: (0,0) unstable focus, $\sqrt{(1+\sqrt{1+4\mu})/2}$ stable cycle.



$-\frac{1}{4} < \mu < 0$



$\mu > 0$



"dangerous" bifurcation.

Above examples of "local" bifurcations (eq. pt. nonhyperbolic at bif. pt.)
Global bifurcations (not local to an eq. pt.) also possible.

Saddle-connection/Homoclinic bifurcation:

$$\left.\begin{aligned}\dot x_1 &= x_2\\ \dot x_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2\end{aligned}\right\} \text{ at eq.} \Rightarrow x_1 = 0 \text{ or } 1,\ x_2 = 0 \quad \text{(note: eq. independent of } \mu)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - 2x_1 + x_2 & \mu + x_1 \end{bmatrix}. \text{ At } (0,0) \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & \mu \end{bmatrix}, \text{ at } (1,0) \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & \mu+1 \end{bmatrix}$$

$$\lambda I - \begin{bmatrix} 0 & 1 \\ 1 & \mu \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ -1 & \lambda - \mu \end{bmatrix} \Rightarrow \det() = \lambda^2 - \mu\lambda - 1 \Rightarrow \lambda = \frac{\mu \pm \sqrt{\mu^2+4}}{2} \Rightarrow \text{saddle}$$

$$\lambda I - \begin{bmatrix} 0 & 1 \\ -1 & \mu+1 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda - \mu - 1 \end{bmatrix} \Rightarrow \det() = \lambda^2 - (\mu+1)\lambda + 1 \Rightarrow \lambda = \frac{(\mu+1) \pm \sqrt{(\mu+1)^2 - 4}}{2} \Rightarrow \text{unstable focus for } -1 < \mu < 1$$