Phase portrait of 2D-linear system (Summary)

CASE	Subcases	Eq. pt. behavior	Comments
$\lambda_1 \neq \lambda_2 \neq 0$	$\lambda_2 < \lambda_1 < 0$ $\lambda_2 > \lambda_1 > 0$ $\lambda_2 < 0 < \lambda_1$	origin stable node origin unstable node origin saddle node	
$\lambda_1 = \alpha + j\beta$ $\lambda_2 = \alpha - j\beta$ $\beta \neq 0$	$\alpha < 0$ $\alpha = 0$ $\alpha > 0$	origin stable focus origin center origin unstable focus	$\beta > 0 \Leftrightarrow$ rotate counterclockwise
$\lambda_1 = \lambda_2 = \lambda \neq 0$	$\lambda < 0$ $\lambda > 0$	origin stable node origin unstable node	$K=0 \Rightarrow$ linear trajectory $K=1 \Rightarrow$ linear + logrithmic trajectory.
$0 = \lambda_1 \neq \lambda_2$	$\lambda_2 < 0$ $\lambda_2 > 0$	Eq. line stable Eq. line unstable	
$\lambda_1 = \lambda_2 = 0$	$K = 0$ $K = 1$	Eq. plane stable Eq. plane unstable	

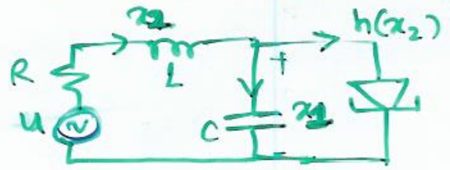
Structural Stability

- Does nature of equilibrium point remain same under parametric perturbations "$A \to A + \Delta A$"?
- Small change in parameter $\Rightarrow$ small change in eigen values (perturbation thy)
  $\Rightarrow$ open half plane eigen values remain in same open half plane under perturbation
  But, imaginary axis eigen values may move to LHP or RHP, i.e, either stable or unstable focus.
- Node/focus : Structurally stable
  Center: Structurally unstable
  (center can change into stable/unstable focus upon perturbation)
- Hyperbolic: No eigenvalue is purely imaginary.
- One eigen value zero $\Rightarrow$ perturbation can cause eq. line to change to a eq. node (saddle or stable/unstable)
- Both eigen value zero $\Rightarrow$ perturbation can cause eq. plane to change to a center/node/focus/saddle.

Muliple Equilibria (of nonlinear system)

- (Eq. set of $\dot{x} = Ax$) = (null space of A) ⇒ an entire subspace, and so only one isolated eq. possible (when $\det(A) \neq 0$).

- Situation is different for nonlinear systems. Recall tunnel diode

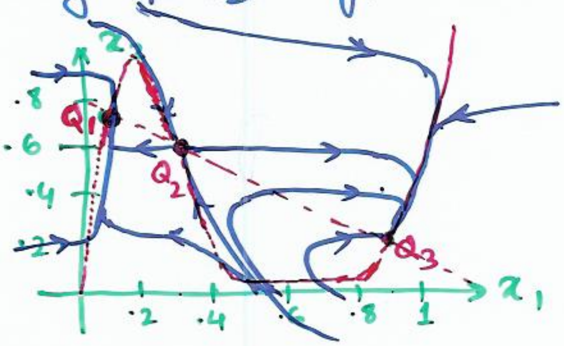


$$C\dot{x}_1 = x_1 - h(x_1)$$
$$L\dot{x}_2 = u - Rx_2 - x_1$$

Choose $u = 1.2V$, $R = 1.5 k\Omega$, $C = 2pF$, $L = 5\mu H$.

Suppose $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$

setting $\dot{x}_1 = \dot{x}_2 = 0$ gives the eq. pts at: $\underbrace{(.063, .758)}_{Q_1}$, $\underbrace{(.285, .61)}_{Q_2}$, $\underbrace{(.884, .21)}_{Q_3}$

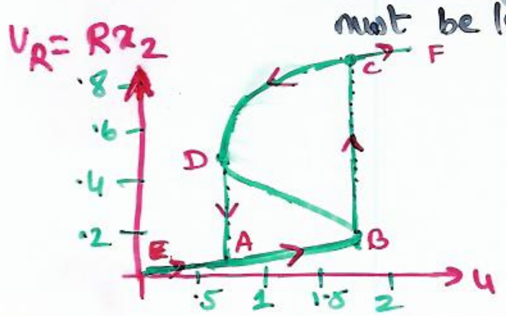


Phase portrait reveals;
Q_1 and Q_2 stable node, Q_3 saddle node

Only two trajectories converge to Q_3, and together divide the plane into regions of stability of Q_1 and Q_3, and hence called separatrix.

In an expt. set up Q_1 and Q_3 will be observed, depending on initial capacitor voltage/inductor current. (Noise will perturb system to not remain at Q_2). Tunnel diode ckt called bistable ckt. Can be used as 1-bit memory device, state which is altered by applying input pulse of sufficient amp. and duration.

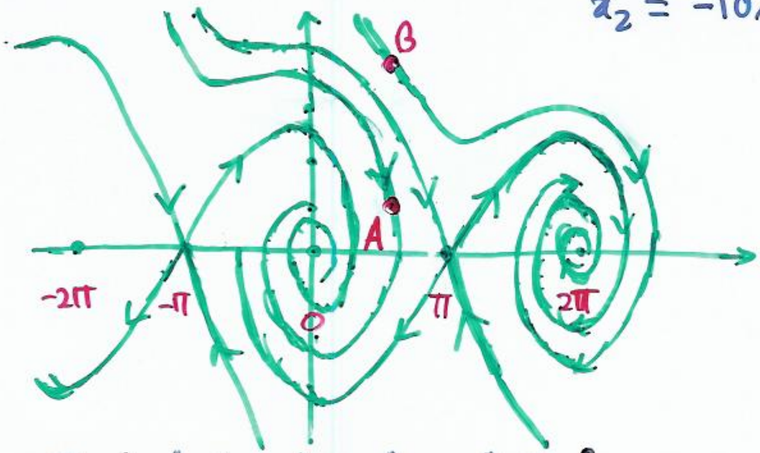
$Q_1 \to Q_3$: Add +ve pulse to existing u so that load line moves up resulting in a single eq. pt. of type Q_3. Duration of pulse must be long enough to all trajectory to move across separatrix.



small u (EA): Single stable eq., Q_1.
medium u (AB): Two stable eq., Q_1 & Q_3
large u (CF): One stable eq., Q_3
As u slowly increased from small to large, jump BC occurs
decreased from large to small, jump DA occurs

Another Example (Pendulum)

- Pendulum with friction: $\dot{x}_1 = x_2$
  $\dot{x}_2 = -10\sin x_1 - x_2$



Eq. pts located at $(\pm n\pi, 0)$

$(\pm 2m\pi, 0)$ stable focus

$(\pm(2m+1)\pi, 0)$ saddle node

Stable trajectories of saddle node form separatrix.

Trajectories from A and B have same initial position but different speed. A stabilizes to $(0,0)$, whereas B stabilizes to $(2\pi, 0) \Rightarrow$ after 1 rotation.

Qualitative behavior near equilibrium

- In tunnel diode ckt: behavior near Q_1 & Q_3 of stable node, near Q_2 of saddle
- In pendulum: Behavior near $(\pm \text{even} * \pi, 0)$ of stable focus, near $(\pm \text{odd} * \pi, 0)$ of saddle

} like linear system.

- This suggests linearization can be used to study behavior near eq. pt.

$$\dot{x} = f(x) \Rightarrow \delta\dot{x} = \left(\frac{\partial f}{\partial x}\bigg|_{x^*}\right)(\delta x) \Leftrightarrow \dot{x} = Ax.$$

- For tunnel diode ckt, $f_1(x_1, x_2) = \frac{1}{C}[-h(x_1) + x_2]$

$$f_2(x_1, x_2) = \frac{1}{L}[-x_1 - Rx_2 + u]$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{1}{C}[-h'(x_1)] & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}h'(x) & \frac{1}{2} \\ -\frac{1}{5} & -\frac{3}{10} \end{bmatrix} \qquad \begin{matrix} C = 2 \\ L = 5 \\ R = 3 \end{matrix}$$

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.74x_1^5$$

$$\Rightarrow h'(x_1) = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4$$

$$\Rightarrow h'(x_1)\Big|_{Q_1} = h'(x_1)\Big|_{.063} = 7.196 \ ; \quad h'(x_1)\Big|_{Q_2} = h'(x_2)\Big|_{.285} = 3.64$$

$$h'(x_1)\Big|_{Q_3} = h'(x_1)\Big|_{.884} = 2.854$$

Qualitative behavior near equilibrium

$A_1 = \begin{bmatrix} -3.598 & .5 \\ -.2 & -.3 \end{bmatrix} \Downarrow \lambda = (-3.57, -.33) \Downarrow$ stable node

$A_2 = \begin{bmatrix} 1.82 & .5 \\ -.2 & -.3 \end{bmatrix} \Downarrow \lambda = (1.77, -.25) \Downarrow$ saddle node

$A_3 = \begin{bmatrix} -1.427 & .5 \\ -.2 & -.3 \end{bmatrix} \Downarrow \lambda = (-1.33, -.4) \Downarrow$ stable node

For pendulum, $\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10\sin x_1 + x_2 \end{aligned}\right\} \Rightarrow \dfrac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & 1 \end{bmatrix}$

@ $(0,0)$, $A_1 = \begin{bmatrix} 0 & 1 \\ -10 & 1 \end{bmatrix} \Rightarrow \lambda = -.5 \pm j3.12 \Rightarrow$ stable focus

@ $(\pi, 0)$, $A_2 = \begin{bmatrix} 0 & 1 \\ +10 & 1 \end{bmatrix} \Rightarrow \lambda = -3.7, 2.7 \Rightarrow$ saddle node

- Linearization near equilibrium has hyperbolic $(\forall \lambda : Re(\lambda) \neq 0)$ equilibrium
  $\Rightarrow$ nonlinear system behaves similar to the linear one near equilibrium.
- Situation is different if $\exists \lambda : Re(\lambda) = 0$.
  (To be expected since non-hyperbolic equilibrium not structurally stable)

Example: $f = \begin{bmatrix} -x_2 - \mu x_1(x_1^2 + x_2^2) \\ x_1 - \mu x_2(x_1^2 + x_2^2) \end{bmatrix}$ $\qquad f = 0 \Rightarrow x_1 = x_2 = 0$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\mu x_1(2x_1) - \mu(x_1^2 + x_2^2) & -1 - 2\mu x_1 x_2 \\ 1 - 2\mu x_2 x_1 & -\mu x_2(2x_2) - \mu(x_1^2 + x_2^2) \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda I - A = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 + 1 \Rightarrow \lambda = \pm j \Rightarrow (0,0) \text{ center.}$$

To see nonlinear behavior, change of variable, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.

$$\Rightarrow r^2 = x_1^2 + x_2^2 \Rightarrow 2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \Rightarrow \dot{r}$$

$$\Rightarrow r\dot{r} = x_1(-x_2 - \mu x_1 r^2) + x_2(x_1 - \mu x_2 r^2) = -\mu r^2(x_1^2 + x_2^2) = -\mu r^4$$

$x_2\cos\theta = x_1\sin\theta \Rightarrow \boxed{\dot{r} = -\mu r^3}$ $\dot{x}_2 \cos\theta - x_2\sin\theta\,\dot{\theta} = \dot{x}_1\sin\theta + x_1\cos\theta\,\dot{\theta}$

Qualitative behavior near eq. pt.

$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \Rightarrow x_1 \sin\theta = x_2\cos\theta \; (= r\sin\theta\cos\theta)$

$\Rightarrow \dot{x}_1 \sin\theta + x_1 \cos\theta\, \dot{\theta} = \dot{x}_2 \cos\theta - x_2 \sin\theta\, \dot{\theta}$

$\Rightarrow (-x_2 - \mu x_1 r^2)\sin\theta + r\cos^2\theta\, \dot{\theta} = (x_1 - \mu x_2 r^2)\cos\theta - r\sin^2\theta\, \dot{\theta}$

$\Rightarrow -r\sin^2\theta - \cancel{\mu r^3 \sin\theta\cos\theta} + r\cos^2\theta\, \dot{\theta} = r\cos^2\theta - \cancel{\mu r^3 \sin\theta\cos\theta} - r\sin^2\theta\, \dot{\theta}$

$\Rightarrow (r\cos^2\theta + r\sin^2\theta)\,\dot{\theta} = (r\cos^2\theta + r\sin^2\theta) \Rightarrow \boxed{\dot{\theta} = 1}$

$\begin{cases} \dot{r} = -\mu r^3 \Rightarrow r(t) \text{ increases/decrease as } t \text{ increases when } \mu>0/\mu<0 \\ \dot{\theta} = 1 \Rightarrow \theta(t) \text{ increases at a const rate of } 1. \end{cases}$

$\left.\begin{array}{l} \mu > 0 \Rightarrow \text{unstable focus} \\ \mu < 0 \Rightarrow \text{stable focus} \end{array}\right\}$ whereas linearized system eq. pt. a "center"

Limit Cycles

- A limit cycle is an isolated periodic orbit, where a periodic orbit is one satisfying, $x(t) = x(t+T)$ for some $T > 0$ (periodic solution excludes "dc" or const. solutions which trivially satisfies $x(t) = x(t+T)$).

- For linear systems, periodic orbit iff imaginary eigenvalues.
  - Such orbits are not structurally stable (or robust)
  - Such orbits are not isolated
  - Amp. depends on initial condition (although freq. doesn't)

$\lambda = \pm j\beta \Rightarrow z_1(t) = r_0 \cos(\beta t + \theta), \quad z_2(t) = r_0 \sin(\beta t + \theta)$

$\Updownarrow \qquad r_0 = \sqrt{z_1^2(0) + z_2^2(0)} \; ; \; \theta = \tan^{-1}\left(\frac{z_2(0)}{z_1(0)}\right).$

$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$

(Linear systems do not possess limit cycles.)

Limit Cycles (in Nonlinear Systems)

- Nonlinear systems possess limit cycles that are structurally stable
- Consider for example -ve resistance ckt.

$$L\frac{di}{dt} = v$$

$$C\frac{dv}{dt} = -h(v) - i$$

$$\tau = \frac{1}{\sqrt{LC}}t \Rightarrow C\dot{v} = C\dot{v}\frac{1}{\sqrt{LC}} = \frac{\dot{v}}{\varepsilon}$$

$$\Rightarrow C\ddot{v} = -h'(v)\dot{v} - \frac{v}{L}$$

$$\tau = \sqrt{\frac{1}{LC}}\,t \Rightarrow d\tau = \sqrt{\frac{1}{LC}}\,dt \Rightarrow d\tau^2 = \frac{1}{LC}dt^2$$

$$\Rightarrow \frac{d^2\tau}{dt^2} = \frac{1}{LC}$$

$$\Rightarrow C\frac{1}{LC}\ddot{v} = -h'(v)\sqrt{\frac{1}{LC}}\,\dot{v} - \frac{v}{L} \Rightarrow \boxed{\ddot{v} = -\sqrt{\frac{L}{C}}\,h'(v)\,\dot{v} - v = -\varepsilon h'(v)\dot{v} - v}$$

$$x_1 = v,\quad x_2 = \dot{v} \Rightarrow \begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -\varepsilon h'(x_1)x_2 - x_1\end{cases}$$

eq. pt. $\dot{x}_1 = \dot{x}_2 = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$ (since $h'(x_1)\big|_0 \neq 0$).

Linearization @ $(0,0) \Rightarrow A = \begin{bmatrix}0 & 1\\ -1-\varepsilon h''(x_1)x_2 & -\varepsilon h'(x_1)\end{bmatrix}$

Near origin $h'(x_1) = \text{const} \Rightarrow h''(x_1) = 0$. So $A = \begin{bmatrix}0 & 1\\ -1 & -\varepsilon h'(0)\end{bmatrix}$

$$\Rightarrow \lambda I - A = \begin{bmatrix}\lambda & -1\\ 1 & \lambda + \varepsilon h'(0)\end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 + \varepsilon\lambda h'(0) + 1$$

$$\Rightarrow \lambda = \frac{-\varepsilon h'(0) \pm \sqrt{\varepsilon^2 h'(0)^2 - 4}}{2}$$

$$h'(0) < 0 \Rightarrow -\varepsilon h'(0) > 0 \Rightarrow \begin{cases}\text{unstable node} & \text{if } |\varepsilon h'(0)| \geq 2\\ \text{unstable focus} & \text{if } |\varepsilon h'(0)| < 2\end{cases}$$

- Thus all trajectories starting from (0,0) move away from (0,0)
- Consider system energy $E = \frac{1}{2}Cv^2 + \frac{1}{2}Li^2 = \frac{1}{2}Cx_1^2 + \frac{1}{2}L\left(-h(x_1) - \frac{x_2}{\varepsilon}\right)^2$

$$= \frac{1}{2}C\left[x_1^2 + (\varepsilon h(x_1) + x_2)^2\right]$$

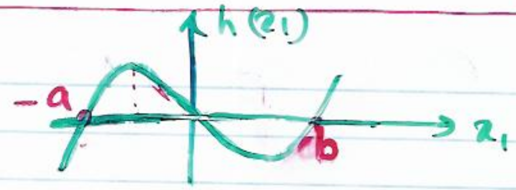
$$\Rightarrow \dot{E} = C\left[x_1\dot{x}_1 + (\varepsilon h(x_1) + x_2)(\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2)\right]$$

$$= C\left[x_1x_2 + (\varepsilon h(x_1) + x_2)(\cancel{\varepsilon h'(x_1)x_2} - \cancel{\varepsilon h'(x_1)x_2} - x_1)\right]$$

$$= -\varepsilon C\,h(x_1)\,x_1 > 0 \text{ (near origin)}$$

Limit Cycle in -ve resistance ckt

$\dot{E} = -\varepsilon C x_1 h(x_1)$



$\Rightarrow \begin{cases} \dot{E} > 0 & \text{for } -a \le x_1 \le b \\ \dot{E} < 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ {System gains energy in strip $-a \le x \le b$ ($x_1 h(x_1) \le 0$ in this strip)
System loses energy outside strip}

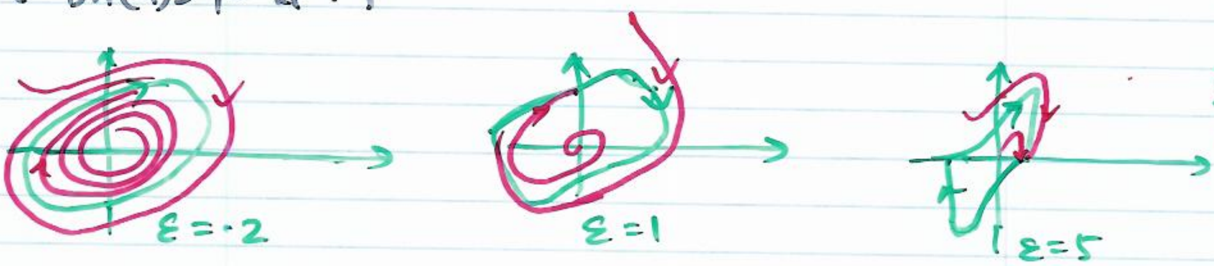
$\Rightarrow$ Stationary oscillation occurs when, along a trajectory, net exchange of energy over a cycle is zero.

__Example__: Van der Pol oscillator:

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + \varepsilon (1 - x_1^2) x_2$

$\left\{ \begin{array}{l} h(x_1) = x_1 - \frac{1}{3} x_1^3 \\ \Rightarrow h'(x_1) = 1 - x_1^2. \end{array} \right.$



Different "stable" limit cycles"

- Phase portrait can be used to verify Van der Pol oscillator has a single stable limit cycle, and origin is unstable focus for small ε, and an unstable node otherwise

- Higher ε shows "jumps" in the orbit near "corner points". This is known as relaxation oscillation.

- Note isolated nature of the orbit. (Not possible for linear system which has continuum of orbits or none.)

- Van der Pol eq. in reverse time: ($t \to -t$)

  $\dot{x}_1 = -x_2$

  $\dot{x}_2 = x_1 - \varepsilon (1 - x_1^2) x_2$

This system has a limit cycle that is unstable.