2^{nd} Order Systems

- 2^{nd} order systems significant since trajectories can be plotted on 2-D plane $\Rightarrow$ visual examination possible
- $x_1 - x_2$ plane : Phase plane or State plane.
- 2^{nd} order $\Rightarrow$

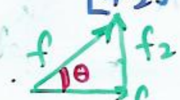
$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\} \text{ assuming TI \& autonomous}$$

$$\Updownarrow$$

$$\dot{x} = f(x)$$

- Trajectory starting at x_0: Locus of $x(t)$ starting at $x(0) = x_0$
- Slope of trajectory in phase plane $= \dfrac{dx_2}{dx_1} = \dfrac{dx_2}{dt} \Big/ \dfrac{dx_1}{dt} = \dfrac{f_2}{f_1}$

The vector $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ has the same slope:



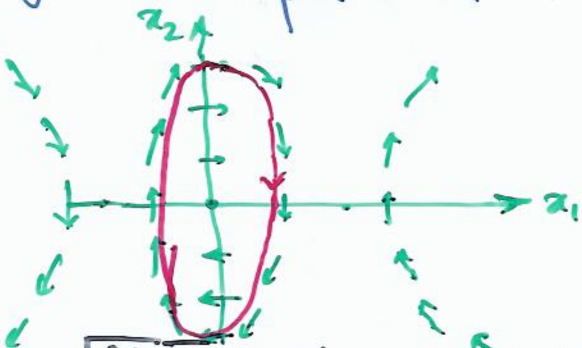
$\Rightarrow \tan\theta = \dfrac{f_2}{f_1}$.

- Thus by plotting vector f at several points in $x_1 - x_2$ plane we can approximately plot trajectory. Such a plot called "vector field".

- **Pendulum's vector field:**

$$\dot{x}_1 = x_2$$

$$x_2 = -10\sin x_1$$



Length of arrow proportion to $\|f\| = \sqrt{f_1^2 + f_2^2}$ at each point.

- Family of all trajectories called phase portrait, which can be obtained by drawing trajectories for several initial states.

- Note a phase portrait shows $x_1 - x_2$ plot, and does not show the "motion" as "t" evolves. So it is "qualitative" in nature. The quantitave information (motion as fn. of time) is not included.

Phase Portrait of 2^{nd} order Linear Systems

- 2^{nd} order linear $\Rightarrow$ $\dot{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x \equiv Ax \equiv (M J_r M^{-1})x$

Equilibrium: $Ax = 0 \Rightarrow x = 0$ if $\det(A) \neq 0$, else eq. a subspace.
J_r : Jordan form can be of these three forms:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

distinct real eigen values | repeated real eigen values ($k = 0$ or 1) | complex eigen values $\lambda_{1,2} = \alpha \pm j\beta$.

M: Matrix of (extended) eigen vectors of A.

- $\dot{x} = M J_r M^{-1} x \Rightarrow M^{-1}\dot{x} = J_r M^{-1} x \Rightarrow \dot{z} = J_r z$ ($z = M^{-1}x$ modal coordinates).

CASE 1 $\lambda_1 \neq \lambda_2 \neq 0$ $\Rightarrow$ $M = [v_1 \ v_2]$ with $Av_i = \lambda_i v_i$
($\Rightarrow \det(A) \neq 0 \Rightarrow x = 0$ eq. pt.)

Also $\dot{z}_i = \lambda_i z_i \Rightarrow z_i(t) = e^{\lambda_i t} z_i(0)$

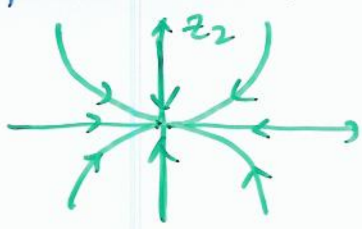
$$\Rightarrow z_2 = c z_1^{\lambda_2/\lambda_1} \quad \left(c = \frac{z_2(0)}{(z_1(0))^{\lambda_2/\lambda_1}} \right)$$

1.1 $\lambda_1, \lambda_2 < 0$: WLOG $\lambda_2 < \lambda_1 < 0$ (λ_2: "faster", λ_1: "slower")

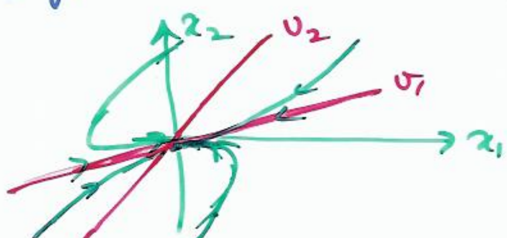
$z_i(t) \to 0$ as $t \to \infty$ $\Rightarrow$ origin stable

$$\frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{(\lambda_2/\lambda_1 - 1)} \quad \begin{array}{l} \longrightarrow 0 \quad \text{as } |z_1| \to 0 \\ \longrightarrow \infty \quad \text{as } |z_1| \to \infty \end{array}$$

$\Rightarrow$ trajectory tangential to z_1 near origin, perpendicular to z_1 (or parallel to z_2) away from origin



Modal plane



Phase plane

1.2 $\lambda_1, \lambda_2 > 0$: WLOG $\lambda_2 > \lambda_1 > 0$

$z_i(t) \to \infty$ as $t \to \infty$ ($\Rightarrow$ origin unstable)

$\frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{(\lambda_2/\lambda_1 - 1)}$ $\Rightarrow$ phase portrait same character but origin being unstable trajectories reversed.

Phase portrait of 2^{nd} order linear systems

1.3 λ_1, λ_2 opposite sign : WLOG $\lambda_2 < 0 < \lambda_1$.

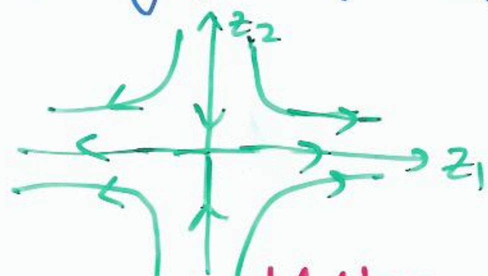
$$\Rightarrow z_1(t) \xrightarrow[t\to\infty]{} \infty \quad , \quad z_2(t) \xrightarrow[t\to\infty]{} 0$$

$$\frac{dz_2}{dz_1} = c\,\frac{\lambda_2}{\lambda_1}\, z_1^{\left(\frac{\lambda_2}{\lambda_1} - 1\right)} \quad \Rightarrow \text{exponent of } z_1 \text{ is } -ve$$

$$\Rightarrow \text{slope} \longrightarrow 0 \quad |z_1| \to \infty$$
$$\text{slope} \longrightarrow \infty \quad |z_1| \to 0$$

$\Rightarrow$ near origin parallel to z_2, away from origin tangent to z_1

phase plane



modal plane

Hyperbolic shape, except along
z_1/v_1 : unstable
z_2/v_2 : stable

Origin in the above case is a "saddle point".

CASE 2 $\lambda_{1,2} = \alpha \pm j\beta$ $\qquad \dot{z}_1 = \alpha z_1 + \beta z_2 \;,\; \dot{z}_2 = \beta z_1 + \alpha z_2$

consider $(r, \theta) = \left(\sqrt{z_1^2 + z_2^2}\,,\, \tan^{-1}\left(\frac{z_2}{z_1}\right)\right)$, i.e., polar coordinates.

$$\Rightarrow r^2 = z_1^2 + z_2^2 \Rightarrow \cancel{2} r\dot{r} = \cancel{2} z_1\dot{z}_1 + \cancel{2} z_2 \dot{z}_2$$
$$= z_1(\alpha z_1 - \beta z_2) + z_2(\beta z_1 + \alpha z_2)$$
$$= \alpha z_1^2 - \beta z_1 z_2 + \beta z_1 z_2 + \alpha z_2^2 = \alpha(z_1^2 + z_2^2)$$
$$= \alpha r^2$$

$$\Rightarrow \boxed{\dot{r} = \alpha r} \Rightarrow r(t) = e^{\alpha t} r(0)$$

Also, $\tan\theta = \frac{z_2}{z_1} \Rightarrow z_1 \sin\theta = z_2 \cos\theta$ ~~$\Rightarrow \dot{z}_1 \sin\theta + z_1 \cos\theta\,\dot{\theta}$~~

$$\Rightarrow \dot{z}_1 \sin\theta + z_1 \cos\theta\,\dot{\theta} = \dot{z}_2 \cos\theta - z_2 \sin\theta\,\dot{\theta}$$
$$\Rightarrow (\alpha z_1 - \beta z_2)\frac{z_2}{z_1} + z_1\dot{\theta} = (\beta z_1 + \alpha z_2) - z_2\frac{z_2}{z_1}\dot{\theta}$$
$$\Rightarrow \cancel{\alpha z_2} - \beta\frac{z_2^2}{z_1} + z_1\dot{\theta} = (\beta z_1 + \cancel{\alpha z_2}) - \frac{z_2^2}{z_1}\dot{\theta}$$
$$\Rightarrow \frac{z_1^2 + z_2^2}{z_1}\dot{\theta} = \frac{z_1^2 + z_2^2}{z_1}\beta \Rightarrow \boxed{\dot{\theta} = \beta} \Rightarrow \theta(t) = \beta t + \theta(0)$$

Phase portrait of 2nd order linear system

$r(t) = e^{\alpha t} r(0)$, $\theta(t) = \beta t + \theta(0)$ $\Rightarrow$ exponential spiral.

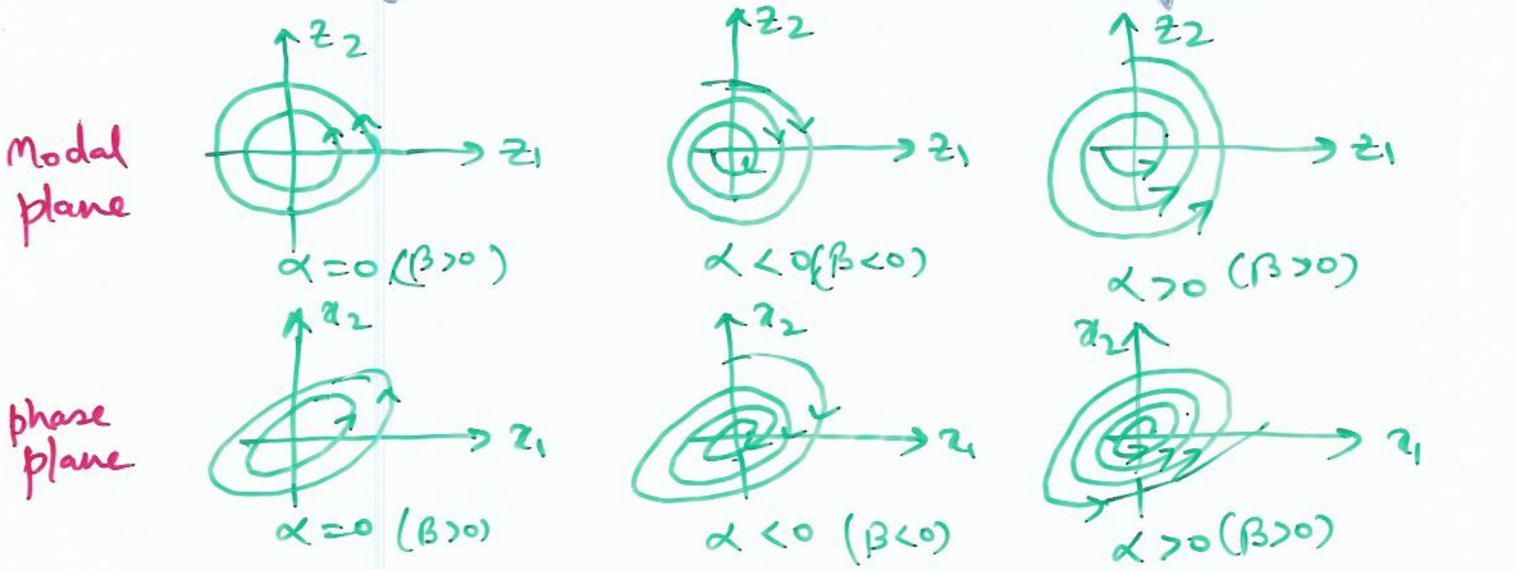
$\alpha = 0 \Rightarrow$ radius stays constant (origin a "center")

$\alpha > 0 \Rightarrow$ radius increases exponentially with time (origin a unstable "focus")

$\alpha < 0 \Rightarrow$ radius decreases exponentially with time (origin an ~~un~~stable "focus")

$\beta > 0 \Rightarrow$ angle rotates counterclockwise linearly with time

$\beta < 0 \Rightarrow$ angle rotates clockwise linearly with time.



CASE III $\lambda_1 = \lambda_2 = \lambda \neq 0$

$\dot{z}_1 = \lambda z_1 + k z_2 \qquad \dot{z}_2 = \lambda z_2$

$\Rightarrow z_2(t) = e^{\lambda t} z_2(0)$, $z_1(t) = e^{\lambda t} z_1(0) + \int_0^t e^{\lambda(t-\tau)} \cdot k e^{\lambda \tau} z_2(0)\, d\tau$

$$= e^{\lambda t} z_1(0) + e^{\lambda t} \int_0^t k z_2(0)\, d\tau$$

$$= e^{\lambda t}\left[z_1(0) + k z_2(0)\, t\right].$$

$\Rightarrow t = \frac{1}{\lambda} \ln \frac{z_2(t)}{z_2(0)}$

Also, $z_1(t) = \frac{z_2(t)}{z_2(0)} \left[z_1(0) + \frac{k}{\lambda} z_2(0) \ln\left(\frac{z_2(t)}{z_2(0)} \right) \right]$

$$\Rightarrow z_1 = z_2 \left[\frac{z_1(0)}{z_2(0)} + \frac{k}{\lambda} \ln \frac{z_2(t)}{z_2(0)} \right]$$

Phase portrait of 2nd order linear system

III.1 $K=0 \Rightarrow z_1 = z_2\left(\dfrac{z_1(0)}{z_2(0)}\right)$

$\lambda<0 \Rightarrow z_1, z_2 \longrightarrow 0$ as $t\to\infty$

$\lambda>0 \Rightarrow z_1, z_2 \longrightarrow \infty$ as $t\to\infty$

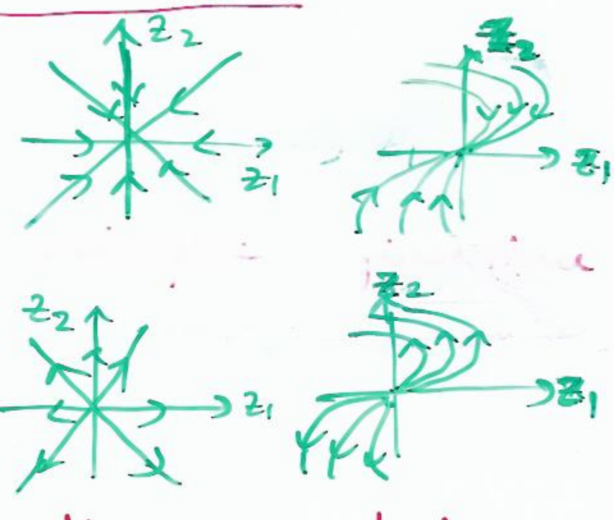
III.2 $K=1 \Rightarrow z_1 = z_2\left(\dfrac{z_1(0)}{z_2(0)}\right) + \dfrac{1}{\lambda}\ln\left(\dfrac{z_2}{z_2(0)}\right)$

$\lambda<0 \Rightarrow z_1, z_2 \longrightarrow 0$ as $t\to\infty$

$\lambda>0 \Rightarrow z_1, z_2 \longrightarrow \infty$ as $t\to\infty$

$\lambda<0$

$\lambda>0$



$k=0$ $\quad$ $k=1$

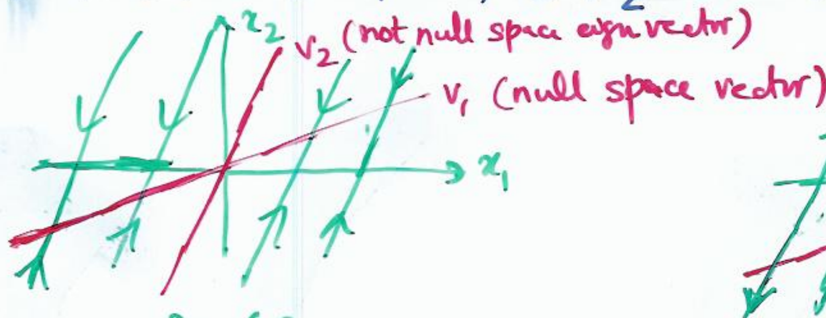
Origin stable node when $\lambda<0$, unstable node when $\lambda>0$.

CASE IV one or both eigen value zero

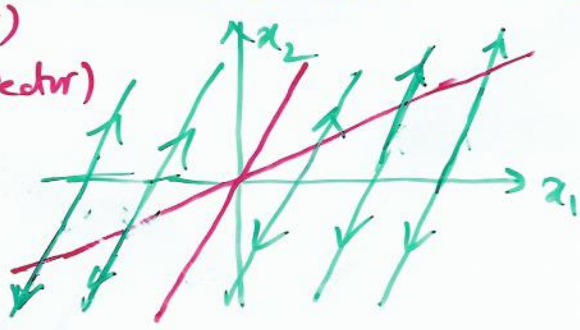
Equilibrium set is subspace with dimension 1 (one eigen value zero)
dimension 2 (both eigen values zero)

$\lambda_1=\lambda_2=0 \Rightarrow \dot z_1 = \dot z_2 = 0 \Rightarrow$ no motion $(z_1(t)=z_1(0),\ z_2(t)=z_2(0))$.

$\lambda_1=0,\ \lambda_2\neq 0 \Rightarrow \dot z_1 = 0,\ \dot z_2 = \lambda_2 z \Rightarrow z_1(t)=z_1(0),\ z_2(t) = e^{\lambda_2 t} z_2(0)$



$\lambda_2<0$



$\lambda_2>0$

$\lambda_2<0 \Rightarrow z_2(t) \xrightarrow[t\to\infty]{} 0$

$\lambda_2>0 \Rightarrow z_2(t) \xrightarrow[t\to\infty]{} \infty$

since z_1 does not change, vertical motion in mode plane (= motion parallel to v_2 in phase plane)

$\lambda_1=\lambda_2=0 \Rightarrow \dot z_1 = z_2,\ \dot z_2 = 0 \Rightarrow z_2(t)=z_2(0)$ and $z_1(t) = z_2(0)\,t$.

$J_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$