Passivity Theory for Stability

A system is passive if it does not generate power. So input power is partly lost and the rest is stored.

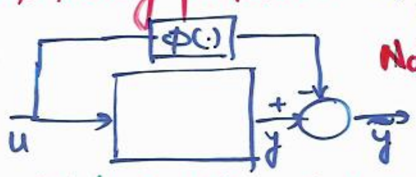
Def:

Time-inv. system with same number of inputs and outputs: $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$

Let V be cont. diff. and ≥ 0.

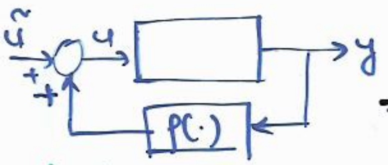
- **passive** if $u^T y \geq \dot{V}$ ($u^T y$: power input, $\dot{V}$: energy storage rate).
- **lossless** if $u^T y = \dot{V}$ (Note: $u^T y$ defined only if same no. of inputs & outputs)
- **input feedforward passive:** $u^T y \geq \dot{V} + u^T \varphi(u)$ for some φ
- **input strictly passive:** $u^T y \geq \dot{V} + u^T \varphi(u)$ and $u^T \varphi(u) > 0$, $\forall u$.
- **output feedback passive:** $u^T y \geq \dot{V} + y^T \rho(y)$ for some ρ
- **output strictly passive:** $u^T y \geq \dot{V} + y^T \rho(y)$ and $y^T \rho(y) > 0$, $\forall y$
- **strictly passive:** $u^T y \geq \dot{V} + \psi(x)$ for some $\psi > 0$.

Note: For memoryless system, no storage possible $\Rightarrow V \equiv 0$.



input feedforward passive

$$\left.\begin{aligned} u^T \tilde{y} &= u^T [y - \varphi(u)] \geq \dot{V} \\ \Rightarrow u^T y &\geq \dot{V} + u^T \varphi(u). \end{aligned}\right\}$$ when $u^T \varphi(u) > 0$, input strict passive



output feedback passive

$$\left.\begin{aligned} \tilde{u}^T y &= [u + \rho(y)]^T y \geq \dot{V} \\ \Rightarrow u^T y &\geq \dot{V} - \rho(y)^T y = \dot{V} + y^T \rho(y). \end{aligned}\right\}$$ when $y^T \rho(y) > 0$, output strict passive.

Results: 1) Passive with +ve definite V $\Rightarrow$ origin is stable for $\dot{x} = f(x,0)$.

2) Output strictly passive with $u^T y \geq \dot{V} + \delta y^T y$ $\Rightarrow$ finite-gain $\mathcal{L}_2$-stable; $\mathcal{L}_2$-gain $\leq 1/\delta$.

3) Strictly passive $\Rightarrow$ 0 asy. stable for $\dot{x} = f(x,0)$

Output strictly passive & zero-state observable $\Rightarrow$ 0 asy. stable for $\dot{x} = f(x,0)$.

zero-state observable: max. inv. subset of $S = \{x \mid h(x,0) = 0\}$ for $\dot{x} = f(x,0)$ is $\{0\}$.

Feedback results: 1) Feedback connection of passive systems is passive.

2) Output strictly passive with $e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i$ $\Rightarrow$ feedback system finite gain $\mathcal{L}_2$-stable and $\mathcal{L}_2$ gain $\leq \dfrac{1}{\min\{\delta_1, \delta_2\}}$.

3) $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i$ $\Rightarrow$ f/b sys. finite-gain $\mathcal{L}_2$ stable if $\varepsilon_i + \delta_j > 0$.

Passivity Theory for Stability

Feedback system results (One system is memoryless):

1) 0 of feedback system asymp. stable if
   - both subsys. strictly passive
   - both output strictly passive & zero-state obs.
   - one strictly passive and other output strictly passive & zero-state obs.

Globally asym. stable when V is radially unbounded.

2) H_1 zero-state obs. & $\exists V > 0$: $e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1)$

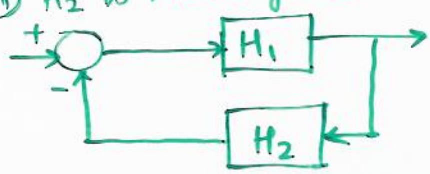
H_2 : $e_2^T y_2 \geq e_2^T \varphi_2(e_2)$.

$\Rightarrow$ 0 of f/b sys. asy. stable if $v^T[\rho_1(v) + \varphi_2(v)] > 0 \quad \forall v \neq 0$

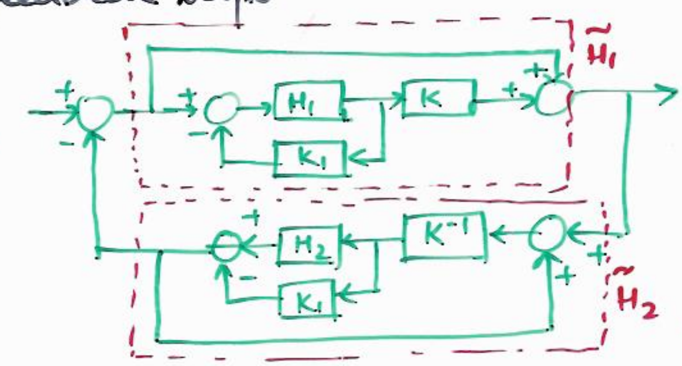
Globally asymp. stable if V_1 radially unbounded.

Loop transformations: Transforms "non-passive" into "passive" by adding input feedforward & output feedback loops.

① H_2 is memoryless.



$\Rightarrow$



- H_2 in sector $[K_1, K_2]$ with $K = K_2 - K_1 > 0$ transformed to $\tilde{H}_2$ in sector $(0, \infty)$.
- Corresponding transformation in H_1

For memoryless systems, passivity can be characterized using "sector criteria":

$[0, \infty]$ sector : $u^T y \geq 0 \iff$ passive

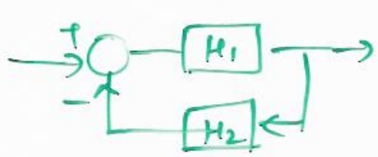
$[K_1, \infty]$ sector : $u^T[y - K_1 u] \geq 0 \Rightarrow$ input feedforward passive with $\varphi(u) = K_1 u$

$[0, K_2]$ sector with $K_2 > 0$: $y^T[y - K_2 u] \leq 0 \Rightarrow$ output strict passive with $\rho(y) = \delta y$ if $K_2 = \delta^{-1} I$

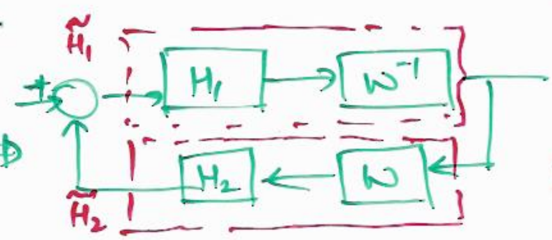
$[K_1, K_2]$ sector with $K_2 - K_1 > 0$: $[y - K_1 u]^T[y - K_2 u] \leq 0$.

A sector $[K_1, K_2]$, with $K_2 - K_1 > 0$, system can be transformed to sector $[0, \infty]$ system by input feedforward followed by output feedback.

② H_2 is dynamical system.



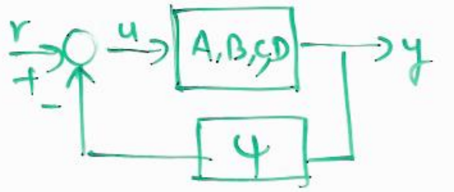
$\Rightarrow$



- Premultiplying H_2 by W canceled by postmultiplying H_1 by W^{-1}.
- WH_2 may be strictly passive
  $H_1 W^{-1}$ may be strictly passive.

Absolute Stability

- Many nonlinear systems are feedback connection of linear system & nonlinear.



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Leftrightarrow G(s) = C(sI-A)^{-1}B + D$$

$$u = -\psi(t, y)$$

- Absolute stability studies stability of origin wrt a class of nonlinearities in a sector. The problem is known as Lure's problem.

Def: (A,B,C,D) with feedback ψ belonging to a sector is absolutely stable if origin globally unif. asym. stable for any nonlinearity in the sector; absolutely stable with a finite domain if origin uniformly asym. stable.

Thm (Circle Criterion):

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1 G(s)]^{-1}$ strictly positive real $\Rightarrow$ absolutely stable
- $\psi \in [K_1, K_2]$ with $K_2 - K_1 > 0$ and $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ strictly +ve real $\Rightarrow$ abs. stable.

Def (strictly +ve real): A proper rational transfer fn. matrix $G(s)$ positive real if.

- poles of all elements of $G(s)$ in LHP
- $j\omega$ not a pole of $G(j\omega)$ $\Rightarrow$ $G(j\omega) + G^T(-j\omega) \geq 0$
- $j\omega$ pole of $G(s)$ $\Rightarrow$ pole is simple and $\lim_{s \to j\omega} (s - j\omega) G(s) \geq 0$ & Hermitian.

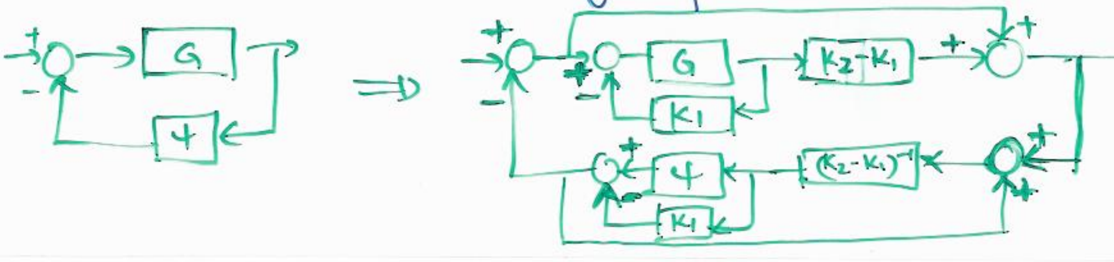
$G(s)$ said to be strictly positive real if $G(s-\varepsilon)$ positive real for some $\varepsilon > 0$.

Thm: $G(s)$ positive real $\Rightarrow$ (A,B,C,D) passive

$G(s)$ strictly +ve real $\Rightarrow$ (A,B,C,D) strictly passive.

Tests for +ve real and strictly +ve real on page 240 (Lemma 6.2, 6.3).

Circle criterion is established by loop transformations.



two outermost loops and gains $K_2 - K_1$, $(K_2 - K_1)^{-1}$ not need when $\psi \in [K, \infty]$

$\tilde{\psi}$ passive.