Introduction

- Finite dimensional inputs/states/outputs ; (lumped parameters)

$$\dot{x}_i = f_i(t, x_1, \dots, x_n; u_1, \dots, u_p)$$
$$y_j = h_j(t, x_1, \dots, x_n; u_1, \dots, u_p)$$

- Vector form : $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ $f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

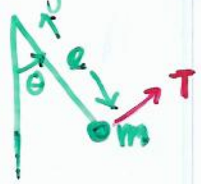
$\dot{x} = f(t, x, u)$ State Equation

$y = h(t, x, u)$ Output Equation

- Time-invariant $\Rightarrow$ $\dot{x} = f(x, u)$ & $y = h(x, u)$ time-inv. dynamics

  Autonomous $\Rightarrow$ $\dot{x} = f(t, x)$ & $y = h(t, \cancel{x})$ input assigned.

- Example (Pendulum): l = length of ~~rod~~, m = mass of rod



Examine motion in tangential direction :

$$\text{velocity} = \frac{\text{incremental distance}}{\text{incremental time}} = \frac{l\,d\theta}{dt} = l\dot{\theta}$$

$$\text{accelration} = \frac{d}{dt}\left[l\frac{d\theta}{dt}\right] = l\ddot{\theta}$$

Newton's 2nd law of motion in tangential direction: $\underbrace{ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta} + \frac{T}{l}}_{\text{autonomous system}}$

$x_1 = \theta$, $x_2 = \dot{\theta}$ $\Rightarrow$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \underbrace{\frac{T}{ml^2}}$$

$\hookrightarrow$ moment of inertia

$\frac{T}{l} = \frac{\text{torque}}{\text{distance}} = $ force.

- Equilibrium point : x^* equilibrium point (or equilibrium state) under input u^* if $\left.\frac{dx}{dt}\right|_{x^*, u^*} = f(t, x^*, u^*) = 0$ (state remains constant).

Pendulum with zero input torque $(T = 0)$ has equilibrium at.

$$0 = x_2^*$$
$$0 = -\frac{g}{l}\sin x_1^* - \frac{k}{m}x_2^* \Rightarrow x_2^* = 0 \ \& \ \sin x_1^* = 0 \Rightarrow (x_1^*, x_2^*) = (n\pi, 0)$$

$(0,0)$ and $(\pi, 0)$ are main eq. points; others are repetitions of these.

Introduction (Linearization)

- Linearization around a point may be used to approximate "local" dynamics

$$\dot{x} = f(t, x, u) \Rightarrow \frac{d}{dt}(x^* + \Delta x) = f(t, x^* + \Delta x, u^* + \Delta u)$$

$$\Rightarrow \cancel{\frac{d}{dt} x^*} + \frac{d}{dt}\Delta x = \cancel{f(t, x^*, u^*)} + \left.\frac{\partial f}{\partial x}\right|_{x^*} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{u^*} \Delta u + \dots$$

$$\Rightarrow \boxed{\frac{d}{dt}\Delta x = A(t)\,\Delta x + B(t)\,\Delta u}$$

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{x^*}, \quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{u^*}$$

$$y = h(t, x, u) \Rightarrow y^* + \Delta y = h(t, x^* + \Delta x, u^* + \Delta u)$$

$$\Rightarrow \cancel{y^*} + \Delta y = \cancel{h(t, x^*, u^*)} + \left.\frac{\partial h}{\partial x}\right|_{x^*} \Delta x + \left.\frac{\partial h}{\partial u}\right|_{u^*} \Delta u + \dots$$

$$\Rightarrow \boxed{\Delta y = C(t)\,\Delta x + D(t)\,\Delta u}$$

$$C(t) = \left.\frac{\partial h}{\partial x}\right|_{x^*}, \quad D(t) = \left.\frac{\partial h}{\partial u}\right|_{u^*}.$$

- $\boxed{\text{For pendulum around } (0,0), u=0}$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 + \frac{u}{m\ell^2} \qquad (u = T)$$

$$A = \begin{bmatrix} \partial f_1/\partial x_1 & \frac{\partial f_1}{\partial x_2} \\ \partial f_2/\partial x_1 & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}$$

$$B = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}$$

For small deviations around $(0,0)$: and small inputs

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u$$

$$\Rightarrow \dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}x_1 - \frac{k}{m}x_2 + \frac{u}{m\ell^2}$$

(Zero input, zero friction $\Rightarrow \ddot{x}_2 = -\frac{g}{\ell}\dot{x}_1 = -\frac{g}{\ell}x_2$ (simple harmonic osc.))

Linearization : Local Approximation

- Linearization local approximation, can be used to study behavior around equilibrium point with small fluctuations

- Nonlinear systems exhibit much richer set of behaviors to be understood via linearization.

- Finite escape time: State becomes infinite in finite time (impossible for linear systems: only exp. growth possible)

- Multiple isolated eq: Linear system can have single isolated equilibrium ($\det A \neq 0$ & $Re(\lambda(A)) < 0$) or a subspace ($\det A = 0$ & $\underbrace{Re(\lambda(A)) < 0 \text{ or } \lambda(A) = 0}$)

Nonlinear system can have multiple isolated eq. points
system settles at one of the points depending on initial condition.

- Limit cycles: For a linear system to have limit cycle, it should possess imaginary pair of poles ($\Rightarrow$ not robust)
  Also, amplitude of osc. fn. of initial condition (freq. not so)

Nonlinear system can have stable limit cycle with amp. & freq. independent of initial condition.

- Subharmonic/harmonic/almost harmonic osc. Stable linear system with periodic input $\Rightarrow$ predic output of same freq.

For nonlinear system, output freq. may be submultiple, multiple, or almost periodic (sum of periodic with periods "irrationally" related)

- Chaos: Only possible in nonlinear systems. Behavior deterministic but random looking (drastically different with slight change in initial condition)

- Multiple modes of behavior: Linear system $\Rightarrow$ 1 isolated eq. or subspace eq. or limit cycles (not isolated).

Nonlinear $\Rightarrow$ Additional modes (multiple isolated, infinite isolated, isolated limit cycles (multiple/infinite))

Mode can "switch" with smooth change in parameter