Synthesis of supervisor under partial observation

- Observability is not preserved union :

$L(G) = pr(a^* b a^*)$ $\quad K_1 = \{b\} \quad K_2 = \{aa\} \Rightarrow K_1, K_2$ observable

$M(a) = M(b) \neq \varepsilon$. $\quad K = K_1 \cup K_2 = \{b, aa\}$ not observable.

- observability not preserved under intersection

$K_3 = \{b, aa, baa, aaa\}$, $K_4 = \{b, aa, ba\} \Rightarrow K_3, K_4$ observable $\Big\}$ (HW)

$K_3 \cap K_4 = \{b, aa\}$ not observable.

- Observability is preserved under union over a chain of increasing languages.

Consider such a chain $\{K_i\}$ with $K_i \subseteq K_{i+1}$, $\forall i \geq 0$.

Pick $s, t \in pr\left[\bigcup_i K_i\right]$, $\sigma \in \Sigma$ s.t. $M(s) = M(t)$, $s\sigma \in pr\left[\bigcup_i K_i\right]$, $t\sigma \in L(G)$.

Since $s, t, s\sigma \in pr\left[\bigcup_i K_i\right] = \bigcup_i pr(K_i)$, there exist i_1, i_2, i_3 such that

$s \in pr(K_{i_1})$, $t \in pr(K_{i_2})$, $s\sigma \in pr(K_{i_3})$.

Choose $i = \max\{i_1, i_2, i_3\}$. Then $s, t, s\sigma \in pr(K_i)$

So from observability of K_i, $t\sigma \in pr(K_i) \subseteq \bigcup_i pr(K_i) = pr\left[\bigcup_i K_i\right]$.

$\Rightarrow \max O(K)$: "maximal obs. sublang. of K" exists.

- Since prefix-closure, relative-closure, controllability all preserved under union,

$\Rightarrow \max PCO(K)$ and $\max RCO(K)$ exist (use on-line computation method).

- Observability of prefix-closed lang. is preserved under intersection

$\Rightarrow \inf \overline{PO}(K)$ exists

Consider $\{K_i\}$ such that each K_i is $\overline{PO}(K)$.

Pick $s, t \in \bigcap_i K_i$, $\sigma \in \Sigma$ such that $s\sigma \in \bigcap_i K_i$, $t\sigma \in L(G)$.

Then $s, t, s\sigma \in K_i, \forall i. \Rightarrow t\sigma \in K_i, \forall i \Rightarrow t\sigma \in \bigcap_i K_i$

Computation of $\inf \overline{PO}(K)$

$\tilde{M}(\varepsilon) = \varepsilon$

$\tilde{M}(s\sigma) = M(s)\sigma$.

$\Rightarrow \tilde{M}^{-1}\tilde{M}(s\sigma) = \{t\sigma \mid M(t) = M(s)\}$

Thm: $\inf \overline{PO}(K) = \underbrace{\sup P\left[\tilde{M}^{-1}\tilde{M}\left(pr(K)\right)\right] \cap L(G)}_{H}$.

Need to show (i) $H \supseteq K$, (ii) $H \in \overline{PO}(K)$, (iii) $H' \in \overline{PO}(K) \Rightarrow H \subseteq H'$.

Proof: (i) $\left.\begin{array}{l} K \subseteq pr(K) = \sup P(pr(K)) \subseteq \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right] \\ K \subseteq L(G) \end{array}\right\} \Rightarrow K \subseteq H$

(ii) H is intersection of two prefix-closed languages $\Rightarrow$ H prefix closed

To see H observable, pick $s, t \in H$, $\sigma \in \Sigma$ s.t. $M(s) = M(t)$, $s\sigma \in H$, $t\sigma \in L(G)$

To show $t\sigma \in H$, it suffices to show $t\sigma \in \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right]$ (since $t\sigma \in L(G)$)

I.e., every prefix of $t\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K))$.

Since $t \in H \subseteq \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right]$, every prefix of $t \in \tilde{M}^{-1}\tilde{M}(pr(K))$.

Suffices to show: $t\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K))$. Follows since, $s\sigma \in H \subseteq \tilde{M}^{-1}\tilde{M}(pr(K))$ (since $\tilde{M}(s\sigma) = \tilde{M}(t\sigma)$)

(iii) Suppose for contradiction, $\exists s \in H - H'$. Then $s \neq \varepsilon$ (since $H' \neq \emptyset$ & closed)

Thus $s = \bar{s}\sigma$, and let s be the smallest trace in $H - H'$. I.e., $\bar{s} \in H'$.

Since $\bar{s}\sigma \in H \subseteq \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right]$; each prefix of $\bar{s}\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K))$

$\Rightarrow \bar{s}\sigma \in \tilde{M}^{-1}\tilde{M}(pr(K)) \Rightarrow \exists \bar{t}\sigma \in pr(K)$ s.t. $\underline{M(\bar{t}) = M(\bar{s})}$.

Since $pr(K) \subseteq H'$, $\bar{t}\sigma \in H'$. ($\Rightarrow \bar{t} \in H'$, since H' prefix-closed)

So, we have $\bar{s}, \bar{t} \in H'$, $\sigma \in \Sigma$ s.t. $\bar{t}\sigma \in H'$, $\bar{s}\sigma = s \in L(G) \supseteq H$, but $\bar{s}\sigma \notin H'$

contradiction to the fact that H' observable.

Modular computation of $\inf \overline{PCO}(K)$

- $\left.\begin{array}{l} \inf \overline{PCO}(K) = K^{M, \Sigma_u} \cap L(G) \\ K^{M, \Sigma_u} = (K^{\Sigma_u})^M \neq (K^M)^{\Sigma_u} \end{array}\right\}$ $K^{M,\Sigma_u} = \inf \overline{PCO}(K)$ when $L(G) = \Sigma^*$

  $\Rightarrow K^{\Sigma_u} = \inf \overline{PC}(K)$ when $L(G) = \Sigma^*$

  $\Rightarrow K^{M} = \inf \overline{PO}(K)$ when $L(G) = \Sigma^*$.

- $\inf \overline{PC}(K) = pr(K)\Sigma_u^* \cap L(G) \Rightarrow K^{\Sigma_u} = pr(K)\Sigma_u^*$
- $\inf \overline{PO}(K) = \quad \Rightarrow K^M = \sup P\left[\tilde{M}^{-1}\tilde{M}(pr(K))\right]$.

Modular computation of $\inf \overline{PCO}(K)$

- $\boxed{K^{\Sigma_u,M} = (K^{\Sigma_u})^M}$ — (1)

($\supseteq$) $K^{\Sigma_u} \subseteq K^{\Sigma_u,M} \Rightarrow (K^{\Sigma_u})^M \subseteq (K^{\Sigma_u,M})^M = K^{\Sigma_u,M}$

($\subseteq$) suffices to show that $(K^{\Sigma_u})^M \in \overline{PCO}(K)$ when $L(G) = \Sigma^*$

Since $(K^{\Sigma_u})^M \in \overline{PO}(K)$ when $L(G) = \Sigma^*$, suffices to show $(K^{\Sigma_u})^M \in \overline{PC}(K)$ when $L(G)=\Sigma^*$.

Pick $s \in (K^{\Sigma_u})^M$, $\sigma \in \Sigma_u \Rightarrow s \in \sup P[\tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})]$

$\Rightarrow \forall t \le s : t \in \tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})$

$\Rightarrow \forall t \le s : \exists u_t \in K^{\Sigma_u}$ s.t. $\tilde{M}(u_t) = \tilde{M}(t)$

Since $u_s \in K^{\Sigma_u}$, $\sigma \in \Sigma_u$, follows that $u_s\sigma \in K^{\Sigma_u}$

Also, $\tilde{M}(u_s) = \tilde{M}(s) \Rightarrow \tilde{M}(u_s\sigma) = \tilde{M}(s\sigma)$. So, $s\sigma \in \tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})$

Also, $\forall t \le s : t \in \tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})$

$s\sigma \in \sup P[\tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})] = (K^{\Sigma_u})^M$.

- $\inf \overline{PCO}(K) = K^{\Sigma_u,M} \cap L(G) \overset{(1)}{=} (K^{\Sigma_u})^M \cap L(G)$

($\subseteq$) $\inf \overline{PCO}(K) \subseteq K^{\Sigma_u,M}$, and $\inf \overline{PCO}(K) \subseteq L(G)$.

($\supseteq$) induction on length of traces.

base step (length of trace $= 0$): $\varepsilon \in \inf \overline{PCO}(K)$ ✓

induction step: Pick $s\sigma \in (K^{\Sigma_u})^M \cap L(G)$. From induction hyp. $s \in \inf \overline{PCO}(K)$.

Since $s\sigma \in (K^{\Sigma_u})^M \cap L(G) \subseteq (K^{\Sigma_u})^M = \sup P[\tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})] \subseteq \tilde{M}^{-1}\tilde{M}(K^{\Sigma_u})$

$\exists t\sigma \in K^{\Sigma_u} = pr(K)\Sigma_u^*$ s.t. $M(t) = M(s)$

Then either $\underbrace{t\sigma \in pr(K)}$ or $\underbrace{t \in pr(K) \text{ and } \sigma \in \Sigma_u}$.

(i) $\underset{\subseteq \inf \overline{PCO}(K)}{t\sigma \in pr(K)}$, $M(t) = M(s)$, $s \in \inf \overline{PCO}(K)$, $s\sigma \in L(G) \underset{(obs.)}{\Rightarrow} s\sigma \in \inf \overline{PO}(K)$

(ii) $\sigma \in \Sigma_u$, $s \in \inf \overline{PCO}(K)$, $s\sigma \in L(G) \underset{(ctrl.)}{\Rightarrow} s\sigma \in \inf \overline{PCO}(K)$

On-line control computation with linear complexity

① Initialize: $\hat{z} := \{z_0\}$, Elist := ordered $(\Sigma - \Sigma_u)$

If $R_{\Sigma_u}(\hat{z}) \not\subseteq z_g$, then ERROR and STOP

② Control computation: (a) $\hat{\Sigma} := \Sigma_u$, $pt := 0$

(b) If $pt = |\text{Elist}|$, then $ACT := \hat{\Sigma}$ and goto 3.

(c) $pt := pt + 1$, $\hat{\Sigma} := \hat{\Sigma} \cup \{\text{Elist}.pt\}$,

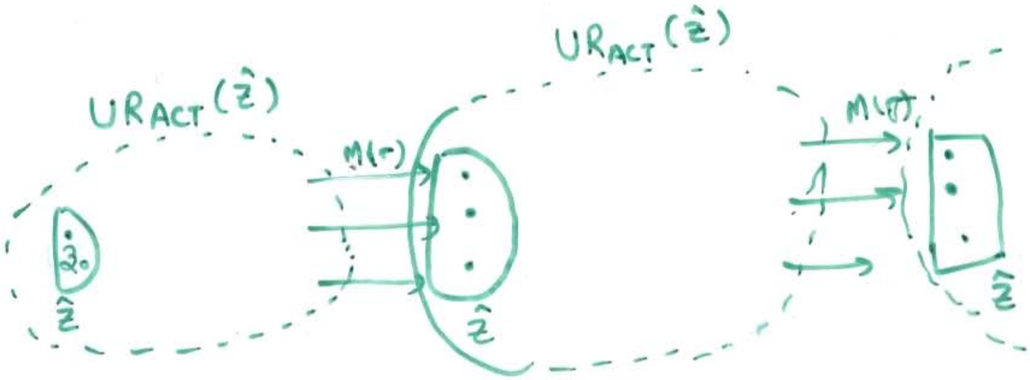
If $UR_{+\hat{\Sigma}}(\hat{z}) \not\subseteq z_g$, then $\hat{\Sigma} := \hat{\Sigma} - \{\text{Elist}.pt\}$. Goto 2(b)

either

(c) $pt := pt+1$, If $UR_{+(\hat{\Sigma} \cup \{\text{Elist}.pt\})}(\hat{z}) \subseteq z_g$, then $\hat{\Sigma} = \hat{\Sigma} \cup \{\text{Elist}.pt\}$. Goto 2(b)

③ Update state-estimate: Wait for observable $M(\sigma)$ to occur, then

$$\hat{z} := \gamma\left(UR_{ACT}(\hat{z}),\ M^{-1}M(\sigma) \right) \text{ and goto } 2.$$



Enable $\Sigma_u \subseteq ACT \subseteq \Sigma$ s.t. $UR_{+ACT}(\hat{z}) \subseteq z_g$.

- ~~If blocking is not an issue (spec. is safety spec. ⇒ prefix closed) then $z_{nb} = z_g$.~~
- No known algorithm for max $\overline{RCO}(K)$.
- Computation of each "ACT" set has $o(mn)$ complexity, and it is a function of only the state estimate $\hat{z}$ (which has cardinality of 2^{mn}).