Computation of $\sup C(K)$

- Remove "bad" states from $G \| \bar{S}$

(a) $Z_0 := \bar{Z} - Z$, $k = 0$

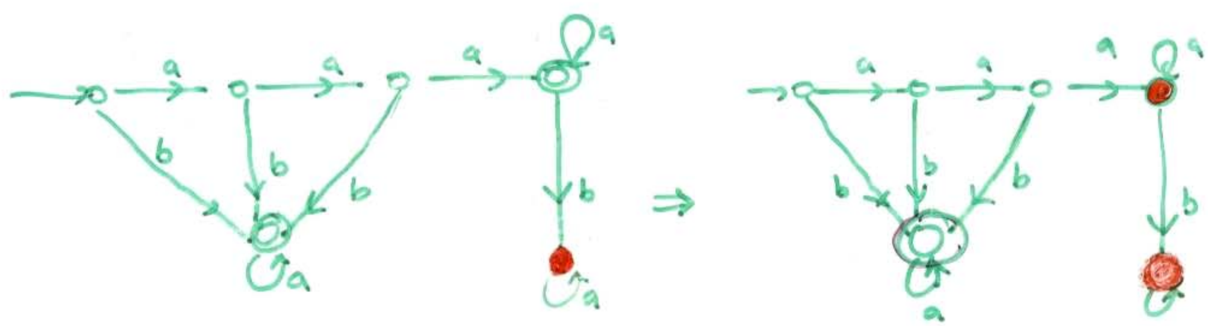
(b) $Z' := Z_k \cup \{ z \in \bar{Z} - Z_k \mid \exists u \in \Sigma_u^* \text{ s.t. } \bar{\gamma}(z,u) \in Z_k \}$

$Z_{k+1} := Z' \cup \{ z \in \bar{Z} - Z' \mid z \text{ does not belong to trim component of } \bar{Z} - Z' \}$

(c) Stop when $Z_{k+1} = Z_k$; delete these states; else $k := k+1$, goto (b).

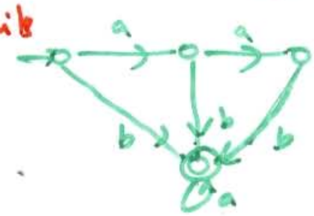
complexity : $O(m^2 n^2)$

- Example:



⇓

legal + uncontrollable
⇓ controllability fails
illegal



Generator for $\sup C(K)$

Example:

G　　　　S

$L_m(G) = pr(ccucu)$　　　$L_m(S) = \{\varepsilon, c, cc, ccuc\}$.

$G \| \bar{S}$　　⇒

legal + (uncontrollable ∨ blocking)
⇓ illegal

⇒

⇒

trim component

⇓

Generator for $\sup C(K)$

(relative)closed / Controllable / Language Classes

$P(K) = \{H \subseteq K \mid pr(H) \subseteq H\}$ $\bar{P}(K) = \{H \supseteq K \mid pr(H) \subseteq H\}$

$R(K) = \{H \subseteq K \mid pr(H) \cap L_m(G) \subseteq H\}$ $\bar{R}(K) = \{H \supseteq K \mid pr(H) \cap L_m(G) \subseteq H\}$

$C(K) = \{H \subseteq K \mid pr(H)\Sigma_u \cap L(G) \subseteq pr(H)\}$ $\bar{C}(K) = \{H \supseteq K \mid pr(H)\Sigma_u \cap L(G) \subseteq pr(H)\}$

$PC(K) = P(K) \cap C(K)$; $RC(K) = R(K) \cap C(K)$; $\overline{PC}(K) = \bar{P}(K) \cap \bar{C}(K)$

$\overline{RC}(K) = \bar{R}(K) \cap \bar{C}(K)$

Closure properties (under union & intersection):

	$\cup$-closed	$\cap$-closed	
$P(K) \rightarrow$	YES	YES	$\leftarrow \bar{P}(K)$
$R(K) \rightarrow$	YES	YES	$\leftarrow \bar{R}(K)$
$C(K) \rightarrow$	YES	NO	$\leftarrow \bar{C}(K)$
$PC(K) \rightarrow$	YES	YES	$\leftarrow \overline{PC}(K)$
$RC(K) \rightarrow$	YES	NO	$\leftarrow \overline{RC}(K)$

Example: ① $C(K)$ is closed under union

For each $\lambda \in \Lambda$, let H_λ be controllable, ie., $pr(H_\lambda)\Sigma_u \cap L(G) \subseteq pr(H_\lambda)$

Then,
$$pr\left(\bigcup_\lambda H_\lambda\right)\Sigma_u \cap L(G) = \left[\bigcup_\lambda pr(H_\lambda)\right]\Sigma_u \cap L(G)$$
$$= \bigcup_\lambda \left[pr(H_\lambda)\Sigma_u \cap L(G)\right]$$
$$\subseteq \bigcup_\lambda pr(H_\lambda) = pr\left[\bigcup_\lambda H_\lambda\right]$$

this last step is where $\cap$ fails.

② $C(K)$ is not closed under intersection: $L(G) = pr[a^* b a^*]$, $\Sigma_u = \{b\}$

$K_1 = \{a, b, ab\}$, $K_2 = \{a, ba, ab\}$ $\Rightarrow$ $K_1 \cap K_2 = \{a, ab\}$

K_1, K_2 controllable, but $K_1 \cap K_2$ not controllable.

③ $\overline{PC}(K)$ is $\cap$-closed: H_λ controllable & prefix closed $\Rightarrow$ $H_\lambda \Sigma_u \cap L(G) \subseteq H_\lambda$.

Also, $pr\left[\bigcap_\lambda H_\lambda\right] = \bigcap_\lambda pr(H_\lambda) = \bigcap_\lambda H_\lambda$.

(Relative) closed / Controllable lang. classes

It follows that following languages exist

$\sup P(K)$, $\inf \overline{P}(K)$
$\sup R(K)$, $\inf \overline{R}(K)$
$\sup C(K)$
$\sup PC(K)$, $\inf \overline{PC}(K)$
$\sup RC(K)$

Notation:
sup $\Leftrightarrow$ supremal sublanguage in given class
inf $\Leftrightarrow$ infimal superlanguage in given class

Definition of $\sup C(K)$: (i) $\sup C(K) \in C(K)$
(ii) $H \in C(K) \Rightarrow H \underset{\substack{\uparrow \\ \text{smaller than}}}{\subseteq} \sup C(K)$.

Definition of $\inf \overline{PC}(K)$: (i) $\inf \overline{PC}(K) \in \overline{PC}(K)$
(ii) $H \in \overline{PC}(K) \Rightarrow H \underset{\substack{\uparrow \\ \text{bigger than}}}{\supseteq} \inf \overline{PC}(K)$.

We can also define maximals and minimals (besides supremal and infimal):

For example, $\min \overline{RC}(K)$: (i) $\min \overline{RC}(K) \in \overline{RC}(K)$
(ii) $H \in \overline{RC}(K) \Rightarrow H \underset{\substack{\uparrow \\ \text{not smaller than}}}{\not\subsetneq} \min \overline{RC}(K)$

(inf $\Rightarrow$ min, since bigger $\Rightarrow$ not smaller)
(sup $\Rightarrow$ max, since smaller $\Rightarrow$ not bigger)

For $\min \overline{RC}(K)$ to exist, if $\{H_\lambda, \lambda \in \Lambda\}$ is a decreasing chain with $H_\lambda \in \overline{RC}(K)$, then $\bigcap_\lambda H_\lambda \in \overline{RC}(K)$. But this does not hold!

Consider $L_m(G) = a^* b$, $L(G) = pr(a^* b)$, $\Sigma_u = \{a\}$
$\dot{=} a^*$, controllable but not relative-closed.
Consider $H_i = a^{\geq i} b$, a decreasing chain, controllable & relative-closed
$\bigcap_i H_i = a^* \dot{=} K$!

Computation of supremal languages

1) $\sup P(K) = K - (\Sigma^* - K)\Sigma^*$

2) $\sup R(K) = K - (L_m(G) - K)\Sigma^*$

3) $\sup C(K) = \begin{cases} K_0 = K \\ K_{n+1} = K_n - [(L(G) - pr(K_n))/\Sigma_u^*]\Sigma^* \end{cases}$ algorithm already discussed

4) $[K = pr(K)] \Rightarrow \sup C(K) = K - ((L(G) - K)/\Sigma_u^*)\Sigma^*$

5) $\sup PC(K) = \sup C(\sup P(K))$. Hint: Show $H \in P(K) \Rightarrow \sup C(H) \in P(K)$

6) $\sup RC(K) = \sup C(\sup R(K))$ Hint: Show $H \in R(K) \Rightarrow \sup C(H) \in R(K)$
show that $H_n \in R(K)\ \forall n \geq 0$.

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1) $\inf \bar{P}(K) = pr(K)$

2) $\inf \bar{R}(K) = pr(K) \cap L_m(G)$

(HW) 3) $\inf \bar{PC}(K) = pr(K)\Sigma_u^* \cap L(G)$

Note: $\inf \bar{C}(K)$ & $\inf \bar{RC}(K)$ don't exist.

Example: $[K = pr(K)] \Rightarrow \sup C(K) = K - \underbrace{((L(G) - K)/\Sigma_u^*)\Sigma^*}_{H}$

(i) $H \subseteq K$ (obvious)

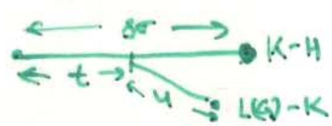
(ii) $H \in C(K)$: Since ($H = K - L\Sigma^*$, K prefix-closed) $\Rightarrow$ H prefix closed (HW)

Pick $s \in H$, $\sigma \in \Sigma_u$ s.t. $s\sigma \in L(G)$. Suppose for contradiction $s\sigma \notin H$

Case I ($s\sigma \in K$): Then since $s\sigma \notin H \Rightarrow s\sigma \in L\Sigma^*$

$\Rightarrow \exists t \leq s$ s.t. $t \in L = (L(G) - K)/\Sigma_u^*$

$\Rightarrow \exists u \in \Sigma_u^*$ s.t. $tu \in L(G) - K$



If $t = s\sigma$, then $tu = s\sigma u \in L(G) - K$

$\Rightarrow s \in [(L(G) - K)/\Sigma_u^*] \subseteq L\Sigma^* \Rightarrow s \notin H$ (*)

$\sigma \in \Sigma_u$

If $t \leq s\sigma$, then $t \leq s$ since $t \in (L(G) - K)/\Sigma_u^* \Rightarrow s \in L\Sigma^* \Rightarrow s \notin H$

Case II ($s\sigma \notin K$): $\Rightarrow s\sigma \in L(G) - K$

$\Rightarrow s \in (L(G) - K)/\Sigma_u^* \subseteq L\Sigma^* \Rightarrow s \notin H$ (*)

(iii) $H' \in C(K) \Rightarrow H' \subseteq H$. For contradiction pick $s \in H' - H$. Since $H' \subseteq K$, $s \in K$.
Since $s \notin H$, and $s \in K$, we must have $s \in L\Sigma^*$.
$\Rightarrow \exists t \leq s$ and $u \in \Sigma_u^*$ s.t. $tu \in L(G) - K \subseteq L(G) - H'$
Since $t \in pr(H')$, this implies H' not controllable (*)

Computation of supremal languages

$\sup PC(K) = \sup C(\sup P(K))$ "modular computation of supPC"

$(\subseteq)$ $\sup PC(K) \subseteq \sup P(K) \Rightarrow \underbrace{\sup C[\sup PC(K)]}_{\sup PC(K)} \subseteq \sup C(\sup P(K))$.

$(\supseteq)$ Since $\sup PC(K)$ is supremal prefix-closed & controllable sublang of K, suffices to show that $\sup C(\sup P(K))$ is a prefix-closed & controllable sublang of K.

Obviously, $\sup C(\sup P(K)) \subseteq K$, and $\sup C(\sup P(K))$ controllable.

Need to show, $\sup C(\sup P(K))$ prefix-closed

<u>Lemma:</u> $[H = pr(H)] \Rightarrow \sup C(H)$ is prefix-closed, i.e., $\sup C(H) = pr[\sup C(H)]$

~~[illegible]~~ consider $pr[\sup C(H)] \subseteq pr(H) = H$

Moreover, $pr[\sup C(H)]$ is controllable (since $\sup C(H)$ is controllable).

thus, $pr[\sup C(H)]$ is a controllable sublang. of H. $\Rightarrow pr[\sup C(H)] \subseteq \sup C(H)$.

$\sup RC(K) = \sup C(\sup R(K))$ "modular computation of supRC"

Just as above, suffices to show, supC operation preserves relative-closure. This requires an inductive proof since supC operation is iteratively computed: Show K_n is relative closed for each $n \geq 0$.

<u>$n=0$</u> : By definition, since $K_0 = K$ is relative-closed.

<u>Claim:</u> $H \in R(K)$, then $H' = (H - L\Sigma^*) \in R(K)$.

$$\begin{aligned} pr(H - L\Sigma^*) \cap L_m(G) &= pr(H) \cap ((L\Sigma^*)^c) \cap L_m(G) \\ &\subseteq pr(H) \cap pr[(L\Sigma^*)^c] \cap L_m(G) \\ &\subseteq H \cap pr[(L\Sigma^*)^c] \quad (H \in R(K)). \\ &= H \cap (L\Sigma^*)^c \quad ((L\Sigma^*)^c \text{ is prefix-closed}) \\ &= (H - L\Sigma^*) \quad \Rightarrow (H - L\Sigma^*) \in R(K). \end{aligned}$$

Computation of supremal languages

Example: $\sup C(K) = \begin{cases} K_0 = K \\ K_{n+1} = K_n - \left[\left(L(G) - pr(K_n)\right)/\Sigma_u^*\right]\Sigma^* \end{cases}$

$s \in (L(G) - pr(K_n)/\Sigma_u^*)\Sigma^* \Leftrightarrow \exists t \leq s,\ u \in \Sigma_u^*$ s.t. $tu \in L(G) - pr(K_n)$.

Suppose termination occurs at step m, ie, $K_m = K_m - \left[\left[L(G) - pr(K_m)\right]/\Sigma_u^*\right]\Sigma^*$

(i) $K_m \subseteq K$ (obvious)

(ii) $K_m \in C(K)$: Pick $s \in pr(K_m), \sigma \in \Sigma_u$ s.t. $s\sigma \in L(G) \overset{?}{\Longrightarrow} s\sigma \in pr(K_m)$

(iii) $H \in C(K) \Rightarrow H \subseteq K_m$ (Hint: show $H \subseteq K_n$ for each $n \geq 0$)

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ii) $K_m = K_m - \left[L(G) - pr(K_m)/\Sigma_u^*\right]\Sigma^* \Leftrightarrow K_m \cap \left(L(G) - pr(K_m)/\Sigma_u^*\right)\Sigma^* = \emptyset$ (*)

Suppose for contradiction, $s\sigma \notin pr(K_m) \Rightarrow s\sigma \in L(G) - pr(K_m)$
$\Rightarrow s \in \left(L(G) - pr(K_m)/\Sigma_u^*\right)$
$\Rightarrow$ any extension of $s \in \left(\left(L(G) - pr(K_m)/\Sigma_u^*\right)\Sigma^*\right.$

Since $s \in pr(K_m) \Rightarrow \exists t \geq s$ s.t. $t \in K_m$
and so, $t \in K_m \cap \left(L(G) - pr(K_m)/\Sigma_u^*\right]\Sigma^*$ — contradiction to (*)

(iii) $H \subseteq K_0 = K$ (base step trivially holds); $H \subseteq K_n \overset{?}{\Longrightarrow} \underbrace{H \subseteq K_{n+1}} = K_n - \left(L(G) - pr(K_n)/\Sigma_u^*\right)\Sigma^*$

Suppose for contradiction, $\exists s \in H - K_{n+1}$. Then since $s \in K_n$ (induction hypothesis),
$s \in \left(L(G) - pr(K_n)/\Sigma_u^*\right)\Sigma^* \subseteq \left(L(G) - pr(H)/\Sigma_u^*\right)\Sigma^*$ (since $H \subseteq K_n$)
$\Rightarrow \exists t \leq s,\ u \in \Sigma_u^*$ s.t. $tu \in L(G) - pr(H)$, Since $s \in H \subseteq pr(H)$, this implies
H not controllable (*)

Note above proof uses: $\left[pr(K)\Sigma_u \cap L(G) \subseteq pr(K)\right] \Leftrightarrow \left[pr(K)\Sigma_u^* \cap L(G) \subseteq pr(K)\right]$

Prove this as h.w. (Hint: $pr(K)\Sigma_u^n \cap L(G) \subseteq pr(K)$, $\forall n \geq 0$).
