Section 5.3
Generalized Permutations and Combinations

Urn models

• We are given set of n objects in an urn (don't ask why it's called an "urn" - probably due to some statistician years ago) .

We are going to pick (select) r objects from the urn in sequence. After we choose an object

• we can replace it-(*selection with replacement*)

• or not -(*selection without replacement*).

If we choose r objects, how many different possible sequences of r objects are there?

Does the order of the objects matter or not?

Permutations

Selection without replacement of r objects from the urn with n objects.

A *permutation* is an arrangement.

Order matters.

After selecting the objects, two different orderings or arrangements constitute different permutations.

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• Choose the first object n ways,

• Choose the second object (since selection is without replacement) (n - 1) ways,

.
.
.

• the rth object (n - r + 1) ways.

By the rule of product,

The number of permutations of n things taken r at a time

$$P(n,r) = n(n - 1)(n - 2) \ldots (n - r + 1)$$

Note:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example:

Let A and B be finite sets and let | A | ≤ | B | .

Count the number of injections from A to B.

Note there are no injections if | A | > | B | (why?)

There are P(| B |, | A |) injections:

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We order the elements of A, {a1, a2, . . . } and assume the urn contains the set B.

- There are | B | ways to choose the image of a1,
- | B | - 1 ways to choose the image of a2,

and so forth.

Selection is without replacement. Otherwise we do not construct an injection.

Combinations

Selection is without replacement but

order does not matter.

It is equivalent to selecting subsets of size r from a set of size n.

Divide out the number of arrangements or permutations of r objects from the set of permutations of n objects taken r at a time:

The number of combinations of n things taken r at a time

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!}$$

Other names for C(n, r):

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- *n choose r*
- *The binomial coefficient*

Example:

How many subsets of size r can be constructed from a set of n objects?

The answer is clearly C(n, r) since once we select the objects (without replacement) the order doesn't matter.

Corollary:

$$\sum_{r=0}^{n} C(n,r) = 2^n$$

Proof:

If we count the number of subsets of a set of size n, we get the cardinality of the power set.

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Example:

Suppose you flip a fair coin n times. How many different ways can you get

- no heads? $C(n, 0)$
- exactly one head? $C(n, 1)$
- exactly two heads? $C(n, 2)$
- exactly r heads? $C(n, r)$
- at least 2 heads? $2^n - C(n, 0) - C(n, 1)$

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