Section 2.3
Functions

Definition: Let A and B be sets. A function (mapping, map) f from A to B, denoted $f : A \to B$, is a subset of A×B such that

$$x[x \quad A \to \quad y[y \quad B \quad < x, y > \quad f]]$$

and

$$[< x, y_1 > \quad f \quad < x, y_2 > \quad f] \to y_1 = y_2$$

Note: *f* associates with each x in A one and only one y in B.

A is called the *domain* and

B is called the *codomain*.

If f(x) = y

- y is called the *image* of x under f
- x is called a *preimage* of y

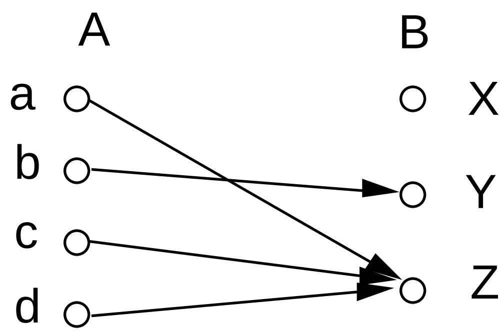
(note there may be more than one preimage of y but there is only one image of x).

The *range* of f is the set of all images of points in A under f. We denote it by f(A).

If S is a subset of A then

$$f(S) = \{f(s) \mid s \text{ in } S\}.$$

Example:



- f(a) = Z
- the image of d is Z
- the domain of f is A = {a, b, c, d}
- the codomain is B = {**X, Y, Z**}
- f(A) = {Y, Z}
- the preimage of Y is b
- the preimages of Z are a, c and d
- f({c,d}) = {**Z**}

Injections, Surjections and Bijections

Let f be a function from A to B.

Definition: f is *one-to-one* (denoted 1-1) or *injective* if preimages are unique.

Note: this means that if a ≠ b then f(a) ≠ f(b).

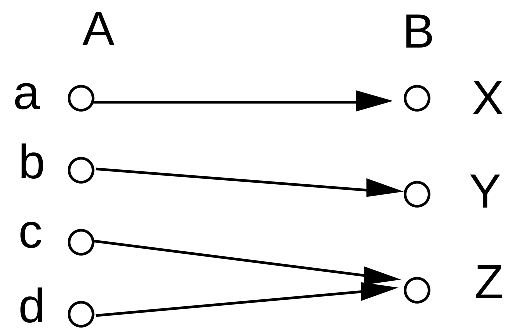
Definition: f is *onto* or *surjective* if every y in B has a preimage.

Note: this means that for every y in B there must be an x in A such that f(x) = y.

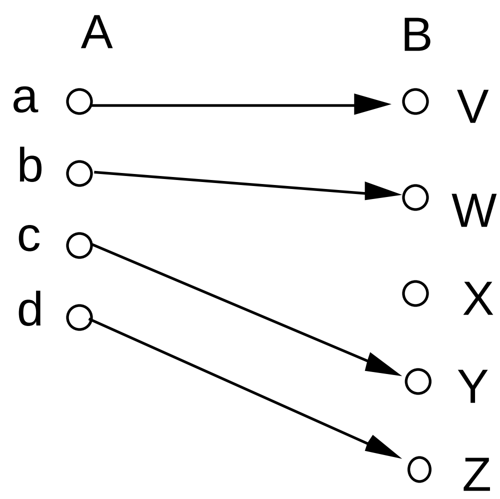
Definition: f is *bijective* if it is surjective and injective (one-to-one and onto).

Examples:

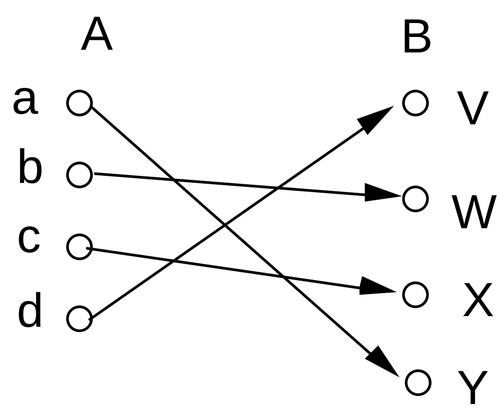
The previous Example function is neither an injection nor a surjection. Hence it is not a bijection.



Surjection but not an injection



Injection but not a surjection



Surjection and an injection, hence a bijection

Note: Whenever there is a bijection from A to B, the two sets must have the same number of elements or

the same *cardinality*.

That will become our *definition*, especially for infinite sets.

Examples:

Let A = B = R, the reals. Determine which are injections, surjections, bijections:

- $f(x) = x$,
- $f(x) = x^2$,
- $f(x) = x^3$,
- $f(x) = x + \sin(x)$,
- $f(x) = |x|$

Let E be the set of even integers {0, 2, 4, 6, ...}.

Then there is a bijection f from *N* to *E* , the even nonnegative integers, defined by

$$f(x) = 2x.$$

Hence, the set of even integers has the <u>same</u> cardinality as the set of natural numbers.

OH, NO! IT CAN'T BE....*E* IS ONLY HALF AS BIG!!!

Sorry! It gets worse before it gets better.

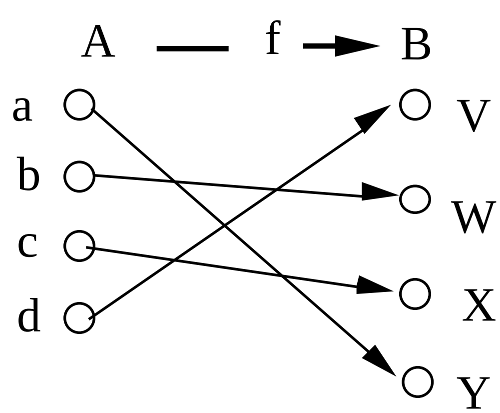
Inverse Functions

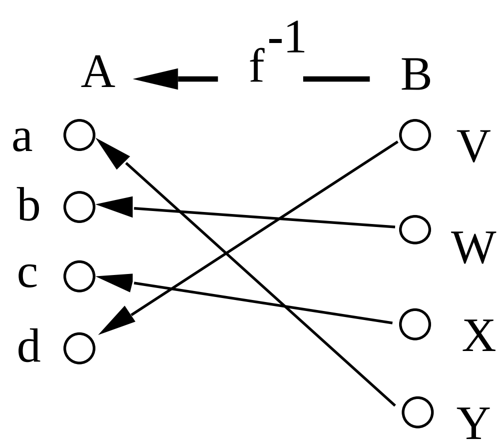
Definition: Let f be a bijection from *A* to *B*. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

Example:

Let f be defined by the diagram:





Note: No inverse exists unless f is a bijection.

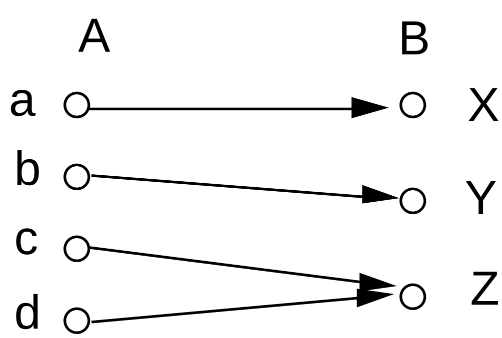
Definition: Let S be a subset of B. Then

$$f^{-1}(S) = \{x \mid f(x) \quad S\}$$

Note: f need not be a bijection for this definition to hold.

Example:

Let f be the following function:



$$f^{-1}(\{Z\}) = \{c, d\}$$

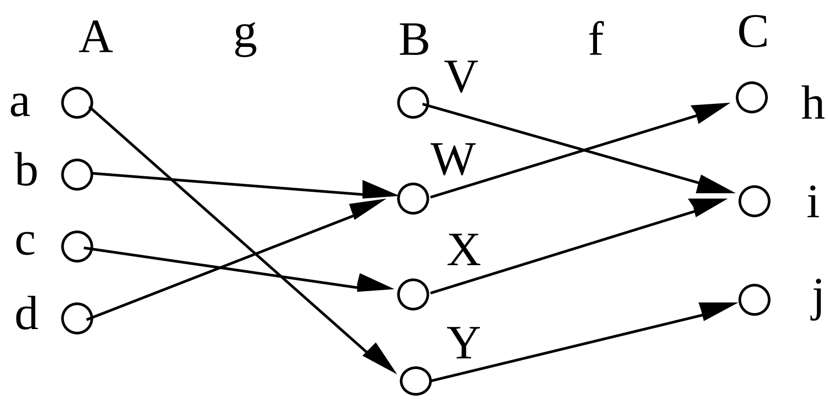
$$f^{-1}(\{X, Y\}) = \{a, b\}$$

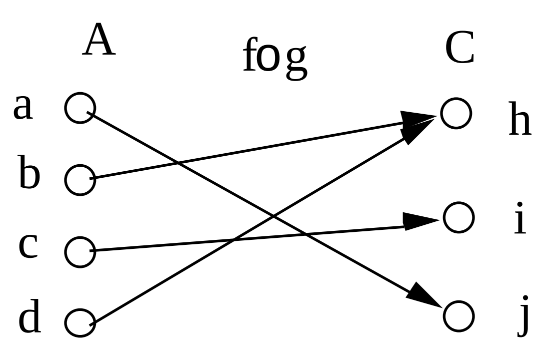
Composition

Definition: Let f: B → C, g: A → B. The *composition of f with g*, denoted f∘g, is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$

Examples:





If $f(x) = x^2$ and $g(x) = 2x + 1$, then $f(g(x)) = (2x+1)^2$ and $g(f(x)) = 2x^2 + 1$

Definition: The

floor function,

denoted $f(x) = \lfloor x \rfloor$ or f(x) = floor(x), is the largest integer less than or equal to x.

The

ceiling function,

denoted $f(x) = \lceil x \rceil$ or f(x) = ceiling(x), is the smallest integer greater than or equal to x.

Examples: $\lfloor 3.5 \rfloor = 3$, $\lceil 3.5 \rceil = 4$.

Note: the floor function is equivalent to truncation for positive numbers.

Example:

Suppose f: B $\rightarrow$ C, g: A $\rightarrow$ B and $f \circ g$ is injective.

What can we say about f and g?

- We know that if a $\neq$ b then f(g(a)) $\neq$ f(g(b)) since the composition is injective.

• Since f is a function, it cannot be the case that g(a) = g(b) since then f would have two different images for the same point.

• Hence, $g(a) \neq g(b)$

It follows that g must be an injection.

However, f need not be an injection (you show).

