Inference rule for "while loop": Invariant & Variant

- Invariant is something that is both pre- & post- condition

$$\frac{\alpha(x)\ \{\text{If } B(x) \text{ then } S\}\ \alpha(x)}{\alpha(x)\ \{\text{while } B(x) \text{ do } S\}\ \alpha(x)}$$

$\alpha(x)$ is loop-invariant.

To establish the premise, $\alpha(x)\ \{\text{If } B(x) \text{ then } S\}\ \alpha(x)$
suffices to establish, $\alpha(x) \wedge B(x)\ \{S\}\ \alpha(x)$
for which it suffices to establish $\alpha(x) \wedge B(x) \longrightarrow \alpha(f(x))\ (= wp_S(\alpha(x)))$

Example:

while $(z \neq x)$ do $z := z+1;\ y := y * z$.

loop-invariant $\equiv [y = z!]$

To show this, we need to show:

$$\underbrace{[y = z!]}_{\alpha(\vec{x})} \wedge \underbrace{[z = x]}_{B(\vec{x})} \longrightarrow \underbrace{[y * z+1 = (z+1)!]}_{\alpha(f(\vec{x}))} \equiv \left[y = \frac{z+1!}{z+1} = z!\right]$$

Obviously, $[y = z!] \wedge [z = x] \rightarrow [y = z!]$.

- Variant is something +ve that decrements each time loop executes.

$$\frac{0 \leq E(x) = T\ \{\text{If } B(x) \text{ then } S\}\ 0 \leq E(x) < T}{0 \leq E(x)\ \{\text{while } B(x) \text{ do } S\}\ (0 \leq E(x)) \wedge B(x)}$$

To establish the above premise suffices to establish

$$(0 \leq E(x) = T) \wedge B(x) \longrightarrow 0 \leq E(f(x)) < T$$

In the factorial example, $E(x) \equiv x - z \Rightarrow E(f(x)) = x - (z+1)$

Need to show,

$$(0 \leq x - z = T) \wedge (z \neq x) \rightarrow 0 \leq x - (z+1) < T$$

$$\left.\begin{array}{l}(x - z \geq 0) \wedge (x - z \neq 0) \longrightarrow x - z \geq 1 \longrightarrow x - (z+1) \geq 0 \\ x - z = T \longrightarrow x - (z+1) = T-1 > T.\end{array}\right\}$$