Limitation of 1^{st}-order logic; higher-order logics

- **One property of 1^{st} order logic is compactness:**

Consider an infinite collection of 1^{st} order formulae, Γ.
If every finite subset of Γ satisfiable, then Γ satisfiable.
Proof: Γ not satisfiable $\Rightarrow$ Assuming Γ we can prove F (completeness)
From completeness, the proof is finite.
Being finite it can only use a finite subset $\Delta \subseteq \Gamma$ of premises.
Then assuming Δ ~~we~~ can prove F.
From soundness Δ is not satisfiable. A contradiction.

- Suppose Path(x,y) a 1^{st} order logic formula.
Define $\Gamma = \{\neg Path_n(x,y) \mid n \geq 0\} \cup \{Path(x,y)\}$.
Clearly Γ not satisfiable, but every subset of Γ satisfiable.
A contradiction to compactness. $\Rightarrow$ Path(x,y) not 1^{st} order formula.
- This limitation of 1^{st} order logic comes because only variables can be quantified. 2^{nd} order logic allows quantification over predicates.
$\exists P \forall Q\ (\forall x \forall y : Q(x,y) \rightarrow Q(y,x)) \rightarrow \forall u \forall v (Q(u,v) \rightarrow P(u,v))$.
- 3^{rd} order logic will allow quantification over predicates of predicates.
- Such higher-order logic must be very carefully constructed; completness & compactness can easily be lost. Soundness can also be violated, such as: $A = \{X \mid X \notin X\}$.
"set of sets X that do not contain themselves."
Frege's higher-order logic allowed expressing such sentences.

- **Gödel's incompleteness result**: Showed that number system if sound must be incomplete by constructing a sentence, $f \Leftrightarrow f$ not provable.

  f true $\Rightarrow$ f not provable (incompleteness)
  f false $\Rightarrow$ f provable, i.e., can prove false (not sound)