Induction: Axiom for Well-founded Sets

- A set well-founded if exists 1-1 map to set of natural numbers. Then elements of the set can be viewed as 0, 1, 2, ...

  Example: Set of all sequences of letters in English alphabet arranged in dictionary ordering.

- **Induction axiom for well-founded sets:**

  Consider seq. of 1^{st}-order formulae $f(0), f(1), \ldots$ defined over a well-founded set. Then,

$$\begin{array}{ll} \text{base} & f(0) \\ \text{indc hyp.} & \forall n \geq 0 : f(n) \rightarrow f(n+1) \\ \hline & \forall n \geq 0 : f(n) \end{array} \qquad \begin{array}{l} f(0) \\ \forall n \geq 0 \, (\forall m \leq n : f(m)) \rightarrow f(n+1) \\ \hline \forall n \geq 0 : f(n) \end{array}$$

1^{st} known as weak induction due to weaker induction hypothesis.
2^{nd} known as strong induction due to stronger induction hypothesis.

- We prove $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ in two ways.

  First without induction, next applying induction.

$$\begin{aligned} S_n &= 1 + 2 + 3 + \cdots + n \\ S_n &= n + n-1 + n-2 + \cdots + 1 \\ \hline 2S_n &= (n+1) + (n+1) + \cdots + (n+1) \\ &= (1 + 1 + \cdots + 1)(n+1) \\ &= n(n+1) \end{aligned}$$

$$\Rightarrow S_n = \frac{n(n+1)}{2}$$