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Solution of the two-dimensional problem of a crack in a uniform field in eddy-current non-destructive evaluation

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Abstract. A two-dimensional, time-harmonic eddy-current problem is examined in which a uniform field is perturbed by a long, surface-breaking crack in a non-magnetic, conducting half-space. The crack is assumed to be ideal in the sense that it has infinitesimal opening and yet forms a perfect barrier to the passage of electric current. A solution is sought which is accurate both at high frequencies, at which the skin depth is small compared with the crack depth, and at intermediate frequencies, where the skin depth and crack depth are of similar magnitude. The Wiener–Hopf technique is used to derive a Fredholm integral equation of the second kind for the scattered magnetic field. An approximate closed form solution of this equation is found as a series of exponentially decreasing terms. At lowest order, local solutions at the buried crack edge and at the corners where the crack meets the surface of the conductor are decoupled. Higher order terms in the series account for the coupling which occurs between the fields perturbed by the crack edge and corners.

1. Introduction

Eddy-current non-destructive evaluation plays an important role in ensuring that metal components are free from defects. Typically, a component under inspection is scanned by an eddy-current probe and a defect is detected by a change in impedance of the probe coil [1]. Although real probes and flaws require a three-dimensional representation, the study of two-dimensional problems is valuable since independent analytical results can be found against which numerical models can be validated in appropriate limits. In addition, such solutions provide physical insight into the interaction between flaws and induced electromagnetic fields. Closely related to the eddy-current method is the AC potential difference method [2] in which electric current can be either 'injected' into the conductor or induced. In this technique the current flow is approximately uniform and approaches more closely the assumed flow in a two-dimensional problem. In addition, although infinitely long cracks do not occur in practice, long machined slots are commonly used in experimental situations and study of the two-dimensional configuration enhances understanding of the behaviour of such systems.

In the high-frequency regime, the electromagnetic skin depth is much smaller than the depth of the crack and current flows uniformly over the crack faces. Only near the buried edge of the crack and in the conductor corners at the crack mouth do the fields depart from this uniform behaviour. Consequently, in the high-frequency regime, the solution for the fields may be found as the sum of distinct contributions from the corners, faces and edges of the crack. This constructive approach was first used by Kahn *et al* [3] and yields good results for cracks of depth greater than about four skin depths. Here, a more general solution is sought which will be valid over a much wider frequency range. To this end, the problem is re-formulated by imaging in the air–conductor interface plane. The Wiener–Hopf method of solution is then employed and yields a Fredholm integral equation of the second kind for the scattered magnetic field. It is possible to solve this equation approximately for high frequencies. The expressions obtained for the scattered magnetic field near the edge and corners of the crack in the high-frequency regime are precisely those deduced in the constructed solution of Kahn *et al* [3]. The way in which the constructed solution arises naturally in the high-frequency limit of the more general Wiener–Hopf solution gives confidence in the validity of the constructive approach. In addition, the Wiener–Hopf method yields higher order terms in the solution for the scattered magnetic field. These higher order terms describe coupling between the field perturbations at the edge and corners of the crack, thereby providing information which cannot be found using the constructive method. This information allows departure from the limit in which the perturbed fields at the crack edge and corners are de-coupled and, consequently, the solution is valid for cracks which are shallower than those for which the edge and corner fields are completely de-coupled.

2. The Wiener–Hopf method in non-destructive evaluation

The Wiener–Hopf technique, published in 1931, was originally developed to solve a particular class of integral equation [4]. The procedure usually depends on use of one of the Fourier–Laplace–Mellin inversion trio in obtaining a complex variable equation which is solved by analytic continuation. In 1952 Jones published a different, but equivalent, approach in which transforms are applied directly to the partial differential equation and the complex variable equation found without the formulation of an integral equation [5]. The relative simplicity of Jones' method led to it being used almost exclusively by Noble [6] in his comprehensive summary of the use of the Wiener–Hopf technique.

The Wiener–Hopf analysis is able to provide both exact and approximate solutions, depending on the specific problem under consideration. It is particularly well suited to the study of wave diffraction at an edge and can, therefore, be used to treat problems involving idealized cracks. It is somewhat surprising that the appearance of this technique in the eddy-current literature is extremely rare. The only example known to the authors is that of Riaziat and Auld [7], who used the technique to solve a simple sub-surface crack problem. A more extensive treatment of a sub-surface crack in a conducting half-space has also been studied as a prelude to the work presented here [8].

There are a few instances in which the Wiener–Hopf method appears in other areas of non-destructive evaluation. For example, the technique has been applied in a straightforward manner by Almond and Lau [9] in a method for defect sizing using transient thermography. The more complex problem of acoustic scattering by an inclined, surface-breaking crack has been solved by Datta [10] in the low-frequency limit. His solution consists of matched asymptotic expansions for the scattered field both near to and far from the crack and uses the Wiener–Hopf technique in conjunction with the Mellin transform to find the terms in the asymptotic series.

The solution to be presented here exploits the power of the Wiener–Hopf method more fully than former work in non-destructive evaluation in which the technique has been used. The closest parallel with this solution is found in the analysis of scattering by a slot in a homogeneous medium by Jones [11]. The configuration of the crack problem examined here is more complicated, however, since the effect of the air–conductor interface must be taken into account.

3. Problem definition and formulation

The chosen orientation of the coordinate axes is shown in figure 1. It is assumed that the crack forms a perfect barrier to the flow of current, that the conductor has the permeability of free space and that the displacement current is negligible. The fields are assumed to be time-harmonic and the uniform incident magnetic field in air, $H_0 \mathrm{e}^{-\mathrm{i}\omega t}$ (where angular frequency $\omega = 2\pi f$), is assumed to have only a y component. The electromagnetic skin depth, δ, is defined by

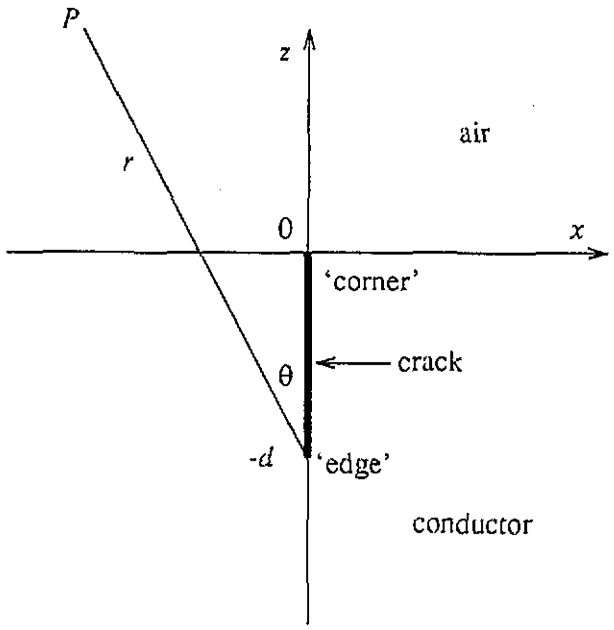

Figure 1. A surface-breaking crack in a conducting half-space showing the position of an arbitrary field point P.

$$\delta = \left(\frac{2}{\omega\mu_0\sigma_0}\right)^{1/2} \tag{1}$$

where μ_0 is the permeability of free space and σ_0 is the conductivity of the conductor.

Let the total magnetic field be written $H = H_0\psi^{(t)}$. From Maxwell's equations it is found that, in the conductor, $\psi^{(t)}$ satisfies the Helmholtz equation:

$$(\nabla^2 + k^2)\psi^{(t)}(x, z) = 0 \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

is a two-dimensional Laplacian and $k^2 = \mathrm{i}\omega\mu_0\sigma_0$. Let

$$\psi^{(t)} = \psi^{(i)} + \psi \tag{4}$$

where $\psi^{(i)}$ represents the incident magnetic field and ψ the magnetic field scattered by the crack. The incident and scattered magnetic fields also satisfy (2) individually. At the air–conductor interface and on the faces of the crack the total magnetic field is constant, with value H_0, which means that $\psi^{(t)}$ satisfies the boundary condition [3]

$$\psi^{(t)} = 1 \qquad \begin{cases} z = 0 & \\ x = 0 & d \le z \le 0. \end{cases} \tag{5}$$

Image theory can be applied to the system shown in figure 1 and the half-space problem converted into a whole-space problem by reflecting the conductor in the plane $z = 0$. Upon reflection, a surface crack of depth d in the plane $x = 0$ occupies the region defined by $|z| \le d$. The incident magnetic field in the extended domain is

$$\psi^{(i)}(x,z) = \begin{cases} \mathrm{e}^{-\mathrm{i}kz} & z < 0 \\ -\mathrm{e}^{\mathrm{i}kz} & z > 0. \end{cases} \tag{6}$$

Note that image theory is used to introduce an equivalent problem in which the y component of the magnetic field is odd in z. A current sheet in the plane $z = 0$ is the source of the field and accounts for the jump in the unperturbed magnetic field. In view of equation (5), the boundary conditions for the scattered magnetic field in the plane of the crack become

$$\psi(0,z) = \begin{cases} 1 - \mathrm{e}^{-\mathrm{i}kz} & -d < z < 0 \\ -(1 - \mathrm{e}^{\mathrm{i}kz}) & 0 < z < d \end{cases} \tag{7}$$

$$\left.\frac{\partial \psi(x,z)}{\partial x}\right|_{x=0} = 0 \qquad |z| > d. \tag{8}$$

Equation (8) follows from the even symmetry of the magnetic field with respect to x. The magnetic field is constructed to be odd with respect to z in order to satisfy the boundary condition

$$\psi(x,0) = 0 \tag{9}$$

on the air–conductor interface. For an ideal crack, the tangential magnetic field is continuous at the crack plane. This means that $\psi(x,z)$ is continuous across the crack at $x = 0$, that is

$$[\psi(x,z)]_{x=0} = \psi(x,z)|_{x=0+} - \psi(x,z)|_{x=0-} = 0. \tag{10}$$

Following Jones' adaptation of the Wiener–Hopf method [5], the bilateral Laplace transform is used to transform the two-dimensional Helmholtz equation for the scattered magnetic field into a complex variable equation. The bilateral Laplace transform is given by

$$\Psi(x,s) = \int_{-\infty}^{\infty} \psi(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z \tag{11}$$

where the complex variable $s = \sigma + \mathrm{i}\tau$. The inverse of (11) is [12, section 9.7]

$$\psi(x,z) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Psi(x,s)\,\mathrm{e}^{sz}\,\mathrm{d}s \tag{12}$$

where c is such that the contour lies in the strip in the complex plane in which Ψ is regular (to be defined). Applying (11) to the Helmholtz equation (2) for the scattered field gives

$$\frac{\partial^2 \Psi(x,s)}{\partial x^2} + \kappa^2 \Psi(x,s) = 0 \tag{13}$$

where $\kappa = (s^2 + k^2)^{1/2}$ is assumed to have a positive imaginary part. Solutions of (13) which decay away from the scatterer are of the form

$$\Psi(x,s) = \begin{cases} A(s)\,\mathrm{e}^{\mathrm{i}\kappa x} & x > 0 \\ B(s)\,\mathrm{e}^{-\mathrm{i}\kappa x} & x < 0. \end{cases} \tag{14}$$

The continuity of $\psi(x,z)$ across $x = 0$ (equation (10)) implies that $\Psi(x,s)$ is also continuous across $x = 0$. This means that $A(s) = B(s)$ and, therefore,

$$\Psi(x,s) = A(s)\,\mathrm{e}^{\mathrm{i}\kappa|x|}. \tag{15}$$

The function $\Psi(x,s)$ is now written as the sum of functions whose regions of regularity are known:

$$\Psi(x,s) = \Psi_+(x,s) + \Psi_-(x,s) + \Psi_0(x,s) \tag{16}$$

where

$$\Psi_0(x,s) = \Psi_{0+}(x,s) + \Psi_{0-}(x,s) \tag{17}$$

$$\Psi_+(x,s) = \int_d^{\infty} \psi(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z \tag{18}$$

$$\Psi_-(x,s) = \int_{-\infty}^{-d} \psi(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z \tag{19}$$

$$\Psi_{0+}(x,s) = \int_0^{d} \psi(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z \tag{20}$$

$$\Psi_{0-}(x,s) = \int_{-d}^{0} \psi(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z. \tag{21}$$

The regions in which Ψ_+ and Ψ_- are analytic may be deduced via an argument given in detail by Noble [6, pp 50–1] in which the scattering by an edge is regarded as being produced by equivalent line sources (line sources are appropriate for two-dimensional problems). It can be deduced that, in the eddy-current case, Ψ_+ is analytic for $\sigma > -k_i$ and Ψ_- is analytic for $\sigma < k_i$, where $k_i = \operatorname{Im} k$. This means that there is a strip defined by $-k_i < \sigma < k_i$ and $-\infty < \tau < \infty$, shown in figure 2, in which both Ψ_+ and Ψ_- are regular. It will be seen that Ψ_0 is regular in this strip and this means that Ψ is also regular there. The contour of (12) must, therefore, be such that $-k_i < c < k_i$ and $-\infty < \tau < \infty$. The explicit forms of Ψ_{0+} and Ψ_{0-} are found by inserting the boundary values (7) for the scattered field into (20) and (21):

$$\Psi_{0+}(0,s) = \mathrm{e}^{-sd}\left(\frac{1}{s} - \frac{\mathrm{e}^{\mathrm{i}kd}}{s - \mathrm{i}k}\right) - \frac{1}{s} + \frac{1}{s - \mathrm{i}k} \tag{22}$$

$$\Psi_{0-}(0,s) = \mathrm{e}^{sd}\left(\frac{1}{s} - \frac{\mathrm{e}^{\mathrm{i}kd}}{s + \mathrm{i}k}\right) - \frac{1}{s} + \frac{1}{s + \mathrm{i}k}. \tag{23}$$

The odd nature of the y component of the magnetic field with respect to z is demonstrated in (22) and (23) since $\Psi_{0+}(0,s) = -\Psi_{0-}(0,-s)$. The bracketed terms in (22) and (23) are associated with the crack edges at $z = d$ and $z = -d$ respectively. The remaining terms are associated with the conductor corners near $z = 0$.

4. Integral equations

Definitions (16)–(21) will now be used in the manipulation of (15) to obtain an equation of suitable form for solution by the Wiener–Hopf approach. The procedure now adopted relates the Laplace transforms of the scattered field and its normal derivative in the plane of the crack through $A(s)$ (equation (15)) which is then eliminated in order to obtain an equation containing only functions whose regions of

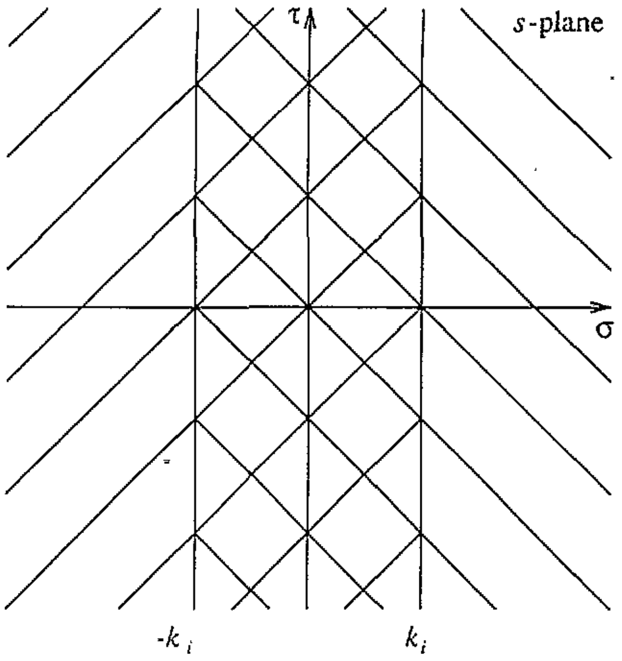

Figure 2. Regions of regularity in the complex plane. Ψ_+ is analytic for $\sigma > -k_i$, Ψ_- for $\sigma < k_i$ and Ψ in the cross hatched strip, $-k_i < \sigma < k_i$, in which the half-planes overlap.

regularity are known. Applying equations (16)–(21) to (15) gives

$$\Psi_+(0,s) + \Psi_-(0,s) + \Psi_0(0,s) = A(s). \qquad (24)$$

The Laplace transform of the normal derivative of the scattered field in the crack plane is defined by

$$\Psi'(x,s) = \int_{-\infty}^{\infty} \frac{\partial\psi(x,z)}{\partial x}\, \mathrm{e}^{-sz}\, \mathrm{d}z. \qquad (25)$$

Noting that $\Psi'(x,s)$ is continuous away from the crack and applying (25) to (15) gives

$$[\Psi_0'(x,s)]_{x=0} = \Psi_0'(x,s)|_{x=0_+} - \Psi_0'(x,s)|_{x=0_-} = 2\mathrm{i}\kappa A(s). \qquad (26)$$

Eliminating $A(s)$ from (24) and (26) and multiplying throughout by $\mathrm{e}^{-sd}/(s-\mathrm{i}k)^{1/2}$ gives

$$2\mathrm{i}(s+\mathrm{i}k)^{1/2}\,\mathrm{e}^{-sd}[\Psi_+(0,s) + \Psi_-(0,s) + \Psi_0(0,s)] = \frac{\mathrm{e}^{-sd}}{(s-\mathrm{i}k)^{1/2}}[\Psi_0'(x,s)]_{x=0}. \qquad (27)$$

This is the equation which will be solved to obtain the unknown functions $\Psi_+(0,s)$ and $\Psi_-(0,s)$ in terms of the known function $\Psi_0(0,s)$.

The right-hand side of (27) is regular in the half-plane $\sigma > -k_i$; the factor e^{-sd} has been introduced to ensure algebraic behaviour as $s \to \infty$ in the positive half-plane. Algebraic behaviour is necessary in order that Liouville's theorem (which plays an important role in the Wiener–Hopf method) can be applied. The function $(s+\mathrm{i}k)^{1/2}\mathrm{e}^{-sd}\Psi_-(0,s)$ is regular and non-zero in the half-plane $\sigma < k_i$. It is possible to split the expression $(s+\mathrm{i}k)^{1/2}\mathrm{e}^{-sd}[\Psi_+(0,s) + \Psi_0(0,s)]$ into the sum of two functions, X_+ and X_-, regular in right and left half-planes respectively. The procedure relies on the use of Cauchy's theorem and is described by Jones [11]. The condition for its application is that both Ψ_+ and Ψ_0 are regular in the strip $\sigma < |k_i|$. We introduce functions $X_+(s)$ and $X_-(s)$ such that

$$(s+\mathrm{i}k)^{1/2}\,\mathrm{e}^{-sd}[\Psi_+(0,s) + \Psi_0(0,s)] = X_+(s) + X_-(s) \qquad (28)$$

where

$$X_+(s) = -\frac{1}{2\pi\mathrm{i}}\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u-s} \times[\Psi_+(0,u) + \Psi_0(0,u)]\,\mathrm{e}^{-ud}\,\mathrm{d}u \qquad \sigma > -c \qquad (29)$$

$$X_-(s) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u-s} \times[\Psi_+(0,u) + \Psi_0(0,u)]\,\mathrm{e}^{-ud}\,\mathrm{d}u \qquad \sigma < c. \qquad (30)$$

Equation (27) may now be re-arranged so that one side of the resulting equation is analytic for $\sigma > -k_i$ and the other is analytic for $\sigma < k_i$:

$$\frac{\mathrm{e}^{-sd}}{(s-\mathrm{i}k)^{1/2}}[\Psi_0'(x,s)]_{x=0} - 2\mathrm{i}X_+(s) = 2\mathrm{i}(s+\mathrm{i}k)^{1/2}\,\mathrm{e}^{-sd}\Psi_-(0,s) + 2\mathrm{i}X_-(s). \qquad (31)$$

By the standard Wiener–Hopf argument, both sides of this equation are equal to an entire function which, from consideration of the s-dependence of the terms in (31), is zero according to the extended form of Liouville's theorem [6]. From the right-hand side of (31) the following relation is obtained:

$$\mathrm{e}^{-sd}\Psi_-(0,s) = -\frac{X_-(s)}{(s+\mathrm{i}k)^{1/2}}. \qquad (32)$$

In (32) the solution for $\Psi_-(0,s)$ is expressed as an integral involving $\Psi_+(0,s)$. The odd nature of ψ with respect to z, which means that Ψ is odd in s, may be exploited by substituting $-s$ for s in (32) and replacing $\Psi_-(0,-s)$ by $-\Psi_+(0,s)$. This yields the following integral equation for $\Psi_+(0,s)$:

$$(s-\mathrm{i}k)^{1/2}\,\mathrm{e}^{sd}\Psi_+(0,s) = P(s) - \frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s}\Psi_+(0,u)\,\mathrm{e}^{-ud}\,\mathrm{d}u \qquad (33)$$

where

$$P(s) = -\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s}\Psi_0(0,u)\,\mathrm{e}^{-ud}\,\mathrm{d}u. \qquad (34)$$

Substituting Ψ_0 (the sum of (22) and (23)) into (34) gives

$$P(s) = -\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s}\left[\frac{1}{u} - \frac{\mathrm{e}^{\mathrm{i}kd}}{u+\mathrm{i}k} + \mathrm{e}^{-ud}\left(\frac{1}{u+\mathrm{i}k} + \frac{1}{u-\mathrm{i}k} - \frac{2}{u}\right) + \mathrm{e}^{-2ud}\left(\frac{1}{u} - \frac{\mathrm{e}^{\mathrm{i}kd}}{u-\mathrm{i}k}\right)\right]\mathrm{d}u. \qquad (35)$$

An equivalent expression for $\Psi_-(0,s)$ can be found but since $\Psi_+(x,s) = -\Psi_-(x,-s)$ it is necessary to consider only $\Psi_+(x,s)$ in detail.

5. Series solution

Equation (33) can be solved for high frequencies by an iterative process. This yields an asymptotic series solution for $\Psi_+(0,s)$ which may be written

$$\Psi_+(0,s) = \sum_{j=0}^{\infty} \Psi_+^{(j)}(0,s) = \Psi_+^{(0)}(0,s) + \Psi_+^{(1)}(0,s) + \Psi_+^{(2)}(0,s) + \cdots \quad (36)$$

where

$$\lim_{k\to\infty} \frac{\Psi_+^{(j+1)}}{\Psi_+^{(j)}} = c_j \, \mathrm{e}^{ikd} \quad (37)$$

and c_j is a coefficient. Since $k = (1+\mathrm{i})/\delta$, the ratio of successive terms is small when $d \gg \delta$.

It is found that the two lowest order terms in the series are contained within $P(s)$, equation (35). The third term is obtained in part from $P(s)$ and in part by substituting the lowest order term, $\Psi_+^{(0)}(0,s)$, into the integral of (33). Terms of higher order are obtained by substituting successive terms into this integral. The lowest order term represents the limit in which there is no coupling between the perturbed fields associated with the corners and edge of the crack. Higher order terms take account of interactions between the perturbed fields at the corners and edge of the crack.

The first three terms in the series solution for $\Psi_+(0,s)$ will now be determined. The integrals to be evaluated in (35) have the general form

$$\mathcal{I} = \frac{1}{2\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{(u+s)(u \pm \mathrm{i}k\cos\phi)} \mathrm{e}^{-ub} \, \mathrm{d}u \qquad \sigma > -c \quad (38)$$

where $\sigma = \mathrm{Re}\, s$. As shown in the appendix, evaluating (38) yields

$$\mathcal{I} = \frac{\mathrm{e}^{\mathrm{i}kb}}{(s \mp \mathrm{i}k\cos\phi)} \Big\{ (s-\mathrm{i}k)^{1/2} w\{\mathrm{i}[b(s-\mathrm{i}k)]^{1/2}\} - [-\mathrm{i}k(1\mp\cos\phi)]^{1/2} w\{[\mathrm{i}kb(1\mp\cos\phi)]^{1/2}\} \Big\} \quad (39)$$

where [13]

$$w(z) = \mathrm{e}^{-z^2} \mathrm{erfc}(-\mathrm{i}z). \quad (40)$$

Special cases of expression (39) will be used in that which follows.

The first two terms in (35) alone contribute to $\Psi_+^{(0)}$:

$$(s-\mathrm{i}k)^{1/2} \mathrm{e}^{sd} \Psi_+^{(0)}(0,s) = -\frac{1}{2\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s} \left(\frac{1}{u} - \frac{\mathrm{e}^{\mathrm{i}kd}}{u+\mathrm{i}k} \right) \mathrm{d}u. \quad (41)$$

The value of the first term in the integral of (41) can be found by putting $\phi = \pi/2$ and $b = 0$ in (39) and that of the second term by putting $\phi = 0$ and $b = 0$ and using the positive sign option in (38). This gives

$$\Psi_+^{(0)}(0,s) = -\mathrm{e}^{-sd} \left(\frac{1}{s} - \frac{\mathrm{e}^{\mathrm{i}kd}}{s-\mathrm{i}k} \right) + \mathrm{e}^{-sd} \frac{(-\mathrm{i}k)^{1/2}}{s(s-\mathrm{i}k)^{1/2}}. \quad (42)$$

The term of next order in the series solution for Ψ_+ is found from the third, fourth and fifth terms in (35):

$$(s-\mathrm{i}k)^{1/2} \mathrm{e}^{sd} \Psi_+^{(1)}(0,s) = -\frac{1}{2\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s} \times \left(\frac{1}{u+\mathrm{i}k} + \frac{1}{u-\mathrm{i}k} - \frac{2}{u} \right) \mathrm{e}^{-ud} \, \mathrm{d}u. \quad (43)$$

Evaluating the integrals using (38) and (39) gives

$$\Psi_+^{(1)}(0,s) = -\mathrm{e}^{\mathrm{i}kd} \mathrm{e}^{-sd} \Big[2\left(\frac{s}{k^2} - \frac{1}{s} \right) w\{\mathrm{i}[d(s-\mathrm{i}k)]^{1/2}\} + 2\frac{(-\mathrm{i}k)^{1/2}}{s(s-\mathrm{i}k)^{1/2}} w[(\mathrm{i}kd)^{1/2}] - \frac{(-2\mathrm{i}k)^{1/2}}{[s-\mathrm{i}k]^{1/2}(s+\mathrm{i}k)} \times w[(2\mathrm{i}kd)]^{1/2} \Big]. \quad (44)$$

The third term in the series solution for $\Psi_+(0,s)$, $\Psi_+^{(2)}(0,s)$, consists of the final two terms in (35) and the result of substituting $\Psi_+^{(0)}(0,s)$ into the integral of (33). Simplification by cancelling terms yields

$$(s-\mathrm{i}k)^{1/2} \mathrm{e}^{sd} \Psi_+^{(2)}(0,s) = -\frac{1}{2\pi} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s} \frac{(-\mathrm{i}k)^{1/2}}{u(u-\mathrm{i}k)^{1/2}} \mathrm{e}^{-2ud} \, \mathrm{d}u. \quad (45)$$

Approximate evaluation of (45) can be performed by noting that the value of the integral stems largely from the region of the branch point at $u = -\mathrm{i}k$. At high frequencies, the branch at $u = \mathrm{i}k$ is sufficiently far from $u = -\mathrm{i}k$ to allow use of the approximation $(u-\mathrm{i}k)^{1/2} \approx (-2\mathrm{i}k)^{1/2}$. Making this approximation and using (39) with $\phi = \pi/2$ gives

$$\Psi_+^{(2)}(0,s) \approx -\frac{\mathrm{e}^{2\mathrm{i}kd} \mathrm{e}^{-sd}}{\sqrt{2}s} \left(w\{\mathrm{i}[2d(s-\mathrm{i}k)]^{1/2}\} - \frac{(-\mathrm{i}k)^{1/2}}{(s-\mathrm{i}k)^{1/2}} w[(2\mathrm{i}kd)]^{1/2} \right). \quad (46)$$

The first three terms in the series solution for $\Psi_+(0,s)$ are thus given by equations (42), (44) and (46). The quantity of greater interest is, however, $A(s)$, from which the solution for the scattered magnetic field in real space may be obtained. We define

$$A_+(s) = \Psi_+(0,s) + \Psi_{0+}(0,s) \quad (47)$$

and $A_-(s)$ similarly so that

$$A(s) = A_+(s) + A_-(s). \quad (48)$$

Then

$$A_+(s) = A_+^{(0)}(s) + A_+^{(1)}(s) + A_+^{(2)}(s) + \cdots \quad (49)$$

where

$$A_+^{(0)}(s) \equiv \Psi_+^{(0)}(0,s) + \Psi_{0+}(0,s)$$
$$A_+^{(1)}(s) \equiv \Psi_+^{(1)}(0,s)$$
$$A_+^{(2)}(s) \equiv \Psi_+^{(2)}(0,s)$$

and so on. From the summation of (22) and (42), we have

$$A_+^{(0)}(s) = \mathrm{e}^{-sd} \frac{(-\mathrm{i}k)^{1/2}}{s(s-\mathrm{i}k)^{1/2}} - \frac{1}{s} + \frac{1}{s-\mathrm{i}k}. \quad (50)$$

From (50), (44) and (46) the ordering in the terms of the solution for $A_+(s)$ is evident;

$$A_+^{(n)}(s) \propto e^{nikd}.$$

At high frequencies $e^{ikd} \to 0$ and $A_+(s) \approx A_+^{(0)}(s)$. Similarly $A(s) \approx A^{(0)}(s)$ at high frequencies, at which $A^{(0)}(s)$ is the lowest order term in the series solution for $A(s)$.

6. Lowest order solution

It will now be shown that the lowest order term for the magnetic field obtained in this analysis is equivalent to the de-coupled edge and corner fields given elsewhere [3]. From (48) and (50) it is found that

$$A^{(0)}(s) = e^{sd}\frac{(ik)^{1/2}}{s(s+ik)^{1/2}} + 2\left(\frac{s}{\kappa^2} - \frac{1}{s}\right) + e^{-sd}\frac{(-ik)^{1/2}}{s(s-ik)^{1/2}}. \quad (51)$$

Through (15) it is clear that the product $A^{(0)}(s)\,e^{i\kappa|x|}$ is the bilateral Laplace transform of the scattered magnetic field in the high-frequency limit, $\Psi^{(0)}(x,s)$. Formally inverting back into real space via (12) gives

$$\psi^{(0)}(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{(ik)^{1/2}}{s(s+ik)^{1/2}}e^{s(z+d)} + 2\left(\frac{s}{\kappa^2} - \frac{1}{s}\right)e^{sz} + \frac{(-ik)^{1/2}}{s(s-ik)^{1/2}}e^{s(z-d)}\right]e^{i\kappa|x|}\,ds. \quad (52)$$

The first term in (52) is the Laplace representation of the Sommerfeld solution for scattering of a uniform incident field by a half-plane with edge at $z = -d$:

$$\psi^{(0)}_{edge}(x,z) = \int_{c-i\infty}^{c+i\infty}\frac{(ik)^{1/2}}{s(s+ik)^{1/2}}e^{i\kappa|x|+s(z+d)}\,ds. \quad (53)$$

The integration can be performed by shifting to the hyperbolic contour described by $s = ik\cos(\theta + it)$, $-\infty < t < \infty$ and shown in figure 3 [6]. It is found that, in complete agreement with the result of Kahn *et al* [3],

$$\psi^{(0)}_{edge}(r,\theta) = \frac{e^{ikr}}{2}\left\{w\left[-(2ikr)^{1/2}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right] + w\left[(2ikr)^{1/2}\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right]\right\} \quad (54)$$

where $x = -r\sin\theta$ and $z + d = r\cos\theta$. The third term in (52) is clearly similar to that just evaluated and is the Laplace representation of the Sommerfeld solution for scattering by a half-plane with edge at $z = d$. This term results from imaging the problem in the plane $z = 0$.

Equation (54) can be re-written to show clearly the way in which current flow is uniform on the crack faces away from the edge. Using the identity [13]

$$w(z) + w(-z) = 2\,e^{-z^2} \quad (55)$$

the first term on the right-hand side of (54) can be re-written to give

$$\psi^{(0)}_{edge} = e^{ikx} + \frac{e^{ikr}}{2}\left\{-w\left[(2ikr)^{1/2}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right] + w\left[(2ikr)^{1/2}\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right]\right\} \qquad x > 0. \quad (56)$$

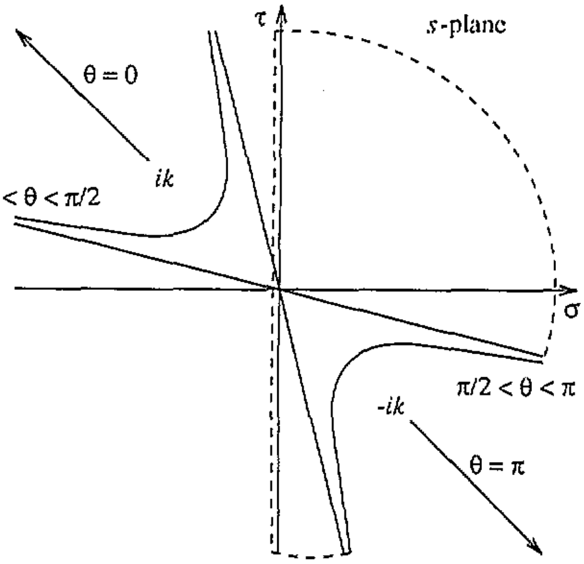

Figure 3. The shift in the s plane to the hyperbolic path described by $s = ik\cos(\theta + it)$.

The asymptotic form of w as r tends to infinity [13] means that, near the crack but distant from the crack edge, $\psi^{(0)}_{edge}$ approaches e^{ikx} which corresponds to uniform current flow over the faces of the crack. (An equivalent form can be found for $x < 0$ by re-writing the second term on the right-hand side of (54) using (55).)

The second term in (52) is clearly very different from the other two and is associated with the perturbed field in the conductor corners near the crack mouth. We denote

$$\psi^{(0)}_{corner}(x,z) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{s}{\kappa^2} - \frac{1}{s}\right)e^{i\kappa|x|+sz}\,ds. \quad (57)$$

The above integral may be evaluated as follows. Firstly, note that

$$\frac{s\,e^{i\kappa|x|+sz}}{\kappa^2} = -\frac{\partial}{\partial z}\int_0^x \frac{e^{i\kappa u+sz}}{i\kappa}\,du + \frac{s\,e^{sz}}{\kappa^2} \qquad x > 0 \quad (58)$$

$$\frac{e^{i\kappa|x|+sz}}{s} = \frac{\partial}{\partial x}\int_0^z \frac{e^{i\kappa x+su}}{i\kappa}\,du + \frac{e^{i\kappa x}}{s} \qquad x > 0. \quad (59)$$

Substituting (58) and (59) into (57) and changing the order of integration gives

$$\pi i\psi^{(0)}_{corner}(x,z) = -\frac{\partial}{\partial z}\int_0^x\int_{c-i\infty}^{c+i\infty}\frac{e^{i\kappa u+sz}}{i\kappa}\,ds\,du - \frac{\partial}{\partial x}\int_0^z\int_{c-i\infty}^{c+i\infty}\frac{e^{i\kappa x+su}}{i\kappa}\,ds\,du + \int_{c-i\infty}^{c+i\infty}\left(\frac{s\,e^{sz}}{\kappa^2} - \frac{e^{i\kappa x}}{s}\right)ds \qquad x > 0. \quad (60)$$

The inner integrals of the first two terms in (60) can be evaluated by transforming to the hyperbolic contour defined by $s = ik\cos(\theta + it)$ and $-\infty < t < \infty$, shown in figure 3. For example,

$$\int_{c-i\infty}^{c+i\infty}\frac{e^{i\kappa u+sz}}{i\kappa}\,ds = -i\int_{-\infty}^{\infty}e^{ik(u^2+z^2)^{1/2}\cosh t}\,dt \quad (61)$$

Using the identity [6]

$$\frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\alpha\cosh t}\,\mathrm{d}t = H_0^{(1)}(k\alpha) \qquad (62)$$

where $H_0^{(1)}$ is the zero order Hankel function of the first kind, it can be shown that

$$\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{\mathrm{i}\kappa u+sz}}{\mathrm{i}\kappa}\,\mathrm{d}s = -\mathrm{i}H_0^{(1)}[k(u^2+z^2)^{1/2}] \qquad (63)$$

and, similarly,

$$\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{\mathrm{i}\kappa x+su}}{\mathrm{i}\kappa}\,\mathrm{d}s = -\mathrm{i}H_0^{(1)}[k(x^2+u^2)^{1/2}]. \qquad (64)$$

Substituting (63) and (64) into (60) gives

$$\psi^{(0)}_{corner}(x,z) = \mathrm{i}[F(x,z)+F(z,x)] + \frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{s\,\mathrm{e}^{sz}}{\kappa^2}-\frac{\mathrm{e}^{\mathrm{i}\kappa x}}{s}\right)\mathrm{d}s \qquad x>0 \qquad (65)$$

where

$$F(x,z) = \frac{\partial}{\partial z}\int_0^x H_0^{(1)}[k(u^2+z^2)^{1/2}]\,\mathrm{d}u. \qquad (66)$$

The two parts of the integrand of the final term in (65) are, respectively, x- and z-independent. They correspond to the uniform current flow in the thin skin near the conductor surface at the air–conductor interface (x-independent) and at the crack faces (z-independent). It can be shown that the x-independent part of the final term in (65) has the same form as the transformed incident field:

$$\Psi^{(i)}(x,s) = \int_{-\infty}^{\infty}\psi^{(i)}(x,z)\,\mathrm{e}^{-sz}\,\mathrm{d}z = -\frac{2s}{\kappa^2} \qquad (67)$$

so that

$$\psi^{(i)}(x,z) = -\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{s\,\mathrm{e}^{sz}}{\kappa^2}\,\mathrm{d}s. \qquad (68)$$

This means that, if (65) and (68) are summed to give the total magnetic field at lowest order, the expression (68) cancels exactly the x-independent part of the final term in (65). It is found that

$$\psi^{(t,0)}_{corner}(x,z) = \mathrm{i}[F(x,z)+F(z,x)] - \frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{\mathrm{i}\kappa x}}{s}\,\mathrm{d}s \qquad x>0 \qquad (69)$$

where $\psi^{(t,0)}_{corner}$ is the lowest order term in the series expansion for the total magnetic field near the crack mouth. The remaining integral in (69) can be evaluated by writing

$$-\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{\mathrm{i}\kappa x}}{s}\,\mathrm{d}s = -\lim_{z\to 0}\left(\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{\mathrm{i}\kappa x+sz}}{s}\,\mathrm{d}s\right) \qquad (70)$$

and again shifting the contour of integration to that shown in figure 3. In this case, $\phi \to \pi/2$ in the limit as $z \to 0$ and

$$-\lim_{z\to 0}\left(\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{\mathrm{i}\kappa x+sz}}{s}\,\mathrm{d}s\right) = -\frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr\cosh t}}{\tanh t}\,\mathrm{d}t \qquad (71)$$

where the contour is now the perpendicular bisector of the line jointing ik and $-$ik and is indented around the pole at $t = 0$ in the usual way. Taking the Cauchy principal value, the contribution to (71) from the straight parts of the contour either side of the pole amounts to zero. All that remains is the contribution of the residue at the pole:

$$-\frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{e}^{\mathrm{i}\kappa x}}{s}\,\mathrm{d}s = -\frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr\cosh t}}{\tanh t}\,\mathrm{d}t = -\mathrm{e}^{\mathrm{i}kx} \qquad x>0. \qquad (72)$$

Substituting (72) into (69) gives, finally,

$$\psi^{(t,0)}_{corner}(x,z) = \mathrm{i}[F(x,z)+F(z,x)] - \mathrm{e}^{\mathrm{i}kx} \qquad x>0. \qquad (73)$$

The first two terms on the right-hand side of (73) correspond exactly with Kahn's [3] solution for the total magnetic field in the conductor corner. The extra term arises here as a consequence of considering the crack as a coherent whole rather than treating the edge and corner perturbations as totally independent problems. Kahn's edge and corner solutions both satisfy the boundary condition $\psi = 1$ on the crack but here, since the lowest order solution is expressed as a sum of terms which dominate in the separate regions near the crack edge and mouth, the $\mathrm{e}^{\mathrm{i}kx}$ term in (73) is needed to cancel that in (56) and thereby ensure that the boundary condition on the crack is satisfied. Summing (56) and (73) allows us to write the lowest order expression for the total magnetic field in the conductor, $\psi^{(t,0)}$:

$$\begin{aligned}\psi^{(t,0)} &= \psi^{(0)}_{edge} + \psi^{(t,0)}_{corner}\\ &= \frac{\mathrm{e}^{\mathrm{i}kr}}{2}\left\{-w\left[(2\mathrm{i}kr)^{1/2}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right] + w\left[(2\mathrm{i}kr)^{1/2}\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right]\right\}\\ &\quad +\mathrm{i}[F(x,z)+F(z,x)] \qquad x>0 \end{aligned} \qquad (74)$$

where $x = -r\sin\theta$, $z + d = r\cos\theta$ and F is defined in (66). Equation (74) satisfies the boundary conditions on the crack and on the air–conductor interface. The terms containing w functions represent the field perturbation by the crack edge and those involving Hankel functions the field perturbation by the conductor corners at the crack mouth.

Through (74) the correspondence between the high-frequency limit of the Wiener–Hopf solution for the field in the neighbourhood of a surface-breaking crack and the constructed solution of Kahn *et al* [3] is clear. The resulting change in impedance of the conductor in this limit due to the presence of the crack (which is usually the quantity of interest in eddy-current non-destructive evaluation) has been calculated numerically by Kahn *et al* [3] and in closed form by Harfield and Bowler [14]. The calculation will not be repeated here.

7. Discussion and conclusion

The Wiener–Hopf method has been used successfully to find a rigorous, closed form solution for the field scattered by a long, surface-breaking crack in a uniform, normally incident electric field. The theory is valid in the high-frequency regime, in which the crack depth is several times greater than the electromagnetic skin depth. Unlike in the constructed solution of Kahn *et al* [3], however, the perturbed parts of the field near the buried crack edge and crack mouth are not completely de-coupled. The solution for the field contains higher order terms which describe these interactions (equations (44) and (46)) and, in principle, the iterative solution of (33) can be extended to even higher order. Although further manipulation of the 'interaction' terms calculated here may only be possible via approximate methods, this theory triumphs in the emergence of the constructed solution of Kahn *et al* [3] in the high-frequency limit of the analysis, throughout which the surface-breaking crack is treated as a coherent whole.

Finally, we note that a promising method by which this problem might be solved to produce tractable terms accounting for the interaction between edge and corner fields is by adapting the uniform asymptotic theory of diffraction [15] for application to eddy-current problems. In this way it should be possible to predict the impedance change produced by surface-breaking cracks of depth as little as one skin depth with high accuracy.

Acknowledgment

The authors acknowledge the significant contribution made by the late Arnold Kahn to the solution of this canonical problem in eddy-current non-destructive evaluation.

Appendix

Integrals of the type

$$\mathcal{I} = \frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{(u+s)(u \pm \mathrm{i}k\cos\phi)}\,\mathrm{e}^{-ub}\,\mathrm{d}u \qquad \sigma > -c \tag{A1}$$

where $\sigma = \mathrm{Re}\, s$, arise in the Wiener–Hopf analysis of this problem through the de-composition of an irregular function into functions regular in different halves of the complex plane. The constant c is chosen such that it lies within the strip for which the integrand is regular, for example, when the pole is positioned at $u = \mathrm{i}k\cos\phi$, $k_i\cos\phi < c < k_i$.

In order to evaluate $\mathcal{I}$, the integrand in (A1) is split by partial fractions to give

$$\mathcal{I} = \frac{1}{2\pi(s \mp \mathrm{i}k\cos\phi)} \times \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(-\frac{(u+\mathrm{i}k)^{1/2}}{u+s} + \frac{(u+\mathrm{i}k)^{1/2}}{u \pm \mathrm{i}k\cos\phi}\right)\mathrm{e}^{-ub}\,\mathrm{d}u. \tag{A2}$$

The two components forming the integrand of (A2) may be evaluated in similar ways since they differ only in the location of the pole. Following the method of Jones [11, section 9.17] the evaluation of the first term in (A2) is assisted by defining:

$$I = \frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{(u+\mathrm{i}k)^{-1/2}}{u+s}\,\mathrm{e}^{-ub}\,\mathrm{d}u \qquad \sigma > -c \tag{A3}$$

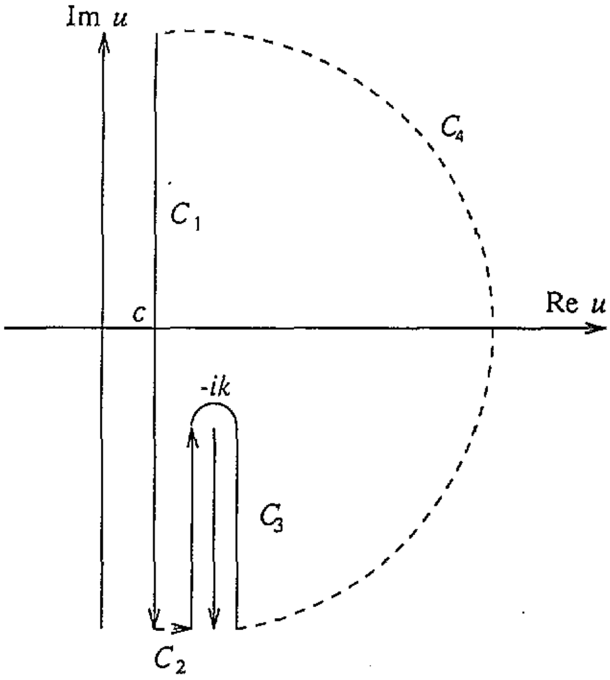

Figure 4. The contour of integration in the u plane around the branch line from $u = -\mathrm{i}k$.

and noting that the form of the integrand in $\mathcal{I}$ is related to I through

$$\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{(u+\mathrm{i}k)^{1/2}}{u+s}\,\mathrm{e}^{-ub}\,\mathrm{d}u = -\left.\frac{\partial I}{\partial b}\right|_{s=0} - (s-\mathrm{i}k)I. \tag{A4}$$

From (A3) it is found that

$$\frac{\partial(\mathrm{e}^{-sb}I)}{\partial b} = -\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(u+\mathrm{i}k)^{-1/2}\,\mathrm{e}^{-b(u+s)}\,\mathrm{d}u. \tag{A5}$$

For k with positive real and imaginary parts, the branch line from $u = -\mathrm{i}k$ can be chosen as shown in figure 4. Since there are no singularities within the contour, the value of the integral around the closed contour is zero. Since, for $b > 0$,

$$\frac{\mathrm{e}^{-b(u+s)}}{(u+\mathrm{i}k)^{1/2}} \to 0$$

as $\mathrm{Re}\, u \to \infty$, the contributions from the parts of the curve denoted by C_2 and C_4 in figure 4 are also zero. This means that the integral of (A5) can be evaluated by integrating around the branch line from $u = -\mathrm{i}k$, namely along C_3. Integration along either side of the branch line leads to the same result so that the branch integral may be evaluated by integrating along one side only and doubling the result.

The change of variable, $u = -\mathrm{i}t - \mathrm{i}k$, in (A5) gives

$$\frac{\partial(\mathrm{e}^{-sb}I)}{\partial b} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\pi}\,\mathrm{e}^{-b(s-\mathrm{i}k)}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}bt}}{\sqrt{t}}\,\mathrm{d}t \tag{A6}$$

which is [16]

$$\frac{\partial(\mathrm{e}^{-sb}I)}{\partial b} = -\frac{\mathrm{e}^{-b(s-\mathrm{i}k)}}{(\pi b)^{1/2}}. \tag{A7}$$

Consider the case in which $b = 0$. The integral I of (A3) may then be evaluated by deforming the contour to the left.

Only the pole at $u = -s$ contributes, resulting in

$$I_{b=0} = \frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{-1/2}}{u+s}\,\mathrm{d}u = \frac{1}{(s-\mathrm{i}k)^{1/2}} \qquad \sigma > -c. \tag{A8}$$

From (A7) and (A8)

$$\mathrm{e}^{-sb} I = -\int_0^b \frac{\mathrm{e}^{-t(s-\mathrm{i}k)}}{(\pi t)^{1/2}}\,\mathrm{d}t + \frac{1}{(s-\mathrm{i}k)^{1/2}}. \tag{A9}$$

It is possible to show that

$$I = \mathrm{e}^{\mathrm{i}kb} \frac{w\{\mathrm{i}[b(s-\mathrm{i}k)]^{1/2}\}}{(s-\mathrm{i}k)^{1/2}} \tag{A10}$$

where [13]

$$w(z) = \mathrm{e}^{-z^2}\frac{2}{\sqrt{\pi}}\int_{-\mathrm{i}z}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = \mathrm{e}^{-z^2}\mathrm{erfc}(-\mathrm{i}z). \tag{A11}$$

Substituting (A10) into (A4) gives

$$\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u+s}\,\mathrm{e}^{-ub}\,\mathrm{d}u = \frac{\mathrm{e}^{\mathrm{i}kb}}{(\pi b)^{1/2}} - (s-\mathrm{i}k)^{1/2}\,\mathrm{e}^{\mathrm{i}kb} w\{\mathrm{i}[b(s-\mathrm{i}k)]^{1/2}\}. \tag{A12}$$

It remains to consider the second integral in (A2). Defining

$$J = \frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{-1/2}}{u \pm \mathrm{i}k\cos\phi}\,\mathrm{e}^{-ub}\,\mathrm{d}u \tag{A13}$$

so that

$$\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u \pm \mathrm{i}k\cos\phi}\,\mathrm{e}^{-ub}\,\mathrm{d}u = -\left.\frac{\partial J}{\partial b}\right|_{\cos\phi=0} + \mathrm{i}k(1 \mp \cos\phi)J \tag{A14}$$

and choosing c so that the closed contour contains no singularities, evaluation of (A13) proceeds along lines parallel to those by which (A3) has been evaluated. By substituting J into (A14) it is found that

$$\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{u \pm \mathrm{i}k\cos\phi}\,\mathrm{e}^{-ub}\,\mathrm{d}u = \frac{\mathrm{e}^{\mathrm{i}kb}}{(\pi b)^{1/2}} - [-\mathrm{i}k(1\mp\cos\phi)]^{1/2}\,\mathrm{e}^{\mathrm{i}kb} w\{[\mathrm{i}kb(1\mp\cos\phi)]^{1/2}\}. \tag{A15}$$

The value of (A1) may now be written down by combining results (A12) and (A15) through (A2):

$$\frac{1}{2\pi}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(u+\mathrm{i}k)^{1/2}}{(u+s)(u \pm \mathrm{i}k\cos\phi)}\,\mathrm{e}^{-ub}\,\mathrm{d}u = \frac{\mathrm{e}^{\mathrm{i}kb}}{(s \mp \mathrm{i}k\cos\phi)}\Big\{(s-\mathrm{i}k)^{1/2} w\{\mathrm{i}[b(s-\mathrm{i}k)]^{1/2}\} - [-\mathrm{i}k(1\mp\cos\phi)]^{1/2} w\{[\mathrm{i}kb(1\mp\cos\phi)]^{1/2}\}\Big\}. \tag{A16}$$

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