Analysis of eddy-current interaction with a surface-breaking crack

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The change in electromagnetic impedance of a conductor due to the presence of a long, perpendicular surface-breaking crack in a normally incident, uniform electric field is calculated in closed form in the high-frequency limit. At high frequencies, where the skin depth is much smaller than the depth of the crack, the fields near the edge and corners of the crack are effectively decoupled. This means that the solution may be formulated as the sum of contributions from the corners, faces, and edge of the crack. Simple analytical expressions for the electric field are found and used to calculate the impedance due to the crack in the high-frequency limit without resorting to numerical methods.

I. INTRODUCTION

A greater understanding of the interaction between induced eddy currents and cracks in metals is important for quantitative nondestructive evaluation. In this paper the interaction between a long, perpendicular, surface-breaking crack in a half-space conductor and a normally incident, uniform electric field is examined. This problem is a special case of the more general problem of a conducting cylinder containing a radial surface-breaking crack of constant depth. The more general problem was solved in 1979[1] in terms of an infinite series whose coefficients must be found by solving a large set of simultaneous equations. We will show that, in this limiting case of a crack in a half-space conductor, a very simple closed-form solution can be found.

At high frequencies, where the skin depth is much smaller than the depth of the crack, current flows uniformly over the crack faces except near the edge (which is buried in the conductor) and the corners (where the crack meets the surface of the conductor). The edge and corner fields are effectively decoupled in this limit and may, therefore, be examined independently. The effective decoupling of the corner and edge fields in the high-frequency limit for a sufficiently deep crack allows the change in probe impedance due to the presence of the crack, ΔZ, to be calculated as a sum of terms

$$\Delta Z = Z_f + Z_e + Z_c. \tag{1}$$

The contribution Z_f arises from the uniform current flow over the faces of the crack, Z_e arises from the perturbation of the fields by the crack edge, and Z_c from the perturbation by the corners. Our approach will be similar to that of Kahn, Spal, and Feldman[2] who used an edge field based on Sommerfeld's analysis of the diffraction of a wave by a half-plane and whose corner solution was expressed as a boundary integral with a Hankel function kernel. Kahn *et al.* proceeded to calculate the components of Eq. (1) numerically. Here, simple expressions for Z_f, Z_e, and Z_c will be found in closed form.

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II. THEORY

The chosen orientation of the coordinate axes is shown in Fig. 1. It is assumed that the crack forms a perfect barrier to the flow of current, that the uniform incident magnetic field, $H_0 e^{-i\omega t}$, has only a y component, and that the displacement currents in the material are negligible. The magnetic field in the conductor satisfies the Helmholtz equation

$$(\nabla^2 + k^2)H(x,z) = 0, \quad k^2 = i\omega\mu_0\sigma, \tag{2}$$

where H is the total magnetic field. At the air-conductor interface and on the faces of the crack the total field satisfies the boundary condition $H = H_0$. The current density, $\mathbf{J}$, and electric field, $\mathbf{E}$, in the conductor may be obtained from the magnetic field by using the following relation:

$$\mathbf{J} = \sigma\mathbf{E} = \nabla \times (\hat{y}H), \tag{3}$$

where $\hat{y}$ is the unit vector perpendicular to the xz plane.

The impedance due to the crack is found by evaluating the following integral:

$$I^2\Delta Z = \int_S [\mathbf{E}^{(p)} \times (\hat{y}H)] \cdot \hat{n}\,dS. \tag{4}$$

In Eq. (4) the Poynting vector formed by the perturbed part of the electric field, $\mathbf{E}^{(p)}$, and the total magnetic field, $\hat{y}H$, is integrated over the surface of the conductor, S. The unit vector normal to S is denoted by $\hat{n}$ and points into the conductor since this is the direction of energy transfer. The surface of integration includes the faces of the crack and is shown by the dashed line in Fig. 1.

The crack edge has been shown to have an effective range of a few skin depths[2] in perturbing the electric field on the crack faces. This means that, when the perturbation due to the corners is ignored, the fields are uniform beyond approximately 2δ from the crack edge (δ is the electromagnetic skin depth). It is the uniform part of the field which gives rise to Z_f in Eq. (1). It is clear that, by retaining this uniform part of the field and integrating the Poynting vector over the two faces of the crack, Z_f and Z_e can be calculated together. (The effect of the corners is ignored in this part of the calculation). Equation (4) reduces to

$$I^2(Z_f + Z_e) = -2H_0 \int_{-d}^{0} E_{ez}(0_+,z)\,dz, \tag{5}$$
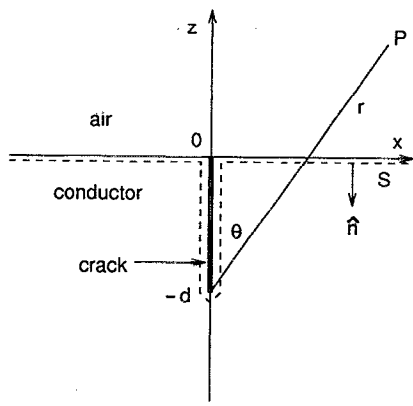
FIG. 1. The geometry of the system showing S, the surface of integration, as a dashed line. The crack is assumed to be of infinite extent in the y dimension so that the system is effectively two dimensional. P is the location of a general field point.

where E_{ez} is the z component of the scattered electric field at the edge and faces of the crack. The integration is performed twice over the crack face at $x=0_+$ for convenience. This operation is equivalent to integrating over the two crack faces separately.

The solution for the field scattered by an insulating half-plane in a conducting medium may be calculated by the Wiener–Hopf technique,[3] yielding the famous result derived originally (in a different way) by Sommerfeld.[4] Cylindrical coordinates centered on the edge of the crack will be used for convenience ($x=r \sin \theta$, $z+d=r \cos \theta$). The crack face at $x=0_+$ corresponds to $\theta=0$. The solution for the total magnetic field in the presence of an insulating half-plane (H_e) is given by

$$H_e(r,\theta)=\frac{H_0 e^{ikr}}{2}\left[w(-\zeta_1)+w(\zeta_2)\right], \tag{6}$$

where[5] $w(z)=\exp(-z^2)\mathrm{erfc}(-iz)$,

$$\zeta_1=\sqrt{2ikr}\,\cos\frac{1}{2}\left(\theta+\frac{1}{2}\,\pi\right)$$

and

$$\zeta_2=\sqrt{2ikr}\,\cos\frac{1}{2}\left(\theta-\frac{1}{2}\,\pi\right).$$

Near the faces of the crack it is found that

$$\frac{\partial H_e(x,z)}{\partial x}\approx\frac{1}{(z+d)}\frac{\partial H_e(r,\theta)}{\partial\theta}, \tag{7}$$

which becomes exact as the crack face is approached. Using this approximation, Eq. (3), and the following relations[6]

$$w(z)=2e^{-z^2}-w(-z),$$

$$\frac{dw(z)}{dz}=-2zw(z)+\frac{2i}{\sqrt{\pi}},$$

the electric field on the crack face can be found

$$E_{ez}(0_+,z)=\frac{ikH_0}{\sigma}\left(\mathrm{erf}[-i\sqrt{ik(z+d)}]+\frac{i}{\sqrt{\pi}}\frac{e^{ik(z+d)}}{\sqrt{ik(z+d)}}\right). \tag{8}$$

For large $(z+d)$ (in practice greater than approximately 2δ)

$$E_z(0_+,z)=\frac{ikH_0^-}{\sigma} \tag{9}$$

since $\mathrm{erf}[-i\sqrt{ik(z+d)}]\to 1$ and $e^{ik(z+d)}/\sqrt{ik(z+d)}\to 0$ as $(z+d)\to\infty$. The field given by Eq. (9) is the uniform part of the field referred to earlier and gives rise to Z_f. Substituting Eq. (8) into Eq. (5) gives

$$Z_f+Z_e=-\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\int_{-d}^0 2ik\left(\mathrm{erf}[-i\sqrt{ik(z+d)}]\right.$$
$$\left.+\frac{i}{\sqrt{\pi}}\frac{e^{ik(z+d)}}{\sqrt{ik(z+d)}}\right)dz. \tag{10}$$

The integral in Eq. (10) can be evaluated by the change of variable $u=-i\sqrt{ik(z+d)}$, giving

$$Z_f+Z_e=-\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\left((2ikd-1)\mathrm{erf}(-i\sqrt{ikd})\right.$$
$$\left.+\frac{2i}{\sqrt{\pi}}\sqrt{ikd}\,e^{ikd}\right). \tag{11}$$

Since $|kd|$ is assumed large, $\mathrm{erf}(-i\sqrt{ikd})\approx 1$ and $\sqrt{ikd}e^{ikd}\approx 0$. Substituting these approximations into Eq. (11) gives

$$Z_f+Z_e=-\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2[2ikd-1]. \tag{12}$$

It remains to calculate Z_c. The solution for the total magnetic field in the corner of a conductor (H_c) can be found by using the method of images, in which the Green's function which vanishes on the conductor surfaces (i.e., the air-conductor interface and the crack face) is found.[7] The result is, for a corner defined by boundaries at $z=0$, $x\geqslant 0$, and $x=0$, $z\leqslant 0$,

$$H_c(x,z)=-iH_0[F(x,z)+F(z,x)], \tag{13}$$

where

$$F(x,z)=\frac{\partial}{\partial z}\int_0^x H_0^{(1)}(k\sqrt{u^2+z^2})du.$$

The zero-order Hankel function of the first kind is represented by $H_0^{(1)}$. As a consequence of the choice of coordinate system, the z coordinate in Eq. (13) is implicitly negative. The other corner in this problem (defined by $z=0$, $x\leqslant 0$, and $x=0$, $z\leqslant 0$) is a mirror image of the first and provides the same contribution to the impedance. The impedance change will be calculated for one surface of the corner only ($x=0$, $z\leqslant 0$) since the contribution from the other surface is identical. The electric field on $x=0$, $z\leqslant 0$ must be found. Using Eq. (3) the electric field is found from Eq. (13)

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$$E_{cz}(0_+,z) = -\frac{iH_0}{\sigma}\left(\frac{\partial}{\partial x}[F(x,z)+F(z,x)]\right)_{x=0_+} . \quad (14)$$

The above may be manipulated by noting that the integral satisfies the Helmholtz equation and that as a consequence

$$\frac{\partial}{\partial x}F(z,x) = \frac{\partial^2}{\partial x^2}\int_0^z H_0^{(1)}(k\sqrt{u^2+x^2})du$$

$$= -\left(\frac{\partial^2}{\partial z^2}+k^2\right)\int_0^z H_0^{(1)}(k\sqrt{u^2+x^2})du. \quad (15)$$

Substituting Eq. (15) into Eq. (14) gives

$$E_{cz}(0_+,z) = \frac{ik^2 H_0}{\sigma}\int_0^z H_0^{(1)}(ku)du \quad (16)$$

which can be evaluated to give an analytical expression for the tangential component of the electric field at the air-conductor interface[8]

$$E_{cz}(0_+,z) = \frac{ik^2 z H_0}{\sigma}\left(H_0^{(1)}(kz)+\frac{\pi}{2}[\mathscr{H}_0(kz)H_1^{(1)}(kz)\right.$$

$$\left. -\mathscr{H}_1(kz)H_0^{(1)}(kz)]\right), \quad (17)$$

where $\mathscr{H}_i$ is the Struve function of ith order. The tangential component of the electric field at the conductor surface, $E_{cx}(x,0)$, would be obtained simply by replacing z by x in the right-hand side of Eq. (17). As with the perturbation of the fields by the crack edge, the field perturbation by the corners has a finite range, which is also of the order of a few skin depths. It can be shown that, sufficiently far from the corner, the field value approaches that of the incident field. From Eq. (16),

$$\lim_{z\to\infty}E_{cz}(0_+,z) = \frac{ik^2 H_0}{\sigma}\int_0^\infty H_0^{(1)}(ku)du = \frac{ikH_0}{\sigma}, \quad (18)$$

where standard integrals[9] have been used to give the value of the incident electric field on the conductor surface, ikH_0/σ. In practice the field is close to this constant value beyond approximately 3δ from the corner. In order to calculate Z_c only the perturbed part of the field, $E^{(p)}$, must be considered. This is achieved by subtracting Eq. (18) from Eq. (16) to give

$$E_{cz}^{(p)}(0_+,z) = -\frac{ik^2 H_0}{\sigma}\int_z^\infty H_0^{(1)}(ku)du. \quad (19)$$

Equation (19) represents the tangential component of the scattered electric field along one semi-infinite surface of a cornerlike conductor ($x=0$, $z\leqslant 0$). Integrating the Poynting vector formed by Eq. (19) and the total magnetic field along this surface provides the contribution to the impedance change by only one boundary of the corner. The impedance change due to the entire corner is found by multiplying the result for one face by a factor of 2. Since in the case of a surface-breaking crack there are two corners, the impedance

change due to one boundary is multiplied by a factor of 4 to find Z_c. Integrating the Poynting vector over the crack face at $x=0_+$ and multiplying by four gives

$$Z_c = \frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 4ik^2\int_0^\infty\int_z^\infty H_0^{(1)}(ku)du\,dz. \quad (20)$$

The limit of integration over the crack face is made infinite for mathematical convenience even though the crack itself is finite. This is permitted since the scattered part of the electric field on the crack face is very small beyond approximately 3δ from the corner. Evaluation of Eq. (20) becomes tractable when the following representation of the Hankel function is used:[10]

$$H_0^{(1)}(\alpha) = \frac{1}{\pi i}\int_{-\infty}^\infty e^{i\alpha\cosh t}dt, \quad (21)$$

where the path of integration is a hyperbola in the complex plane. Substituting this form into Eq. (20), changing the order of integration with respect to u and t, and integrating with respect to u yields

$$Z_c = \frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\frac{4ik}{\pi}\int_0^\infty\int_{-\infty}^\infty\frac{e^{ikz\cosh t}}{\cosh t}dt\,dz. \quad (22)$$

The evaluation of Eq. (22) is straightforward when the order of integration is changed. The result is

$$Z_c = -\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\frac{8}{\pi}. \quad (23)$$

III. DISCUSSION AND CONCLUSION

Equations (12) and (23) together give the total impedance change for a crack of depth greater than four or five skin depths (the sum of the effective ranges of influence of the edge and corners) in the high-frequency limit

$$\Delta Z = -\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\left(2ikd-1+\frac{8}{\pi}\right). \quad (24)$$

The first term in Eq. (24) may be thought of as being the contribution from the faces of the crack since it depends on their combined length, $2d$, and otherwise looks like the impedance per unit length of a plane conductor surface. The second term embodies the effect of the presence of the crack edge and the third the effect of the corners. These latter two agree with terms used in calculating an approximate expression for the impedance of an open, surface-breaking crack[11] (1.0 and -2.56, respectively) and also with the appropriate cases of a solution for arbitrary corner angle found using the Kontorovich–Lebedev transform.[12]

The real part of the impedance change is given by

$$\mathrm{Re}(\Delta Z) = \frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 2\left(\frac{d}{\delta}+\frac{1}{2}-\frac{4}{\pi}\right). \quad (25)$$

It can be seen that the edge and corners effectively reduce the depth of the crack by $(1/2-4/\pi)\delta$. This improves upon the value of -0.78δ calculated numerically by Kahn et al.[2,13]

In this paper a number of analytical results have been presented which were formerly found numerically. In Eqs.

(8) and (17) expressions for the electric field on the conductor surface near the edge and corner of a perpendicular surface-breaking crack in a normally incident, uniform electric field are given. The results are valid in the high-frequency limit in which the fields perturbed by the crack edge and corners are effectively decoupled. The exact values of the contribution to the change in probe impedance due to the presence of the crack edge and corners are given in Eqs. (12) and (23), enabling the total impedance change due to the presence of the crack to be given for a crack of depth greater than four or five skin depths [Eq. (24)].

In conclusion it should be noted that this form of constructed solution can be developed further, for example by the introduction of terms accounting for the scattering between the edge of the crack and its image. The Wiener–Hopf method of problem solving lends itself to the determination of approximate values for these terms.

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