JOURNAL OF APPLIED PHYSICS VOLUME 96, NUMBER 8 15 OCTOBER 2004

Electric field due to alternating current injected at the surface of a metal plate

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(Received 7 May 2004; accepted 27 July 2004)

Closed form analytical expressions for the electric field interior and exterior to a metal plate, due to alternating current injected at its surface, are derived. Assuming that the current is injected and extracted by wires oriented perpendicular to the surface of the plate, the problem is decomposed into two cylindrically symmetric systems in which a single current-carrying wire lies on the coordinate axis. This simplified problem is formulated in terms of a single magnetic potential and the solution obtained by use of the Hankel transform. The resulting expression for the electric field in the plate takes the form of an analytic series. In exterior regions, the electric field is expressed in terms of Hankel transforms. The result for the physical system with two current-carrying wires is obtained by superposition. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1793332]

I. INTRODUCTION

Analytical expressions for the electric field interior and exterior to a metal plate, due to alternating current injected at its surface, are derived. A schematic diagram of the system is shown in Fig. 1. This work is motivated by applications of the alternating-current potential difference (ACPD) method on metal plates, in which a four-point probe is used. Of the four contact points, two inject and extract alternating current and two form part of a high impedance circuit which measures potential drop. An accurate description of the electric field, interior and exterior to the plate, is necessary for proper interpretation of ACPD measurements.

This work extends that of a previous article[1] in which an analytic expression for the electric field in a half-space conductor was derived. Here, it is shown that the introduction of a second surface (the back plane of the conductor) leads to an analytic series expression for the electric field in the conductor, of which the first term is the solution for the half space. In addition, an expression for the electric field exterior to the conductor is derived here. In ACPD measurements it is important to consider the effect of inductance in the pick-up circuit since, being proportional to frequency, this contribution dominates when the frequency is sufficiently high. Knowledge of the electric field in region of the probe (air) permits evaluation of the inductive contribution to the ACPD measurement.

The form of the electric field external to a conductor is rarely considered in the literature. Dyakin and Kaibicheva present a general formulation based on solving for a δ-function distribution of harmonically varying source current situated outside a metal region.[2] Solutions for a number of current-carrying elements may be summed to give a specific configuration. Particular examples given in Ref. 2 are: a vertical semi-infinite thin wire in contact with a conductive half space, a vertical semi-infinite wall in contact with a half space, and a vertical cylinder connected to a half space. Penchenkov and Shcherbinin analyze a system with two current-carrying wires in perpendicular contact with a conductive half space.[3] The half space in which the wires are situated is also permitted to be conductive. Fourier-space representations for the electric and magnetic fields in both regions are obtained. In a complementary problem, the electric field external to a conductive spherical shell excited by an external dipole is examined by Mrozynski and Baum.[4]

II. FORMULATION

The electric field problem is formulated as a superposition of two cylindrically symmetric systems. In one, current flows into the plate by means of a wire contact perpendicular to the surface of the conductor, Fig. 2. In the second, the current flows out of the conductor through a similar wire. The total electric field $\mathbf{E}_j^T$ is then determined according to

$$\mathbf{E}_j^T(\mathbf{r}) = \mathbf{E}_j(\mathbf{r}_+)\text{-}\mathbf{E}_j(\mathbf{r}_-), \tag{1}$$

where the subscript j denotes either region 1, 2, or 3, and $r_\pm = \sqrt{(x\pm S)^2+y^2+z^2}$. Analysis of the problem shown in Fig. 2 follows the method described in Ref. 1, in which an expression for the electric field in a half-space conductor, due to a similar excitation, was derived. The analysis is simplified by expressing the electric field in terms of a single, transverse magnetic, potential.

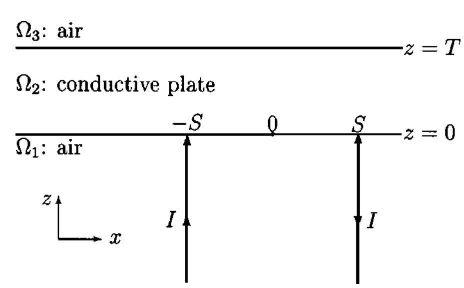

FIG. 1. Two wires carrying current I, in contact with one surface of a conductive plate.

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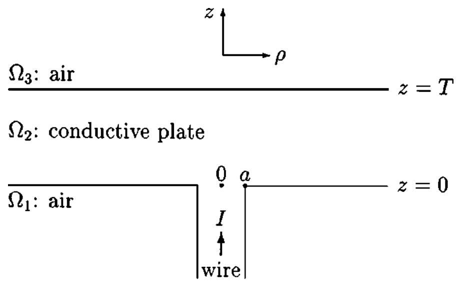

FIG. 2. Cross section of a wire, radius a, carrying current I, in contact with a conductive plate. The system in cylindrically symmetric.

Consider a time-harmonic current source varying as the real part of $\mathbf{J}(\mathbf{r})\exp(-i\omega t)$, where ω is the angular frequency of the excitation. In this case the source is essentially a wire carrying current I as shown in Fig. 2. It is assumed that the material properties are linear and that the conductor has conductivity σ_2 and scalar permeability μ_2. From Maxwell's equations, the electric field in each of the three regions Ω_1, Ω_2, and Ω_3 is a solution of

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E}_1(\mathbf{r}) = i\omega\, \mu_0 \mathbf{J}(\mathbf{r}), \quad z \leqslant 0,$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E}_2(\mathbf{r}) - k^2\mathbf{E}_2(\mathbf{r}) = 0, \quad 0 \leqslant z \leqslant T, \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E}_3(\mathbf{r}) = 0, \quad z \geqslant T,$$

where μ_0 is the permeability of free space and $k^2 = i\omega\, \mu_2\sigma_2$. As discussed in Refs. 1 and 5, the electric field will be written in terms of two scalar potentials defined with respect to the direction perpendicular to the air-conductor interface:

$$\mathbf{E}_j(\mathbf{r}) = i\omega\, \mu_j \boldsymbol{\nabla} \times [\hat{z}\psi_j'(\mathbf{r}) - \boldsymbol{\nabla} \times \hat{z}\psi_j''(\mathbf{r})]. \tag{3}$$

In Eq. (3), μ_j is the scalar permeability, $\hat{z}$ is a unit vector in the z direction, ψ' is a transverse electric (TE) potential, and ψ'' is a transverse magnetic (TM) potential.

As described in Ref. 5, uncoupled equations for the potentials may be obtained by substituting the expression for the electric field, given in Eq. (3), into Eq. (2). It is found that, in a case where the source is directed in the z direction alone,

$$\mathbf{J} = J_z\hat{z}, \quad z < 0, \tag{4}$$

only the TM potential is required to describe the field.[1] Equation (4) is certainly true sufficiently far form the conducting plate but will be assumed true as $z \to 0-$. As $z \to 0-$, Eq. (4) is valid for the problem under consideration if the radius of the wire a is sufficiently small. Practically, the assumption expressed in Eq. (4) is reasonable if $a \ll 2S$, the separation between the wires.

Having established that the electric field may be fully described by the TM potential alone, the governing equations are

$$\nabla^2\nabla_z^2\psi_1''(\mathbf{r}) = -\hat{z}\cdot\mathbf{J}(\mathbf{r}), \quad z \leqslant 0, \tag{5}$$

$$(\nabla^2 + k^2)\nabla_z^2\psi_2''(\mathbf{r}) = 0, \quad 0 \leqslant z \leqslant T, \tag{6}$$

$$\nabla^2\nabla_z^2\psi_3''(\mathbf{r}) = 0, \quad z \geqslant T, \tag{7}$$

in which the transverse differential operator $\boldsymbol{\nabla}_z$ is defined as

$$\boldsymbol{\nabla}_z \equiv \boldsymbol{\nabla} - \hat{z}\frac{\partial}{\partial z}.$$

It is assumed that ψ_j'' vanishes as $|\mathbf{r}| \to \infty$. For Eq. (6) to be satisfied it is sufficient that ψ_2'' satisfies the Helmholtz equation. Similarly, ψ_3'' satisfies the Laplace equation, as does ψ_1'' in source-free regions.

In order to simplify the solution, a new potential is defined as follows:

$$\Psi_j = \nabla_z^2\psi_j''. \tag{8}$$

Equations (5) to (7) become

$$\nabla^2\Psi_1(\mathbf{r}) = -\hat{z}\cdot\mathbf{J}(\mathbf{r}), \quad z \leqslant 0, \tag{9}$$

$$(\nabla^2 + k^2)\Psi_2(\mathbf{r}) = 0, \quad 0 \leqslant z \leqslant T. \tag{10}$$

$$\boldsymbol{\nabla}^2\Psi_3(\mathbf{r}) = 0, \quad z \geqslant T. \tag{11}$$

From Eq. (3), retaining only the TM potential, the two components of the electric field can be expressed as

$$E_{zj}(\mathbf{r}) = i\omega\mu_j\Psi_j(\mathbf{r}), \tag{12}$$

$$E_{\rho j}(\mathbf{r}) = -i\omega\mu_j\frac{\partial^2\psi_j''(\mathbf{r})}{\partial\rho\,\partial z}, \tag{13}$$

where ρ and z are coordinates of the cylindrical system. It is not convenient to express E_ρ in terms of Ψ. Rather, E_ρ will be obtained from Eq. (13) by means of relationship (8).

In this article, expressions for the electric field in regions Ω_1 (away from the current source), Ω_2, and Ω_3 will be derived.

III. SCALAR POTENTIAL IN THE PLATE

A. Governing equation and boundary conditions

Helmholtz equation (10) will now be solved for the scalar potential Ψ_2 in the conductor, subject to certain boundary conditions at its surface. Assuming that Ψ_2 is independent of azimuthal angle ϕ, Eq. (10) may be written as

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + k^2\right)\Psi_2(\rho,z) = 0. \tag{14}$$

As argued in Ref. 1, consideration of the normal component of the current density at the surface $z=0$ gives rise to the following boundary condition:

$$\Psi_2(\rho,0) = C(\rho), \tag{15}$$

where

$$C(\rho) = \begin{cases} \dfrac{I}{\pi(ka)^2}, & \rho < a, \\ 0, & \rho > a. \end{cases} \tag{16}$$

Implicit in Eq. (16) is the assumption that the current density in the wire is uniform with respect to the radial coordinate ρ. This is a reasonable assumption provided that the radius of the wire is somewhat smaller than the electromagnetic skin depth in the wire. In the limit $a \to 0$, to be taken later, it is reasonable to assume uniform current density in the wire for arbitrary frequency.

At $z=T$, the component of current density normal to the air-conductor interface is zero everywhere. Hence

$$\Psi_2(\rho,T)=0. \tag{17}$$

B. Solution

The solution of Eq. (14), subject to the boundary conditions expressed in Eqs. (15), (16), and (17), proceeds along similar lines to that for the half-space conductor.[1] The radial variable ρ is removed by application of the Hankel transform. The Hankel transform of order m of a function $f(\rho)$ is given by[6,7]

$$\tilde{f}(\kappa)=\int_0^\infty f(\rho)J_m(\kappa\rho)\rho d\rho, \tag{18}$$

with the inverse being of the same form. Now apply the zero-order Hankel transform to Eq. (14), making use of the following identity[6]

$$\int_0^\infty \left[\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)f(\rho)\right]J_0(\kappa\rho)\rho d\rho \equiv -\kappa^2\tilde{f}(\kappa), \tag{19}$$

where $f(\rho)$ is assumed to be such that the terms $\rho J_0(\kappa\rho)\,\partial f(\rho)/\partial\rho$ and $\rho f(\rho)\,\partial J_0(\kappa\rho)/\partial\rho$ vanish at both limits. The result is a one-dimensional Helmholtz equation:

$$\left(\frac{\partial^2}{\partial z^2}-\gamma^2\right)\tilde{\Psi}_2(\kappa,z)=0, \quad 0 \leqslant z \leqslant T, \tag{20}$$

wherein $\gamma^2=\kappa^2$-k^2. For γ the root with positive real part is taken.

The general solution of Eq. (20), to which the inverse Hankel transform has been applied, is

$$\Psi_2(\rho,z)=\int_0^\infty [A(\kappa)e^{-\gamma z}+B(\kappa)e^{\gamma z}]J_0(\kappa\rho)\kappa d\kappa. \tag{21}$$

The relationship between $A(\kappa)$ and $B(\kappa)$ is found from the boundary condition at $z=T$, Eq. (17):

$$B(\kappa)=-A(\kappa)e^{-2\gamma T}. \tag{22}$$

Hence,

$$\Psi_2(\rho,z)=\int_0^\infty A(\kappa)[e^{-\gamma z}-e^{\gamma(z-2T)}]J_0(\kappa\rho)\kappa d\kappa. \tag{23}$$

$A(\kappa)$ will now be sought from the boundary condition given in Eqs. (15) and (16). At $z=0$,

$$C(\rho)=\int_0^\infty A(\kappa)(1-e^{-2\gamma T})J_0(\kappa\rho)\kappa d\kappa. \tag{24}$$

$A(\kappa)$ is extracted from Eq. (24) by using the Fourier-Bessel integral (Ref. 8, result 6.3.62). Multiply both sides of Eq. (24) by $\int_0^\infty J_0(\kappa'\rho)\rho d\rho$. Reverse the order of integration on the right-hand side and simplify. This yields

$$A(\kappa)(1-e^{-2\gamma T})=\frac{I}{\pi(ka)^2}\int_0^a J_0(\kappa\rho)\rho d\rho. \tag{25}$$

Evaluation of the integral in Eq. (25) (Ref. 9, result 9.1.30) results in the following expression for $A(\kappa)$:

$$A(\kappa)=\frac{I}{\pi k^2}\frac{J_1(\kappa a)}{\kappa a(1-e^{-2\gamma T})}. \tag{26}$$

Now insert the above expression for $A(\kappa)$ into Eq. (23) to obtain

$$\Psi_2(\rho,z)=\frac{I}{\pi k^2 a}\int_0^\infty \frac{[e^{-\gamma z}-e^{\gamma(z-2T)}]}{(1-e^{-2\gamma T})}J_1(\kappa a)J_0(\kappa\rho)d\kappa. \tag{27}$$

The integral in Eq. (27) cannot be evaluated analytically for arbitrary z, but putting $z=0$ it is found that (Ref. 10, result 6.512.3)

$$\Psi_2(\rho,0)=\begin{cases}\dfrac{I}{\pi(ka)^2}, & \rho<a,\\ 0, & \rho>a,\end{cases} \tag{28}$$

in accordance with boundary condition (15).

In order to make further progress, the limit $a \to 0$ is now taken. This is justified by noting that, practically, the inequality $a \ll 2S$ usually holds. As discussed in Ref. 1, $\lim_{z\to 0}[J_1(z)/z] \sim 1/2$. Hence,

$$\lim_{a\to 0} A(\kappa) \sim \frac{I}{2\pi k^2(1-e^{-2\gamma T})}. \tag{29}$$

If $A(\kappa)$, as given in Eq. (29), is now inserted into Eq. (23), the following expression for Ψ_2 is obtained:

$$\Psi_2(\rho,z)=\frac{I}{2\pi k^2}\int_0^\infty e^{-\gamma z}\left[\frac{1-e^{2\gamma(z-T)}}{1-e^{-2\gamma T}}\right]J_0(\kappa\rho)\kappa d\kappa, \tag{30}$$

wherein the limit $a \to 0$ has been taken. If $T\to\infty$, the term in square brackets tends to unity and the resulting integral is identical to that obtained in the case of a half-space conductor.[1]

It is possible to evaluate the integral in Eq. (30) analytically by expanding the term in the denominator as a binomial series (Ref. 9, result 3.6.10):

$$(1-e^{-2\gamma T})^{-1}=1+e^{-2\gamma T}+e^{-4\gamma T}+e^{-6\gamma T}+e^{-8\gamma T}+\cdots$$
$$=\sum_{n=0}^\infty e^{-2n\gamma T}. \tag{31}$$

Multiplying the right-hand side of Eq. (31) by the factor $e^{-\gamma z}[1-e^{2\gamma(z-T)}]$, and substituting the result into Eq. (30), yields

$$\Psi_2(\rho,z)=\frac{I}{2\pi k^2}\sum_{n=0}^\infty\int_0^\infty \{e^{-\gamma(z+2nT)} - e^{\gamma[z-2(n+1)T]}\}J_0(\kappa\rho)\kappa d\kappa, \tag{32}$$

where the order of summation and integration has been reversed. The first term in braces in the integrand of Eq. (32), $e^{-\gamma z}$, gives rise to the result for the TM potential in a half-space conductor, which has been derived elsewhere:[1]

$$\Psi_2(\rho,z) = -\frac{I}{2\pi}\frac{ikz}{(ikr)^3}e^{ikr}(1-ikr), \quad z>0, \quad T\to\infty, \tag{33}$$

with $r^2=\rho^2+z^2$. The second term $-e^{\gamma(z-2T)}$ accounts for the primary reflection of the field from the back surface of the plate at $z=T$. Higher terms deal with multiple reflections between the surfaces of the plate. By analogy with the result for the half-space conductor, Eq. (33), or by multiple use of the analytic result given in Ref. 7 (result 8.2.23), the terms in Eq. (32) can be integrated. It is found that

$$\Psi_2(\rho,z) = -\frac{I}{2\pi}\sum_{n=0}^{\infty}\left\{\frac{ik(z+2nT)}{(ikr_n)^3}e^{ikr_n}(1-ikr_n) + \frac{ik[z-2(n+1)T]}{(ikr_n')^3}e^{ikr_n'}(1-ikr_n')\right\}, \quad 0<z\leqslant T, \tag{34}$$

with $r_n=\sqrt{\rho^2+(z+2nT)^2}$ and $r_n'=\sqrt{\rho^2+[z-2(n+1)T]^2}$.

C. Truncation of the series

For the purpose of practical computation, the infinite series of result (34) must be suitably truncated. The accuracy with which the boundary conditions are fulfilled can be used as a guide in this process. Consider first the boundary condition at $z=0$, Eqs. (15) and (16). The form of result (34) reveals that $\Psi_2(\rho,0)$ is identically zero for $\rho>0$ (away from the current source) provided that the series has an odd number of terms. Explicitly, a suitable form for the truncated series is

$$\Psi_2(\rho,z) \approx -\frac{I}{2\pi}\left(\sum_{n=0}^{N}\left\{\frac{ik(z+2nT)}{(ikr_n)^3}e^{ikr_n}(1-ikr_n)\right\} + \sum_{n=0}^{N-1}\left\{\frac{ik[z-2(n+1)T]}{(ikr_n')^3}e^{ikr_n'}(1-ikr_n')\right\}\right), \quad 0<z\leqslant T. \tag{35}$$

This choice gives greatest accuracy near the conductor surface at $z=0$. With this truncation the residual error in the potential Ψ_2 at $z=T$ is given by

$$\Psi_2(\rho,T) = -\frac{I}{2\pi}\frac{(2N+1)ikT}{(ikr_{NT})^3}e^{ikr_{NT}}(1-ikr_{NT}), \tag{36}$$

where $r_{NT}^2=\rho^2+[(2N+1)T]^2$. The form of Eq. (36) reveals that boundary condition (17) is matched most closely by expression (35) when T is large, $\rho\to\infty$, and $N\to\infty$. This behavior is shown in Figs. 3 and 4.

Conversely, an exact match with the boundary condition at $z=T$ can be achieved by truncating the series in Eq. (34) to an even number of terms. Then, a residual error in the value of Ψ_2 exists at $z=0$.

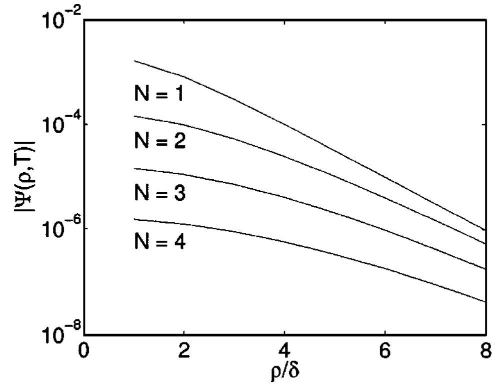

FIG. 3. $|\Psi_2(\rho,T)|$ as a function of ρ for various values of maximum index N. $T=\delta$. $|\Psi_2(\rho,T)|$ reduces when N and ρ increase.

IV. ELECTRIC FIELD IN THE PLATE

For the purpose of deriving expression for the electric field in air, following section, it is useful to express the electric field in the plate in integral form. From Eqs. (30), (12), and (13),

$$E_{z2}(\rho,z) = \frac{I}{2\pi\sigma_2}\int_0^{\infty}\kappa e^{-\gamma z}\left[\frac{1-e^{2\gamma(z-T)}}{1-e^{-2\gamma T}}\right]J_0(\kappa\rho)d\kappa, \quad 0\leqslant z\leqslant T, \tag{37}$$

and

$$E_{\rho2}(\rho,z) = \frac{I}{2\pi\sigma_2}\int_0^{\infty}\gamma e^{-\gamma z}\left[\frac{1+e^{2\gamma(z-T)}}{1-e^{-2\gamma T}}\right]J_1(\kappa\rho)d\kappa, \quad 0\leqslant z\leqslant T. \tag{38}$$

Next, following the method of Ref. 1, or by analogy with the results for the half-space conductor, real-space analytic forms for the two components of the electric field in the plate can be obtained. It is a trivial matter to obtain E_z from results (35) and (28) since E_{zj} and Ψ_j are simply related by the factor $i\omega\mu_j$, Eq. (12);

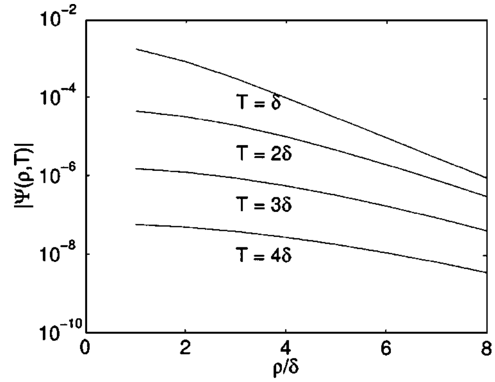

FIG. 4. $|\Psi_2(\rho,T)|$, given in Eq. (36), as a function of ρ for various values of plate thickness T. $N=1$. It can be seen that $|\Psi_2(\rho,T)|$ reduces when T and ρ increase.

$$E_{z2}(r) = -\frac{i\omega\mu_2 I}{2\pi}\left(\sum_{n=0}^{N}\left\{\frac{ik(z+2nT)}{(ikr_n)^3}e^{ikr_n}(1-ikr_n)\right\} + \sum_{n=0}^{N-1}\left\{\frac{ik[z-2(n+1)T]}{(ikr'_n)^3}e^{ikr'_n(1-ikr'_n)}\right\}\right),$$
$$0 < z \leq T \tag{39}$$

and

$$E_{z2}(\rho,0) = 0, \quad \rho > 0. \tag{40}$$

To obtain $E_{\rho 2}$ via relations (13) and (8) requires some manipulation.[1] The result is

$$E_{\rho 2}(\mathbf{r}) = -\frac{ikI}{2\pi\sigma_2\rho}\left(\sum_{n=0}^{N}\left\{e^{ik(z+2nT)} - \frac{e^{ikr_n}}{ikr_n}\left[1+\frac{[ik(z+2nT)]^2}{ikr_n}\left(1-\frac{1}{ikr_n}\right)\right]\right\} + \sum_{n=0}^{N-1}\left\{e^{-ik[z-2(n+1)T]} - \frac{e^{ikr'_n}}{ikr'_n}\left[1+\frac{\{ik[z-2(n+1)T]\}^2}{ikr'_n}\left(1-\frac{1}{ikr'_n}\right)\right]\right\}\right),$$
$$0 \leq z \leq T. \tag{41}$$

Considering the forms of E_{z2} and $E_{\rho 2}$ given in Eqs. (39) and (41), respectively, it can be seen that the electric field exhibits correct behavior in certain simple cases. E_{z2} is symmetric with respect to ρ and both E_{z2} and $E_{\rho 2} \to 0$ as $r \to \infty$. On the z axis, $E_{\rho 2}(0,z)=0$ whereas

$$E_{z2}(0,z) \to \frac{I}{2\pi\sigma_2 z^2} \quad \text{as } z \to 0.$$

In the far field, the electric field is dominated by terms of the form e^{ikz}/ρ in Eq. (41). The associated current density is

$$J_\rho(\mathbf{r}) \approx -\frac{ikI}{2\pi\rho}\left(\sum_{n=0}^{N-1}\{e^{ik(z+2nT)} + e^{-ik[z-2(n+1)T]}\} + e^{ik(z+2NT)}\right),$$
$$\text{as } r \to \infty. \tag{42}$$

If the far-field current density, given in Eq. (42), is integrated over a cylindrical surface of large radius extending from $z=0$ to T, the result is $I[1+e^{ik(2N+1)T}]$. This expression tends to I as $N \to \infty$, as it should. For a field point in a thin plate, it is often the case that T and, consequently, z are much smaller than the other variables x and y defining the position of the point of interest. Under these circumstances, the far-field approximation given in Eq. (42) is also applicable in this "thin plate" regime. In the case in which $T \to 0$ and the thickness of the plate becomes infinitesimal, the electric field in the conductor is divergent. This behavior is shown in the divergence of the series summations in Eqs. (39), (41), and, indeed, (42). Finally, in the static limit of direct current, $k \to 0$ and the following expressions for the current densities J_z and J_ρ are obtained from Eqs. (39) and (41):

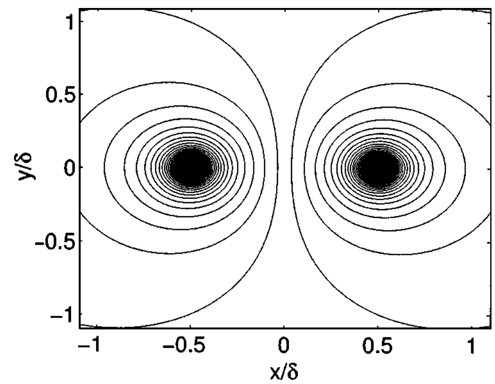

FIG. 5. Contour plot of $|E_\rho^T|$ on the surface of a conductive plate, conductivity $\sigma_2 = 1.1 \times 10^7$ S/m. Frequency, $f = 1$ kHz and $T = \delta$. Current is injected/extracted at $x = \pm 0.5\delta$.

$$J_z(\mathbf{r}) = \frac{I}{2\pi}\left\{\sum_{n=0}^{N}\frac{(z+2nT)}{r_n^3} + \sum_{n=0}^{N-1}\frac{[z-2(n+1)T]}{(r'_n)^3}\right\}, \tag{43}$$

$$J_\rho(\mathbf{r}) = \frac{I}{2\pi}\rho\left(\sum_{n=0}^{N}\frac{1}{r_n^3} + \sum_{n=0}^{N-1}\frac{1}{(r'_n)^3}\right). \tag{44}$$

The term containing r_0 is the solution for the half-space conductor, in which the current density radiates uniformly from the point of injection and $J_r(\mathbf{r}) = I/(2\pi r^2)$. Higher terms represent contributions to the current density due to internal reflections from the plate surfaces. These contributions act as through originating at image sources located at $(0, 0, \pm 2nT)$, with $n = 1, \ldots, N$.

For interest, contour plots of $|E_\rho^T|$ on the conductor surface ($z=0$) and of $|E_z^T|$ in the plane $y=0$, Eq. (1), are shown in Figs. 5 and 6, for a representative case. The components of $\mathbf{E}$, Eqs. (39) and (41), have been combined according to Eq. (1). Lengths are normalized to the electromagnetic skin depth in the conductor; $\delta = (2/\omega\mu_2\sigma_2)^{1/2}$.

V. ELECTRIC FIELD IN AIR

In region Ω_1, Fig. 2, there are two contributions to the electric field. One is from the current flowing the wire, $\mathbf{E}^w$, and the other is from the current density in the half-space conductor, $\mathbf{E}^c$. These contributions will be analyzed separately and then combined to give the electric field in region Ω_1, $\mathbf{E}_1$. Again, it is assumed that $a \to 0$.

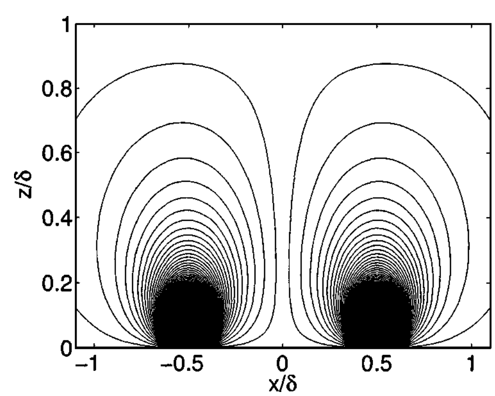

FIG. 6. Contour plot of $|E_z^T|$ in the plane $y=0$. Parameters are as for Fig. 5.

$$\mathbf{E}_1 = \mathbf{E}^w + \mathbf{E}^c, \quad z \leqslant 0. \tag{45}$$

An expression for the magnetic field in air due to the wire, $\mathbf{H}^w$, can be obtained by applying the well-known integral form of Ampere's law in the region $z \leqslant 0$. It is found that

$$\mathbf{H}^w = \frac{I}{2\pi\rho}\hat{\phi}, \quad \rho > 0, \quad z \leqslant 0. \tag{46}$$

Symmetry dictates that $\mathbf{E}^w$ has only a $\hat{z}$ component. Continuity of the tangential electric field at the surface of the wire dictates that $\mathbf{E}^w$ has the same direction as the current density $\mathbf{J}$ in the wire. Applying Faraday's law it is found that

$$\mathbf{E}^w = \hat{z}\frac{i\omega\mu_1 I}{2\pi}\ln \rho, \quad \rho > 0, \quad z \leqslant 0, \tag{47}$$

for $\mathbf{J} = \hat{z}J_z$. Clearly the electric field expressed in Eq. (47) diverges as $\rho \to 0$ and as $\rho \to \infty$. Divergence in the former case is a consequence of the assumption that the radius of the wire is infinitesimal. Divergence in the latter case is a consequence of the fact that only one current-carrying wire is considered at this stage in the analysis. Closing the current loop by superposing the fields due to two wires carrying opposing currents, as expressed in Eq. (1), yields the nondivergent field which is obtained in practice as $\rho \to \infty$.

A solution for the electric field in air (due to the current in the metal plate), $\mathbf{E}^c$, is obtained by solving for the modified transverse magnetic potential Ψ_1 defined in Eq. (8). The boundary conditions on $\mathbf{E}^c$ are

$$E^c_{\rho 1}(\rho,0) = E_{\rho 2}(\rho,0) \tag{48}$$

and

$$|\mathbf{E}^c(\rho,z)| \to 0 \quad \text{as } z \to -\infty. \tag{49}$$

Hence, through Eq. (48), the solution for the electric field in the conductor is needed to determine the field in air.

The potential Ψ_1 obeys Laplace's equation in source-free regions, Eq. (9). If Eq. (9) is written in cylindrical coordinates and Ψ_1 is independent of azimuthal angle ϕ, then

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial^2}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right)\Psi_1(\rho,z) = 0, \quad \rho > 0, \quad z \leqslant 0. \tag{50}$$

Now apply the zero-order Hankel transform to Eq. (50). The result is a one-dimensional Helmholtz equation,

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2\right)\tilde{\Psi}_1(\kappa,z) = 0, \quad z \leqslant 0, \tag{51}$$

the solution of which is

$$\tilde{\Psi}_1(\kappa,z) = A(\kappa)e^{-\kappa z} + B(\kappa)e^{\kappa z}, \quad z \leqslant 0. \tag{52}$$

Here $A(\kappa)$ is zero since Ψ_1 must remain finite as $z \to -\infty$, Eq. (49). Applying the inverse transform to $\tilde{\Psi}_1$ yields

$$\Psi_1(\rho,z) = \int_0^\infty B(\kappa)e^{\kappa z}J_0(\kappa\rho)\kappa d\kappa. \tag{53}$$

The coefficient $B(\kappa)$ will be sought from the continuity condition on the tangential component of the electric field at the air-conductor interface, $z=0$, Eq. (48). This will be done with the electric field written as an infinite integral with respect to κ. To express $E^c_{\rho 1}$ in this form, first note that [Ref. 1, Eq. (35)]

$$\tilde{\psi}''_j(\kappa,z) = -\frac{\tilde{\Psi}_j(\kappa,z)}{\kappa^2}. \tag{54}$$

This relation is obtained by applying the Hankel transform to Eq. (8). Then, from relations (53), (54), and (13),

$$E^c_{\rho 1}(\rho,z) = -i\omega\mu_1\int_0^\infty B(\kappa)e^{\kappa z}J_1(\kappa\rho)\kappa d\kappa. \tag{55}$$

Putting $z=0$ in Eqs. (38) and (55) and equating gives the following result for $B(\kappa)$:

$$B(\kappa) = -\frac{I}{2\pi k^2}\frac{\mu_2}{\mu_1}\frac{\gamma}{\kappa}\coth(\gamma T). \tag{56}$$

On substituting this expression for $B(\kappa)$ into Eq. (53) it is found that

$$\Psi_1(\rho,z) = -\frac{I}{2\pi k^2}\frac{\mu_2}{\mu_1}\int_0^\infty \gamma\coth(\gamma T)e^{\kappa z}J_0(\kappa\rho)d\kappa,$$
$$z \leqslant 0. \tag{57}$$

From relation (12),

$$E^c_{z1}(\rho,z) = -\frac{I}{2\pi\sigma_2}\int_0^\infty \gamma\coth(\gamma T)e^{\kappa z}J_0(\kappa\rho)d\kappa, \quad z \leqslant 0. \tag{58}$$

Finally an expression for $E^c_{\rho 1}$ is obtained by substituting $B(\kappa)$ into Eq. (55):

$$E^c_{\rho 1}(\rho,z) = \frac{I}{2\pi\sigma_2}\int_0^\infty \gamma\coth(\gamma T)e^{\kappa z}J_1(\kappa\rho)d\kappa, \quad z \leqslant 0. \tag{59}$$

Comparing Eqs. (30), (37), and (38), for the potential and electric field in the conductor, with Eqs. (57), (58), and (59), for the potential and electric field in air, it is clear that the exponential decay in the $\hat{z}$ direction is governed by γ in the conductor and κ in air. Comparing Eqs. (38) and (59) it is obvious that boundary condition (48) is satisfied. While analytic forms for the electric field in the conductor were obtained previously by evaluating the integrals in Eqs. (37) and (38) to give Eqs. (39) and (41),[1] it is not possible to evaluate the integrals in Eqs. (58) and (59) analytically. The electric field in region Ω_1 is hence given by Eqs. (47), (58), and (59). The terms in Eq. (47) and (58) are summed to give the full $\hat{z}$ component while Eq. (59) alone describes the $\hat{\rho}$ component of $\mathbf{E}$.

For completeness, expressions for electric field in region Ω_3 can be obtained following the same method by which Eqs. (58) and (59) were obtained. The result is

$$E_{z3}(\rho,z) = \frac{I}{4\pi\sigma_2}\int_0^\infty \gamma\,\mathrm{csch}(\gamma T)e^{-\kappa(z-T)}J_0(\kappa\rho)d\kappa,$$
$$z \geqslant T, \tag{60}$$

$$E_{\rho 3}(\rho,z) = \frac{I}{4\pi\sigma_2}\int_0^\infty \gamma \operatorname{csch}(\gamma T) e^{-\kappa(z-T)} J_1(\kappa\rho) d\kappa, \quad z \geqslant T. \tag{61}$$

The result for a half-space conductor is obtained by taking the limit $T \to \infty$ in Eqs. (58) and (59). Then coth $(\gamma T) \to 1$ and the results are equivalent to those reported in Ref. 2.

VI. CONCLUSION

Analytic expressions for the electric field due to alternating current injected at the surface of a metal plate have been derived. In the plate, the electric field is expressed in terms of a real-space series expansion whose terms originate in internal reflections at the plate surfaces. In air, the $\hat{\rho}$ and $\hat{z}$ components of the electric field are expressed as first- and zero-order Hankel transforms, respectively. The results facilitate proper interpretation of four-point ACPD measurements since, by integrating the electric field around the loop of the pick-up circuit, all contributions to the measured voltage can be evaluated. This is the subject of a future article.

ACKNOWLEDGMENTS

This work was supported by the NSF Industry/University Cooperative Research program. The author would like to thank Yongqiang Huang for assistance with preparing the diagrams and Theodoros Theodoulidis for useful discussions and references.

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