Electrodynamics of Continuous Media

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Landau and Lifshitz Course of Theoretical Physics

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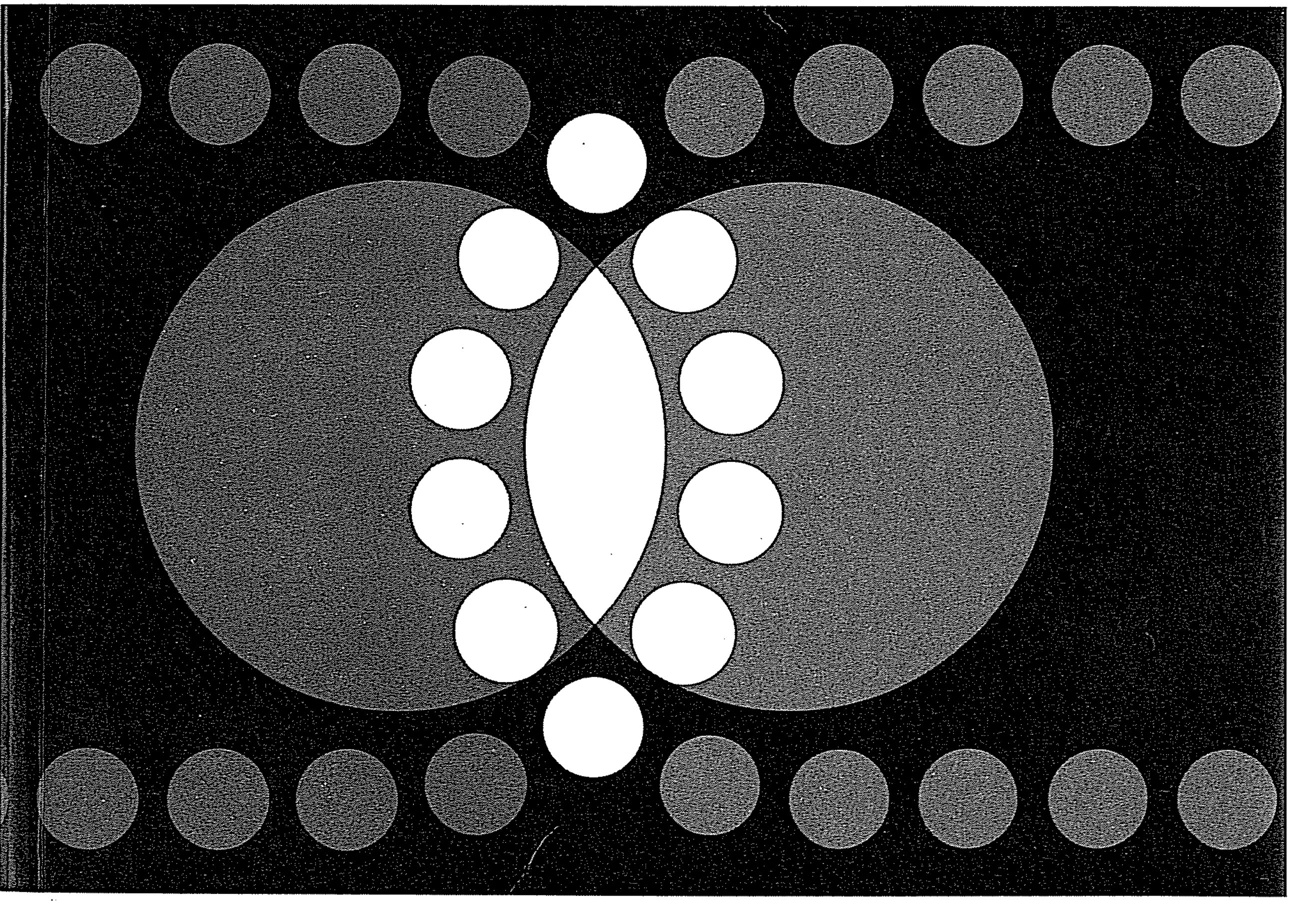

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The first term is the solution of the homogeneous equation $\triangle\phi = 0$, and the coefficient is chosen so as to give continuity of the potential, and therefore of $\mathbf{E}_t$, at the surface of the sphere. Substituting (3) and (4) in (2), we obtain

$$D_{ik} = -\frac{a^5}{c}\frac{3(\varepsilon\mu-1)}{(3+2\varepsilon)(2+\mu)}[\mathfrak{H}_i\Omega_k+\mathfrak{H}_k\Omega_i-\tfrac{2}{3}\delta_{ik}\mathfrak{H}\cdot\boldsymbol{\Omega}]. \tag{5}$$

Thus a quadrupole electric field is formed near the rotating sphere, and the quadrupole moment of the sphere is given by formula (5). In particular, if the axis of rotation (the z-axis) is parallel to the external field, D_{ik} has only the diagonal components

$$D_{zz} = -\frac{a^5}{c}\frac{4(\varepsilon\mu-1)}{(3+2\varepsilon)(2+\mu)}\mathfrak{H}\cdot\boldsymbol{\Omega}, \qquad D_{xx} = D_{yy} = -\tfrac{1}{2}D_{zz}.$$

Similarly, a quadrupole magnetic field occurs near a sphere rotating in a uniform electric field. The magnetic quadrupole moment is given by (5) if the sign is changed and ε, μ, $\mathfrak{H}$ are replaced by μ, ε, $\mathfrak{E}$ respectively.

PROBLEM 2. A magnetized sphere rotates uniformly in a vacuum about its axis, which is parallel to the direction of magnetization. Determine the resulting electric field near the sphere.†

SOLUTION. The magnetic field inside the sphere is uniform, and is expressed in terms of the constant magnetization $\mathbf{M}$ by the equations $\mathbf{B}^{(i)}+2\mathbf{H}^{(i)} = 0$ (cf. (8.1)) and $\mathbf{B}^{(i)}-\mathbf{H}^{(i)} = 4\pi\mathbf{M}$, whence $\mathbf{B}^{(i)} = 8\pi\mathbf{M}/3$, $\mathbf{H}^{(i)} = -4\pi\mathbf{M}/3$. The second of formulae (76.9) does not hold in this case, because the formula $\mathbf{B} = \mu\mathbf{H}$ is not valid for a ferromagnet at rest; from the first of (76.9) we have, inside the sphere,

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E}+\varepsilon\mathbf{v}\times\mathbf{B}/c-\mathbf{v}\times\mathbf{H}/c\\ &= \varepsilon\mathbf{E}+4\pi(2\varepsilon+1)\,\mathbf{v}\times\mathbf{M}/3c.\end{aligned}$$

The potential of the resulting electric field outside the sphere satisfies the equation $\triangle\phi^{(e)} = 0$, and that inside the sphere satisfies $\triangle\phi^{(i)} = 8\pi(2\varepsilon+1)M\Omega/3c\varepsilon$.

The boundary condition that D_n be continuous at the surface of the sphere gives

$$-\varepsilon\left[\frac{\partial\phi^{(i)}}{\partial r}\right]_{r=a}+\frac{4\pi(2\varepsilon+1)}{3c}a\Omega M\sin^2\theta = -\left[\frac{\partial\phi^{(e)}}{\partial r}\right]_{r=a},$$

where θ is the angle between the normal $\mathbf{n}$ and the direction of $\boldsymbol{\Omega}$ and $\mathbf{M}$ (the z-axis). We seek $\phi^{(e)}$ and $\phi^{(i)}$ in the forms

$$\phi^{(e)} = \frac{D_{ik}n_in_k}{2r^3} = \frac{D_{zz}}{4r^3}(3\cos^2\theta-1),$$

$$\phi^{(i)} = \frac{r^2}{4a^5}D_{zz}(3\cos^2\theta-1)+\frac{4\pi(2\varepsilon+1)}{9c\varepsilon}M\Omega(r^2-a^2).$$

From the boundary condition we obtain the following expressions for the electric quadrupole moment of the rotating sphere:

$$D_{zz} = -\frac{4(2\varepsilon+1)}{3c(2\varepsilon+3)}a^2\Omega\mathscr{M}, \qquad D_{xx} = D_{yy} = -\tfrac{1}{2}D_{zz},$$

where $\mathscr{M}$ is the total magnetic moment of the sphere. For a metal sphere we must take $\varepsilon\to\infty$, giving

$$D_{zz} = -4\Omega\mathscr{M}a^2/3c.$$

§77. The dispersion of the permittivity

Let us now go on to study the important subject of rapidly varying electromagnetic fields, whose frequencies are not restricted to be small in comparison with the frequencies

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† If the direction of magnetization is not the same as that of the axis of rotation, the problem is considerably changed, since the sphere then emits electromagnetic waves.

which characterize the establishment of the electric and magnetic polarization of the substances concerned.

An electromagnetic field variable in time must necessarily be variable in space also. For a frequency ω, the spatial periodicity is characterized by a wavelength $\lambda \sim c/\omega$. As the frequency increases, λ eventually becomes comparable with the atomic dimensions a. The macroscopic description of the field is thereafter invalid.

The question may arise whether there is any frequency range in which, on the one hand, dispersion phenomena are important but, on the other hand, the macroscopic formulation still holds good. It is easy to see that such a range must exist. The most rapid manner of establishment of the electric or magnetic polarization in matter is the electronic mechanism. Its relaxation time is of the order of the atomic time a/v, where v is the velocity of the electrons in the atom. Since $v \ll c$, even the wavelength $\lambda \sim ac/v$ corresponding to these times is large compared with a.

In what follows we shall assume the condition $\lambda \gg a$ to hold.† It must be borne in mind, however, that this condition may not be sufficient: for metals at low temperatures there is a range of frequencies in which the macroscopic theory is inapplicable, although the inequality $c/\omega \gg a$ is satisfied (see §87).

The formal theory given below is equally applicable to metals and to dielectrics. At frequencies corresponding to the motion of the electrons within the atoms (*optical frequencies*) and at higher frequencies, there is, indeed, not even a quantitative difference in the properties of metals and dielectrics.

It is clear from the discussion in §75 that Maxwell's equations

$$\operatorname{div}\mathbf{D} = 0, \qquad \operatorname{div}\mathbf{B} = 0, \tag{77.1}$$

$$\mathbf{curl}\,\mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t, \qquad \mathbf{curl}\,\mathbf{H} = (1/c)\,\partial\mathbf{D}/\partial t \tag{77.2}$$

remain formally the same in arbitrary variable electromagnetic fields. These equations are, however, largely useless until the relations between the quantities $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{H}$ which appear in them have been established. At the high frequencies at present under consideration, these relations bear no resemblance to those which are valid in the static case and which we have used for variable fields in the absence of dispersion.

First of all, the principal property of these relations, namely the dependence of $\mathbf{D}$ and $\mathbf{B}$ only on the values of $\mathbf{E}$ and $\mathbf{H}$ at the instant considered, no longer holds good. In the general case of an arbitrary variable field, the values of $\mathbf{D}$ and $\mathbf{B}$ at a given instant are not determined only by the values of $\mathbf{E}$ and $\mathbf{H}$ at that instant. On the contrary, they depend in general on the values of $\mathbf{E}(t)$ and $\mathbf{H}(t)$ at every previous instant. This expresses the fact that the establishment of the electric or magnetic polarization of the matter cannot keep up with the change in the electromagnetic field. The frequencies at which dispersion phenomena first appear may be completely different for the electric and the magnetic properties of the substance.

In the present section we shall refer to the dependence of $\mathbf{D}$ on $\mathbf{E}$; the specific features of the dispersion of magnetic properties will be discussed in §79.

The polarization vector $\mathbf{P}$ has been introduced in §6 by means of the definition $\bar{\rho} = -\operatorname{div}\mathbf{P}$, ρ being the true (microscopic) charge density. This equation expresses the

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† The effects (called the *natural optical activity*) resulting from terms of the next order in the small ratio a/λ will be considered in §§104–106.

electric neutrality of the body as a whole, and together with the condition $\mathbf{P} = 0$ outside the body it shows that the total electric moment of the body is $\int \mathbf{P}\, dV$. This derivation is evidently valid for variable as well as for static fields. Thus in any variable field, even if dispersion is present, the vector $\mathbf{P} = (\mathbf{D} - \mathbf{E})/4\pi$ retains its physical significance: it is the electric moment per unit volume.

In rapidly varying fields, the field strengths involved are usually fairly small. Hence the relation between **D** and **E** can always be taken to be linear.† The most general linear relation between $\mathbf{D}(t)$ and the values of the function $\mathbf{E}(t)$ at all previous instants can be written in the integral form

$$\mathbf{D}(t) = \mathbf{E}(t) + \int_0^\infty f(\tau)\, \mathbf{E}(t-\tau)\, d\tau. \tag{77.3}$$

It is convenient to separate the term $\mathbf{E}(t)$, for reasons which will become evident later. In equation (77.3) $f(\tau)$ is a function of time and of the properties of the medium. By analogy with the electrostatic formula $\mathbf{D} = \varepsilon \mathbf{E}$, we write the relation (77.3) in the symbolic form $\mathbf{D} = \hat{\varepsilon}\mathbf{E}$, where $\hat{\varepsilon}$ is a linear integral operator whose effect is shown by (77.3).

Any variable field can be resolved by a Fourier expansion into a series of monochromatic components, in which all quantities depend on time through the factor $e^{-i\omega t}$. For such fields the relation (77.3) between **D** and **E** becomes

$$\mathbf{D} = \varepsilon(\omega)\, \mathbf{E}, \tag{77.4}$$

where the function $\varepsilon(\omega)$ is defined as

$$\varepsilon(\omega) = 1 + \int_0^\infty f(\tau) e^{i\omega\tau}\, d\tau. \tag{77.5}$$

Thus, for periodic fields, we can regard the permittivity (the coefficient of proportionality between **D** and **E**) as a function of the field frequency as well as of the properties of the medium. The dependence of ε on the frequency is called its *dispersion relation.*

The function $\varepsilon(\omega)$ is in general complex. We denote its real and imaginary parts by ε' and ε'':

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega). \tag{77.6}$$

From the definition (77.5) we see at once that

$$\varepsilon(-\omega) = \varepsilon^*(\omega). \tag{77.7}$$

Separating the real and imaginary parts, we have

$$\varepsilon'(-\omega) = \varepsilon'(\omega), \qquad \varepsilon''(-\omega) = -\varepsilon''(\omega). \tag{77.8}$$

Thus $\varepsilon'(\omega)$ is an even function of the frequency, and $\varepsilon''(\omega)$ is an odd function.

For frequencies which are small compared with those at which the dispersion begins, we can expand $\varepsilon(\omega)$ as a power series in ω. The expansion of the even function $\varepsilon'(\omega)$ contains only even powers, and that of the odd function $\varepsilon''(\omega)$ contains only odd powers. In the limit

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† Here we assume that **D** depends linearly on **E** alone, and not on **H**. In a constant field, a linear dependence of **D** on **H** is excluded by the requirement of invariance with respect to time reversal. In a variable field, this condition no longer applies, and a linear relation between **D** and **H** is possible if the substance possesses symmetry of various kinds. It is, however, a small effect of the order of a/λ, of the kind mentioned in the last footnote.

as $\omega \to 0$, the function $\varepsilon(\omega)$ in dielectrics tends, of course, to the electrostatic permittivity, which we here denote by ε_0. In dielectrics, therefore, the expansion of $\varepsilon'(\omega)$ begins with the constant term ε_0, while that of $\varepsilon''(\omega)$ begins, in general, with a term in ω.

The function $\varepsilon(\omega)$ at low frequencies can also be discussed for metals, if it is defined in such a way that, in the limit $\omega \to 0$, the question

$$\mathbf{curl\, H} = (1/c)\, \partial \mathbf{D}/\partial t$$

becomes the equation

$$\mathbf{curl\, H} = 4\pi\sigma\, \mathbf{E}/c$$

for a static field in a conductor. Comparing the two equations, we see that for $\omega \to 0$ we must have $\partial \mathbf{D}/\partial t \to 4\pi\sigma\, \mathbf{E}$. But, in a periodic field, $\partial \mathbf{D}/\partial t = -i\omega\varepsilon\, \mathbf{E}$, and we thus obtain the following expression for $\varepsilon(\omega)$ in the limit of low frequencies:

$$\varepsilon(\omega) = 4\pi i \sigma/\omega. \tag{77.9}$$

Thus the expansion of the function $\varepsilon(\omega)$ in conductors begins with an imaginary term in $1/\omega$, which is expressed in terms of the ordinary conductivity σ for steady currents.† The next term in the expansion of $\varepsilon(\omega)$ is a real constant, although for metals this constant does not have the same electrostatic significance as it does for dielectrics.‡

Moreover, this term of the expansion may again be devoid of significance if the effects of the spatial non-uniformity of the field of the electromagnetic wave appear before those of its periodicity in time.

§78. The permittivity at very high frequencies

In the limit as $\omega \to \infty$, the function $\varepsilon(\omega)$ tends to unity. This is evident from simple physical considerations: when the field changes sufficiently rapidly, the polarization processes responsible for the difference between the field **E** and the induction **D** cannot occur at all.

It is possible to establish the limiting form of the function $\varepsilon(\omega)$ at high frequencies, which is valid for all bodies, whether metals or dielectrics. The field frequency is assumed large compared with the frequencies of the motion of all, or at least the majority, of the electrons in the atoms forming the body. When this condition holds, we can calculate the polarization of the substance by regarding the electrons as free and neglecting their interaction with one another and with the nuclei of the atoms.

The velocities v of the motion of the electrons in the atoms are small compared with the velocity of light. Hence the distances v/ω which they traverse during one period of the electromagnetic wave are small compared with the wavelength c/ω. For this reason we can assume the wave field uniform in determining the velocity acquired by an electron in that field.

The equation of motion is $m\, d\mathbf{v}'/dt = e\mathbf{E} = e\mathbf{E}_0 e^{-i\omega t}$, where e and m are the electron charge and mass, and $\mathbf{v}'$ is the additional velocity acquired by the electron in the wave field.

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† The imaginary part of the function $\varepsilon(\omega)$ is sometimes represented in the form (77.9) for all frequencies; this amounts to introducing a new function $\sigma(\omega)$, which has no physical significance apart from its relationship to $\varepsilon''(\omega)$.

‡ To avoid misunderstanding, we should point out a slight change in notation in comparison with §75. In equation (75.10) for poor conductors, $\varepsilon(\omega)$ is $(4\pi i\sigma/\omega) + \varepsilon$.

From (81.5), for a small change in the capacitance,

$$\delta\omega/\omega = -\tfrac{1}{2}\delta C/C. \tag{81.8}$$

This change consists of two parts:

$$\delta C = (\delta C)_{st} + (\mathrm{d}C/\mathrm{d}\omega)\delta\omega. \tag{81.9}$$

The first term is the static part, related to the strain as in the static case; here it is important that, in the presence of dispersion, the capacitance $C(\omega)$ is expressed in terms of $\varepsilon(\omega)$ in the same way as in the static case. The second term depends only on the frequency change. From (81.8) and (81.9) we find as the static part

$$(\delta C)_{st} = -\frac{1}{\omega^2}\frac{\mathrm{d}(\omega^2 C)}{\mathrm{d}\omega}\delta\omega. \tag{81.10}$$

When (81.6) is substituted in (81.7), and (81.10) is used, $\mathrm{d}C/\mathrm{d}\omega$ disappears, and the energy variation becomes

$$\delta\overline{\mathscr{U}} = -\tfrac{1}{2}\overline{\phi^2}\,(\delta C)_{st} = -\tfrac{1}{2}(\overline{e^2}/C^2)(\delta C)_{st}, \tag{81.11}$$

which is in fact the same as the averaged second term in (81.3).

The disappearance from $\delta\overline{\mathscr{U}}$ of the terms involving the derivative with respect to ω is quite general and does not depend on a specific manner of change in the state of the body (in this case, the capacitor). In particular, for a dispersive medium formula (14.1) (with $\mathbf{E}^2$ replaced by $\overline{\mathbf{E}^2}$) remains valid for the change in the free energy due to a small change in ε:

$$\delta\mathscr{F} = -\int \delta\varepsilon(\omega)\overline{\mathbf{E}^2}\,\mathrm{d}V/8\pi. \tag{81.12}$$

Knowing the stress tensor, we can use (75.17) to find the force on unit volume of the dielectric. The terms containing spatial derivatives coincide with the corresponding terms in the time-averaged expression (75.18), in which we must put $\mu = 1$. The term in the time derivative (the Abraham force) is not the same. This term arises as the difference

$$\frac{1}{4\pi c}\left\{\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{H}) - \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H})\right\},$$

which is now to be averaged over time. To do so, we express **D**, **E** and **H** in complex form (i.e. replace them by $\tfrac{1}{2}(\mathbf{D}+\mathbf{D}^*)$ and so on), and then use (80.10) for $\partial\mathbf{D}/\partial t$. This gives the Abraham force in the form

$$\frac{1}{8\pi c}(\varepsilon-1)\,\mathrm{re}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}^*) + \frac{1}{8\pi c}\omega\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\,\mathrm{re}\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{H}^*\right) \tag{81.13}$$

(H. Vashina and V. I. Karpman, 1976).

The stress tensor in a variable field is significant for absorbing as well as transparent media, unlike the internal energy, for which the problem can be formulated only by neglecting the dissipation. There is, however, reason to suppose that in an absorbing medium the stress tensor cannot be expressed in terms of the permittivity alone, and therefore cannot be derived in a general form by macroscopic methods.

§82. The analytical properties of $\varepsilon(\omega)$

The function $f(\tau)$ in (77.3) is finite for all values of τ, including zero.† For dielectrics it tends to zero as $\tau\to\infty$. This simply expresses the fact that the value of $\mathbf{D}(t)$ at any instant cannot be appreciably affected by the values of $\mathbf{E}(t)$ at remote instants. The physical agency underlying the integral relation (77.3) consists in the processes of the establishment of the electric polarization. Hence the range of values in which the function $f(\tau)$ differs appreciably from zero is of the order of the relaxation time which characterizes these processes.

The above statements are true also of metals, the only difference being that the function $f(\tau)-4\pi\sigma$, rather than $f(\tau)$ itself, tends to zero as $\tau\to\infty$. This difference arises because the passage of a steady conduction current, though it does not cause any actual change in the physical state of the metal, in our equations leads formally to the presence of an induction **D** such that $(1/c)\partial\mathbf{D}/\partial t = 4\pi\sigma\mathbf{E}/c$ or

$$\mathbf{D}(t) = \int\limits_{-\infty}^{t} 4\pi\sigma\,\mathbf{E}(\tau)\mathrm{d}\tau = 4\pi\sigma\int\limits_{0}^{\infty}\mathbf{E}(t-\tau)\,\mathrm{d}\tau.$$

We have defined the function $\varepsilon(\omega)$ by (77.5):

$$\varepsilon(\omega) = 1 + \int\limits_{0}^{\infty} e^{i\omega\tau}f(\tau)\,\mathrm{d}\tau. \tag{82.1}$$

It is possible to derive some very general relations concerning this function by regarding ω as a complex variable ($\omega = \omega' + i\omega''$). These relations could be formulated immediately, since the dielectric susceptibility $[\varepsilon(\omega)-1]/4\pi$ is one of the generalized susceptibilities already discussed in *SP* 1, §123. We shall nevertheless repeat here some of the arguments and results, both to assist the reader and to emphasize certain differences between dielectrics and metals. From the definition (82.1) and the above-mentioned properties of the function $f(\tau)$, it follows that $\varepsilon(\omega)$ is a one-valued function which nowhere becomes infinite (i.e. has no singularities) in the upper half-plane. For, when $\omega''>0$, the integrand in (82.1) includes the exponentially decreasing factor $e^{-\omega''\tau}$ and, since the function $f(\tau)$ is finite throughout the region of integration, the integral converges. The function $\varepsilon(\omega)$ has no singularity on the real axis ($\omega''=0$), except possibly at the origin (where, for metals, $\varepsilon(\omega)$ has a simple pole).

In the lower half-plane, the definition (82.1) is invalid, since the integral diverges. Hence the function $\varepsilon(\omega)$ can be defined in the lower half-plane only as the analytical continuation of formula (82.1) from the upper half-plane, and in general has singularities.

The function $\varepsilon(\omega)$ has a physical as well as a mathematical significance in the upper half-plane: it gives the relation between **D** and **E** for fields whose amplitude increases as $e^{\omega'' t}$. In the lower half-plane, this physical interpretation is not possible, if only because the presence of a field which is damped as $e^{-|\omega''|t}$ implies an infinite field for $t\to-\infty$.

It is useful to notice that the conclusion that $\varepsilon(\omega)$ is regular in the upper half-plane is,

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† It was to ensure this that the term $\mathbf{E}(t)$ was separated in (77.3), since otherwise the function $f(\tau)$ would have a delta-function singularity at $\tau = 0$.

physically, a consequence of the causality principle. The integration in (77.3) is, on account of this principle, taken only over times previous to t, and the region of integration in formula (82.1) therefore extends from 0 to ∞ rather than from $-\infty$ to ∞.

It is evident also from the definition (82.1) that

$$\varepsilon(-\omega^*) = \varepsilon^*(\omega). \tag{82.2}$$

This generalizes the relation (77.7) for real ω. In particular, for purely imaginary ω we have

$$\varepsilon(i\omega'') = \varepsilon^*(i\omega'') \tag{82.3}$$

i.e. the function $\varepsilon(\omega)$ is real on the positive imaginary axis.†

It should be emphasized that the property (82.2) merely expresses the fact that the operator relation $\mathbf{D} = \hat{\varepsilon}\,\mathbf{E}$ must give real values of $\mathbf{D}$ for real $\mathbf{E}$. If the function $\mathbf{E}(t)$ is given by the real expression

$$\mathbf{E} = \mathbf{E}_0\, e^{-i\omega t} + \mathbf{E}_0{}^*\, e^{i\omega^* t}, \tag{82.4}$$

then, applying the operator $\hat{\varepsilon}$ to each term, we have

$$\mathbf{D} = \varepsilon(\omega)\,\mathbf{E}_0\, e^{-i\omega t} + \varepsilon(-\omega^*)\,\mathbf{E}_0{}^*\, e^{i\omega^* t},$$

and the condition for this to be real is just (82.2).

According to the results of §80, the imaginary part of $\varepsilon(\omega)$ is positive for positive real $\omega = \omega'$, i.e. on the right-hand half of the real axis. Since, by (82.2), $\operatorname{im}\varepsilon(-\omega') = -\operatorname{im}\varepsilon(\omega')$, the imaginary part of $\varepsilon(\omega)$ is negative on the left-hand half of this axis. Thus

$$\operatorname{im}\varepsilon \gtrless 0 \text{ for } \omega = \omega' \gtrless 0. \tag{82.5}$$

At $\omega = 0$, im ε changes sign, passing through zero for dielectrics and through infinity for metals. This is the only point on the real axis for which im $\varepsilon(\omega)$ can vanish.

When ω tends to infinity in any manner in the upper half-plane, $\varepsilon(\omega)$ tends to unity. This has been shown in §78 for the case where ω tends to infinity along the real axis. The general result is seen from formula (82.1): if $\omega \to \infty$ in such a way that $\omega'' \to \infty$, the integral in (82.1) vanishes because of the factor $e^{-\omega''\tau}$ in the integrand, while if ω'' remains finite but $|\omega'| \to \infty$ the integral vanishes because of the oscillating factor $e^{i\omega'\tau}$.

The above properties of the function $\varepsilon(\omega)$ are sufficient to prove the following theorem: the function $\varepsilon(\omega)$ does not take real values at any finite point in the upper half-plane except on the imaginary axis, where it decreases monotonically from $\varepsilon_0 > 1$ (for dielectrics) or from $+\infty$ (for metals) at $\omega = i0$ to 1 at $\omega = i\infty$. Hence, in particular, it follows that the function $\varepsilon(\omega)$ has no zeros in the upper half-plane. We shall not repeat here the proof of these results given in *SP* 1, §123; it need only be remembered that the generalized susceptibility is $\varepsilon(\omega) - 1$, not $\varepsilon(\omega)$.

We shall also not repeat the derivation of the relations between the imaginary and real parts of $\varepsilon(\omega)$, but give only the final formulae, with the notation appropriately modified. We write the function $\varepsilon(\omega)$ of the real variable ω, as in §77, in the form $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$. If $\varepsilon(\omega)$ relates to a dielectric, the relations in question are

$$\varepsilon'(\omega) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon''(x)}{x - \omega}\,\mathrm{d}x, \tag{82.6}$$

$$\varepsilon''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon'(x) - 1}{x - \omega}\,\mathrm{d}x \tag{82.7}$$

(H. A. Kramers and R. de L. Kronig, 1927). It should be emphasized that the only important property of the function $\varepsilon(\omega)$ used in the proof is that it is regular in the upper half-plane. Hence we can say that Kramers and Kronig's formulae, like this property of $\varepsilon(\omega)$, are a direct consequence of the causality principle.

Using the fact that $\varepsilon''(x)$ is an odd function, we can rewrite (82.6) as

$$\varepsilon'(\omega) - 1 = \frac{2}{\pi} P \int_{0}^{\infty} \frac{x\,\varepsilon''(x)}{x^2 - \omega^2}\,\mathrm{d}x. \tag{82.8}$$

If a metal is concerned, the function $\varepsilon(\omega)$ has a pole at the point $\omega = 0$, near which $\varepsilon = 4\pi i\sigma/\omega$ (77.9). This gives an additional term in (82.7) (cf. *SP* 1, (123.18)):

$$\varepsilon''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon'(x)}{x - \omega}\,\mathrm{d}x + \frac{4\pi\sigma}{\omega}, \tag{82.9}$$

but (82.6) and (82.8) remain unchanged. A further remark is also necessary as regards metals. We have said at the end of §77 that there may be ranges of frequency for metals in which the function $\varepsilon(\omega)$ becomes physically meaningless on account of the spatial non-uniformity of the field. In the formulae given here, however, the integration must be taken over all frequencies. In such cases $\varepsilon(\omega)$ must be taken, in the frequency ranges concerned, as the function obtained by solving the formal problem of the behaviour of the body in a fictitious uniform periodic electric field (and not in the necessarily non-uniform field of the electromagnetic wave).

Formula (82.8) is of particular importance: it makes possible a calculation of the function $\varepsilon'(\omega)$ if the function $\varepsilon''(\omega)$ is known even approximately (for example, empirically) for a given body. It is important to note that, for any function $\varepsilon''(\omega)$ satisfying the physically necessary condition $\varepsilon'' > 0$ for $\omega > 0$, formula (82.8) gives a function $\varepsilon'(\omega)$ consistent with all physical requirements, i.e. one which is in principle possible (the sign and magnitude of ε' are subject to no general physical restrictions). This makes it possible to use formula (82.8) even when the function $\varepsilon''(\omega)$ is approximate. Formula (82.7), on the other hand, does not give a physically possible function $\varepsilon''(\omega)$ for an arbitrary choice of the function $\varepsilon'(\omega)$, since the condition that $\varepsilon''(\omega) > 0$ is not necessarily fulfilled.

In dispersion theory the expression for $\varepsilon'(\omega)$ is customarily written in the form

$$\varepsilon'(\omega) - 1 = -\frac{4\pi e^2}{m} P \int_{0}^{\infty} \frac{f(x)}{\omega^2 - x^2}\,\mathrm{d}x, \tag{82.10}$$

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† This is not in general true for the negative imaginary axis. Here the function $\varepsilon(\omega)$ may have branch points, and a cut along the negative axis may be necessary in order to define it as an analytic function in the lower half-plane. Equation (82.2) then signifies only that $\varepsilon(\omega)$ has complex conjugate values on the two sides of the cut.

where e and m are the charge and mass of the electron, and $f(\omega)\,\mathrm{d}\omega$ is called the *oscillator strength* in the frequency range $\mathrm{d}\omega$. According to (82.8), this quantity is related to $\varepsilon''(\omega)$ by

$$f(\omega) = \frac{m}{2\pi^2 e^2}\,\omega\,\varepsilon''(\omega). \tag{82.11}$$

For metals, $f(\omega)$ tends to a finite limit as $\omega \to 0$.

For sufficiently large ω, x^2 can be neglected in comparison with ω^2 in the integrand in (82.8). Then

$$\varepsilon'(\omega) - 1 = -\frac{2}{\pi\omega^2}\int_0^\infty x\,\varepsilon''(x)\,\mathrm{d}x.$$

For the permittivity at high frequencies, on the other hand, formula (78.1) holds, and a comparison shows that we have the *sum rule*

$$\frac{m}{2\pi^2 e^2}\int_0^\infty \omega\,\varepsilon''(\omega)\mathrm{d}\omega = \int_0^\infty f(\omega)\,\mathrm{d}\omega = N, \tag{82.12}$$

where N is the total number of electrons per unit volume.

If $\varepsilon''(\omega)$ is regular at $\omega = 0$, we can take the limit $\omega \to 0$ in formula (82.8), obtaining

$$\varepsilon'(0) - 1 = \frac{2}{\pi}\int_0^\infty \frac{\varepsilon''(x)}{x}\,\mathrm{d}x. \tag{82.13}$$

If the point $\omega = 0$ is a singularity of $\varepsilon''(\omega)$ (as in metals), the limit of the integral (82.8) as $\omega \to 0$ is not what is obtained by simply deleting the term in ω. To calculate the limit, we must first replace $\varepsilon''(x)$ in the integrand by $\varepsilon''(x) - 4\pi\sigma/x$; the value of the integral is unchanged, because

$$P\int_0^\infty \frac{\mathrm{d}x}{x^2 - \omega^2} \equiv 0.$$

For a dielectric, formula (82.13) can be rewritten as

$$\varepsilon_0 - 1 = \frac{4\pi e^2 N}{m}\,\overline{\omega^{-2}}, \tag{82.14}$$

where the bar denotes averaging with respect to the oscillator strength:

$$\overline{\omega^{-2}} = \frac{1}{N}\int_0^\infty \frac{f(\omega)}{\omega^2}\,\mathrm{d}\omega.$$

The expression (82.14) may be useful in estimating ε_0.

The following formula† relates the values of $\varepsilon(\omega)$ on the upper half of the imaginary axis to those of $\varepsilon''(\omega)$ on the real axis:

$$\varepsilon(i\omega) - 1 = \frac{2}{\pi}\int_0^\infty \frac{x\varepsilon''(x)}{x^2 + \omega^2}\,\mathrm{d}x. \tag{82.15}$$

Integrating this relation over all ω, we obtain

$$\int_0^\infty [\varepsilon(i\omega) - 1]\,\mathrm{d}\omega = \int_0^\infty \varepsilon''(\omega)\,\mathrm{d}\omega. \tag{82.16}$$

All the above results are applicable, apart from slight changes, to the magnetic permeability $\mu(\omega)$. The differences are due principally to the fact that the function $\mu(\omega)$ ceases to be physically meaningful at relatively low frequencies. Hence, for example, Kramers and Kronig's formulae must be applied to $\mu(\omega)$ as follows. We consider not an infinite but a finite range of ω (from 0 to ω_1), which extends only to frequencies where μ is still meaningful but no longer variable, so that its imaginary part may be taken as zero; let the real quantity $\mu(\omega_1)$ be denoted by μ_1.‡ Then formula (82.8) must be written as

$$\mu'(\omega) - \mu_1 = \frac{2}{\pi}P\int_0^{\omega_1} \frac{x\mu''(x)}{x^2 - \omega^2}\,\mathrm{d}x. \tag{82.17}$$

Unlike ε_0, the value μ_0 of $\mu(0)$ may be either less than or greater than unity. The variation of $\mu(\omega)$ along the imaginary axis is again a monotonic decrease, from μ_0 to $\mu_1 < \mu_0$.

Lastly, it may be noted that the analytical properties of $\varepsilon(\omega)$ derived in this section are also possessed by $\eta(\omega) \equiv 1/\varepsilon(\omega)$. For example, $\eta(\omega)$ is analytic in the upper half-plane, because $\varepsilon(\omega)$ is analytic and has no zeros in that half-plane. The same Kramers–Kronig relations (82.6) and (82.7) apply to $\eta(\omega)$ as to $\varepsilon(\omega)$.

§83. A plane monochromatic wave

Maxwell's equations (77.2) for a monochromatic field are

$$i\omega\mu(\omega)\,\mathbf{H} = c\,\mathbf{curl}\,\mathbf{E}, \qquad i\omega\varepsilon(\omega)\,\mathbf{E} = -c\,\mathbf{curl}\,\mathbf{H}. \tag{83.1}$$

These equations as they stand are complete, since equations (77.1) follow from (83.1) and so do not require separate consideration. Assuming the medium homogeneous, and eliminating $\mathbf{H}$ from equations (83.1), we obtain the second-order equation

$$\triangle\mathbf{E} + \varepsilon\mu(\omega^2/c^2)\,\mathbf{E} = 0; \tag{83.2}$$

elimination of $\mathbf{E}$ gives a similar equation for $\mathbf{H}$.

Let us consider a plane electromagnetic wave propagated in an infinite homogeneous medium. In a plane wave in a vacuum, the spatial dependence of the field is given by a factor $e^{i\mathbf{k}\cdot\mathbf{r}}$, with a real wave vector $\mathbf{k}$. In considering wave propagation in matter, however, it is in

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† Derived in *SP* 1, §123.

‡ In fact, ω_1 must be such that $\omega_1\tau \gg 1$, where τ is the shortest relaxation time for ferromagnetic and paramagnetic processes in a magnetic material.