Recursive Sparse Recovery and Applications in Dynamic Imaging

Namrata Vaswani

Department of Electrical and Computer Engineering
Iowa State University
Web: http://www.ece.iastate.edu/~namrata

(portions joint work with Wei Lu and Chenlu Qiu)

Talk Outline

Background on Sparse Recovery

Recursive Reconstruction of Sparse Signal Sequences (RecSparsRec)

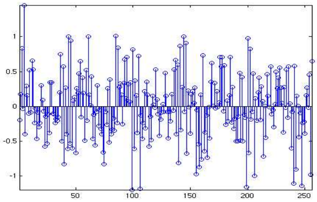
- The problem, motivation and applications, key ideas
- Modified-CS: noise-free case and exact recovery result
- Modified-CS: noisy case and time-invariant error bounds (stability)

Rec Robust PCA $\Leftrightarrow$ RecSparsRec in Large but Correlated Noise

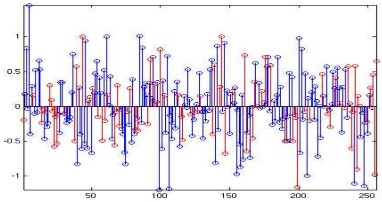
- Video Surveillance – Background subtraction application

Sparse Recovery: the question

- Can I recover a 256-length signal from only 80 samples?



(a) the unknown signal



(b) its 80 time samples (red)

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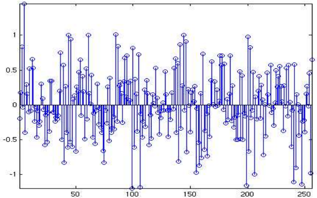
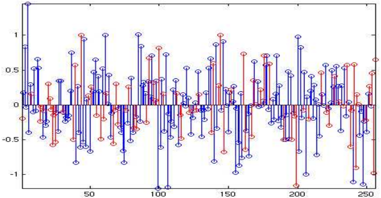


(c) the unknown signal



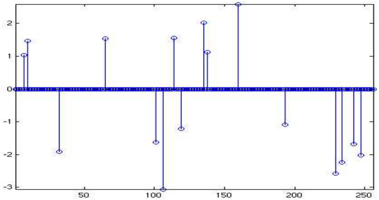
(d) its 80 time samples (red)

- Under certain situations: YES!
  - if it is bandlimited – use Nyquist
  - or if it is a weighted sum of only a few sinusoids – use sparsity

Example taken from L1-Magic webpage of Candes,Romberg,Tao

Sparse Recovery: the answer

- This signal satisfies the latter – it is Fourier sparse



(e) DFT of original signal

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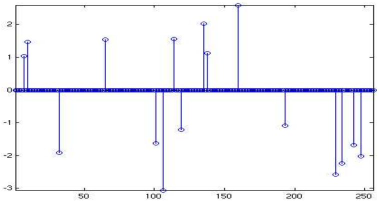
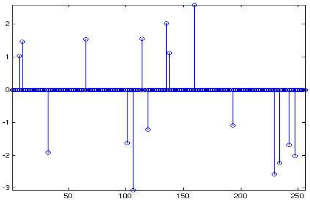


(g) DFT of original signal



(h) recovered DFT: exact!

- We used its Fourier sparsity and ℓ_1 minimization to recover its DFT exactly!
  - one-to-one mapping between a signal and its DFT

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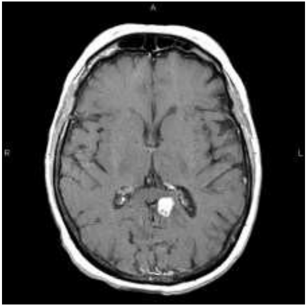
Sparse (or Compressible) Signals



a brain image:
wavelet compressible

- Sparse vector: only a few nonzero elements
- Compressible vector: approx sparse vector (most energy lies in only a few elements)
- Sparse (compressible) signal: either the signal or a linear transform of it is sparse (compress.)

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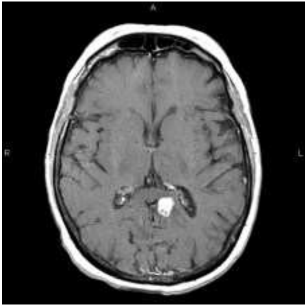

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- Sparse vector: only a few nonzero elements
- Compressible vector: approx sparse vector (most energy lies in only a few elements)
- Sparse (compressible) signal: either the signal or a linear transform of it is sparse (compress.)
- Support: set of indices of the nonzero (non-negligible) elements of the vector,
  - e.g. 99%-energy support: set containing indices of the largest elements that make up 99% of the total energy

Sparse recovery [Mallat et al'93],[Chen,Donoho'95],[Candes,Romberg,Tao'05],[Donoho'05]

- Reconstruct a sparse signal x, with support N, from $y := Ax$,
  - when A has more columns than rows (underdetermined sys)

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  - and any $S = 2|N|$ columns of A are linearly independent
  - but combinatorial search – $O(m^{|N|})$ complexity
- Practical approaches (polynomial complexity in m)
  - convex relaxation approaches [Chen,Donoho'95], ..., [Candes,Tao'06],...: ℓ_1 minimization
    - replace ℓ_0 norm by ℓ_1 norm – convex problem
  - greedy methods [Mallat,Zhang'93], [Pati et al'93], [Dai,Milenkovic'09], [Needell,Tropp'09]

Sparse recovery and Compressive Sensing

- Compressed Sensing (CS) literature [Candes,Romberg,Tao'05], [Donoho'05]
  - provides exact reconstruction conditions and error bounds for the practical approaches – much stronger results than earlier ones based on mutual coherence

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  - random Gaussian matrices: if $n = O(S \log m)$, $\delta_S(A) < b < 1$ holds w.h.p. (prob $\to 0$ as $m \to \infty$)
    - similar results for Rademacher and partial Fourier matrices

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    - similar results for Rademacher and partial Fourier matrices
- this talk: sparse recovery ⇔ CS ⇔ ℓ_1 minimization

Recursive Sparse Recovery [Vaswani,ICIP'08][1]

- Recursive approaches for causally reconstructing a time sequence of sparse signals
- from a greatly reduced number of measurements at each time.
- "recursive": use only current measurement vector and the previous reconstructed signal to reconstruct the current signal

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- Sparsity patterns can change with time, but the changes are gradual

- Existing work: mostly batch CS approaches – expensive

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Potential Applications

- Dynamic medical imaging for real-time apps, e.g.
  - MRI-guided interventional radiology, MRI-guided surgery,
  - real-time functional MRI
- Video surveillance or denoising or fMRI based active region detection
  - track one or more moving objects/regions when the background scene itself is changing – foreground is sparse

Why "reduced" measurements?

- Projection Imaging, e.g. MRI or CT or single-pixel camera
  - Fourier transform or Radon transform or random-projections of the region-of-interest acquired sequentially
    - Fewer measurements $\Rightarrow$ faster scanning – needed for real-time imaging for fast changing phenomena

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  - The full image is acquired in one go, but it can have more than one layers, e.g. foreground and background
    - both change, how can I estimate both?

Why "causal" and "recursive"

- Why causal?
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- Why causal?
  - reconstruct as soon as get data for current frame – desirable for real-time (or at most allow small buffering)
- Why recursive?
  - one way to ensure computational and storage complexity is comparable to CS for one image (simple CS)
  - much faster and lower on memory than both causal and offline implementations of batch CS
  - recursive CS at time t v/s causal batch CS at time t
    - time: $O(1)$ v/s $O(t^3)$
    - memory: $O(1)$ v/s $O(t)$
    - $O(1)$: time taken or memory reqd for CS for one image

Problem Formulation [Vaswani,ICIP'08] (KF-CS)

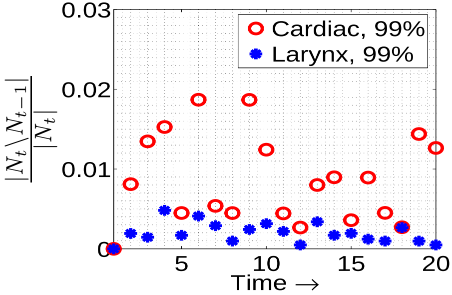
- Measure

$$y_t := Ax_t + w_t$$

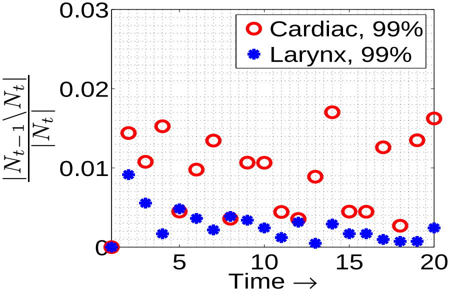
  - $A = H\Phi$ (given): $n \times m$, $n < m$
    - H: measurement matrix, Φ: sparsity basis matrix
    - e.g. in MRI: H = partial Fourier, Φ = inverse wavelet
  - y_t: measurements (given)
  - x_t: sparsity basis vector
  - N_t: support set of x_t (set of indices of nonzero elements of x_t)
- Goal: recursively reconstruct x_t from $y_0, y_1, \ldots y_t$,
  - i.e. use only $\hat{x}_{t-1}$ and y_t for reconstructing x_t
- Assumptions:
  - support set of x_t, N_t, changes slowly over time
  - also use slow signal value change where valid

Slow sparsity pattern change in medical image sequences [Qiu, Lu, Vaswani,ICASSP'09]

image sequences: `http://www.ece.iastate.edu/~luwei/modcs`



(a) slow support changes (adds)



(b) slow support changes (removals)

- N_t: 99%-energy support set of x_t, where
- x_t: wavelet transform of cardiac or larynx image at time t
- Notice: all support changes are less than 2% of support size

Slow signal value change in medical seq's (common tracking assumption) [Lu,Vaswani,ArXiv]

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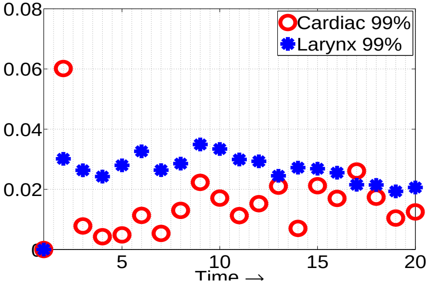

- Plot of $\frac{\|(x_t - x_{t-1})\|_2}{\|(x_t)\|_2}$ against time, t
- x_t: wavelet transform of cardiac or larynx image at time t
- Notice: almost all changes are less than 4%

Questions we answer

1. How to solve RecSparsRec *while not increasing reconstruction algorithm speed or memory requirement w.r.t. simple CS?*

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2. When does it achieve exact recovery?
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   - (critical question for a recursive approach)
   - are the conditions required weaker than those for simple CS?
4. How much better do our algorithms do compared to existing work for real experimental data?

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4. How much better do our algorithms do compared to existing work for real experimental data?
5. RecSparsRec in large but correlated noise

Related Work

- Simple CS (CS done at each time separately)
- CS-diff (CS on difference meas's) [Cevher et al,ECCV'08]: works only if
  - first frame reconstructed very accurately, and
  - difference signal sparser or signal values change very slowly
- Kalman Filtered CS (KF-CS) & LS-CS [Vaswani,ICIP'08,T-SP'10]
  - defined RecSparsRec problem; proposed an efficient solution
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- Work with different goals than ours
  - homotopy methods: speed up optimization but not reduce n [Asif,Romberg'08,09]
  - recover *one* signal recursively as more meas's come in [Sanghavi et al,'08], [Angelosante et al'09], [Asif,Romberg'09], [Ghaoui et al'09]
  - batch methods: much slower, need a lot more memory [Wakin et al'06(video)],[Gamper et al'08 (MRI)], [Angelosante et al'09 (dyn Lasso)]

Least Squares CS and Kalman Filtered CS [Vaswani,ICIP'08][2], [Vaswani,IEEE Trans. SP,Aug'10][3]

At each time t,

- Let $T = \hat{N}_{t-1}$ be previous support estimate
- Compute LS (or KF) estimate assuming T is current support
  - LS estimate: $(\mu)_T = {A_T}^\dagger y_t$, $(\mu)_{T^c} = 0$
- CS on Residual
  - CS-residual: $\hat{\beta} = \arg\min \|\beta\|_1$ s.t. $\|(y_t - A\mu) - A\beta\|_2 \leq \epsilon$
  - Compute $\hat{x}_t = \hat{\beta} + \mu$
- Estimate support $\hat{N}_t = \{i : |(\hat{x}_t)_i| > \alpha\}$
- Final LS (or KF) using $\hat{N}_t$

[2] N. Vaswani, Kalman Filtered Compressed Sensing, ICIP, 2008

[3] N. Vaswani, "LS-CS-residual (LS-CS): Compressive Sensing on the Least Squares Residual", IEEE Trans. Sig. Proc., Aug. 2010.

Least Squares CS and Kalman Filtered CS [Vaswani,ICIP'08][4], [Vaswani,IEEE Trans. SP,Aug'10][5]

- Have same complexity and memory requirement as simple-CS
  - but accurate recovery with much fewer noisy measurements
- Proved LS-CS error "stability" (time-invariant error bound) under mild assumptions [Vaswani,TSP,Aug'10]:
- BUT: could not achieve exact recovery with fewer measurements

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  1. support changes every-so-often and delay b/w support change times is large enough;
  2. support change size, S_a, and support size, S_0, small enough (for a given A);
  3. newly added elements' either added at a large-enough value or their value increases at least at a certain rate, r
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CS with partially known support [Vaswani,Lu, ISIT'09, T-SP, Sept'10][6]

- Reconstruct a sparse signal, x, with support, N, from $y := Ax$
  - given partial and possibly erroneous support knowledge: T

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- Reconstruct a sparse signal, x, with support, N, from $y := Ax$
  - given partial and possibly erroneous support knowledge: T
- Rewrite the true support, N, as

$$N = T \cup \Delta \setminus \Delta_e$$

  - T: erroneous support estimate (use $T = \hat{N}_{t-1}$ at time t)
  - $\Delta := N \setminus T$: errors (misses) in T – unknown
  - $\Delta_e := T \setminus N$: errors (extras) in T – unknown

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Modified-CS idea

- If Δ_e empty: above ⇔ find signal that is sparsest outside T

$$\min_{\beta} \|(\beta)_{T^c}\|_0 \; s.t. \; y = A\beta$$

  - the unknowns are Δ, $(\beta)_\Delta$ **and** $(\beta)_T$

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- Exact recovery: if every set of $(|T| + 2|\Delta|) = (|N| + |\Delta_e| + |\Delta|)$ columns of A are linearly independent

- Compare: ℓ_0-CS needs this to hold for every set of $2|N|$ columns

- Under slow support change, $|\Delta| \ll |N|$ and $|\Delta_e| \ll |N|$

Modified-CS [Vaswani,Lu, ISIT'09, T-SP,Sept'10][7]

- Modified-CS

$$\min_{\beta} \|(\beta)_{T^c}\|_1 \ s.t. \ y = A\beta$$

  - we obtained exact recon cond's for Modified-CS; argued they are weaker than CS

- Other related parallel/later work:
  - [vonBorries et al, TSP'09]: no exact recon conditions or expts
  - [Khajenejad et al, ISIT'09]: probab. prior on support, studies exact recon
  - Later: [Jacques, Elsev.Sig.Proc'10]: error bounds for noisy mod-CS

[7] N. Vaswani and W. Lu, "Modified-CS: Modifying Compressive Sensing for Problems with Partially Known Support", IEEE Trans. Sig. Proc., Sept. 2010. (shorter version in ISIT'09)

Exact reconstruction result [Vaswani,Lu, ISIT'09, T-SP,Sept.'10]

$$\min_{\beta} \|\beta_{T^c}\|_1 \ s.t. \ y = A\beta \quad \text{(modified-CS)}$$

Theorem (simplified condition)

x is the unique minimizer of (modified-CS) if

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|N|+|\Delta_e|-|\Delta|} + \delta^2_{|N|+|\Delta_e|} + 2\delta^2_{|N|+|\Delta_e|+|\Delta|} < 1$$

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- *δ_S: RIP constant – smallest real number s.t. singular values of any S-column sub-matrix of A lie in $[\sqrt{1-\delta_S}, \sqrt{1+\delta_S}]$* [Candes,Tao,T-IT'05]
  - non-increasing function of n (# of measurements)

recall: $\Delta := N \setminus T$: misses in T, $\Delta_e := T \setminus N$: extras in T

Proof Outline [Vaswani,Lu, ISIT'09, T-SP,Sept.'10][8]

Use overall approach of [Candes,Tao,Decoding by LP,T-IT,Dec'05]

- Obtain conditions on the Lagrange multiplier, w, to ensure that x is a *unique* minimizer
- Find sufficient conditions under which such a w can be found
  - key lemma: create a w that satisfies most reqd conditions
  - apply this lemma recursively to get a final w that satisfies *all* reqd conditions.

[8] N. Vaswani and W. Lu, "Modified-CS: Modifying Compressive Sensing for Problems with Partially Known Support", IEEE Trans. Sig. Proc., Sept. 2010. (shorter version in ISIT'09)

Comparison with best sufficient cond's for CS

- CS gives exact reconstruction if [Candes'08, Candes-Tao'06]

$$\delta_{2|N|} < \sqrt{2} - 1 \quad \text{or} \quad \delta_{2|N|} + \delta_{3|N|} < 1$$

- Modified-CS gives exact reconstruction if

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|N|+|\Delta_e|-|\Delta|} + \delta^2_{|N|+|\Delta_e|} + 2\delta^2_{|N|+|\Delta_e|+|\Delta|} < 1$$

- If $|\Delta| = |\Delta_e| = 0.02|N|$ (typical in medical sequences),
  - **sufficient condition for CS** to achieve exact recovery:

$$\delta_{0.04|N|} < 0.004$$

  - **sufficient condition for Mod-CS** to achieve exact recovery:

$$\delta_{0.04|N|} < 0.008$$

- **Mod-CS sufficient condition is weaker (needs fewer meas's)**

recall: $\Delta := N \setminus T, \quad \Delta_e := T \setminus N$

Simulations: exact reconstruction probability

Simulation setup:

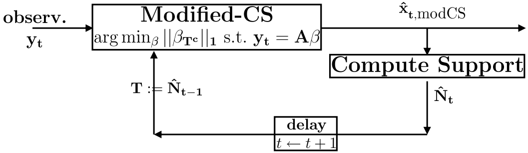
- signal length, $m = 256$, supp size, $|N| = 0.1m$
- supp error sizes, $|\Delta| = |\Delta_e| = 0.08|N|$
- used random-Gaussian A, varied n
- we say "works" (gives exact recon) if $\|x - \hat{x}\|_2 < 10^{-5}\|x\|_2$

Conclusions:

- With 19% measurements:
  - mod-CS "works" w.p. 99.8%, CS "works" w.p. 0
- With 25% measurements:
  - mod-CS "works" w.p. 100%, CS "works" w.p. 0.2%
- CS needs 40% measurements to "work" w.p. 98%

recall: Δ: errors (misses) in T, Δ_e: errors (extras) in T

Modified-CS for time sequences



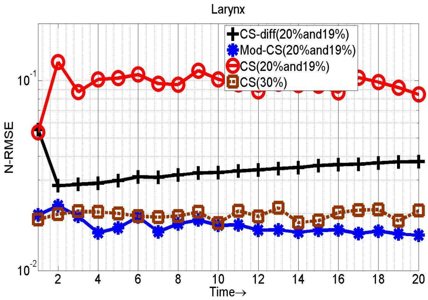
Support Estimation: use thresholding

$$\hat{N}_t := \{i : |(\hat{x}_{t,\text{modCS}})_i| > \alpha\}$$

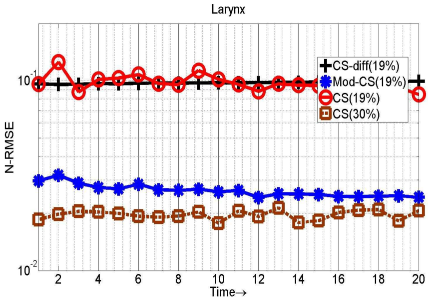
Initial time ($t = 0$):

- use T_0 from prior knowledge, e.g. wavelet approximation coeff's
- may need more measurements at $t = 0$

Simulated MRI of an actual larynx (vocal tract) sequence: noise-free case



(c) $n_0 = 20\%$, $n = 19\%$



(d) $n_0 = 19\%$, $n = 19\%$

- A real image sequence: only compressible (approx sparse)
- With only $n = 19\%$ MRI meas's, Mod-CS error is small and stable at 2-3%, CS-diff error is unstable or large, simple-CS error is large

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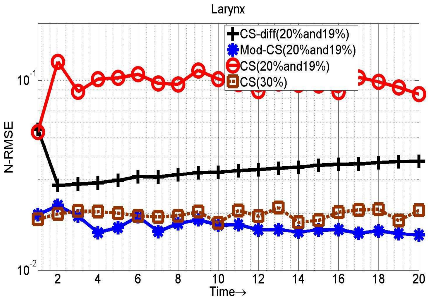
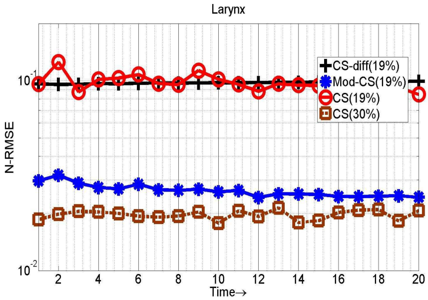
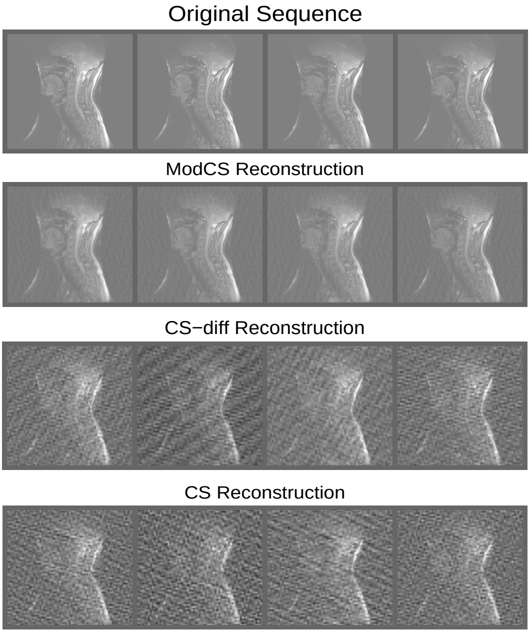


(e) $n_0 = 20\%$, $n = 19\%$



(f) $n_0 = 19\%$, $n = 19\%$

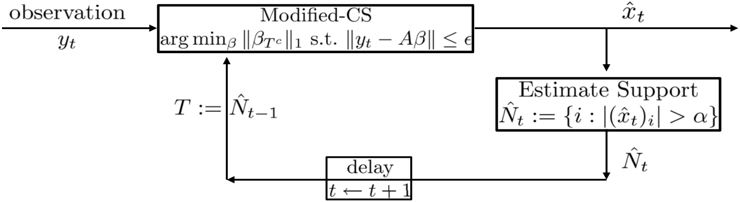
- A real image sequence: only compressible (approx sparse)
- With only $n = 19\%$ MRI meas's, Mod-CS error is small and stable at 2-3%, CS-diff error is unstable or large, simple-CS error is large
  - simple CS needs $n = 30\%$ to achieve small error



A larynx sequence (not sparsified)

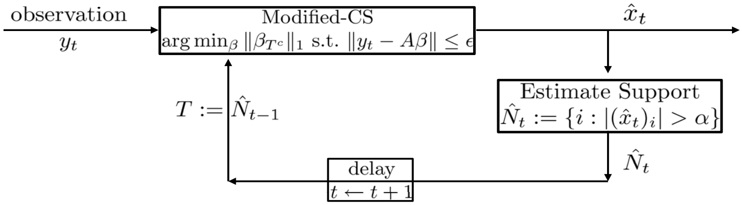
- 99%-support size $\sim$ 7%,
- supp change $\sim$ 2%
- using only 19% MRI measurements at all times
- simple CS needs $n = 30\%$ for same error
- http://www.ece.iastate.

Modified-CS for noisy measurements



- Difficulty:
  - along T^c: solution is biased towards zero
  - along T: no cost and only data constraint – solution can be biased away from zero

Modified-CS for noisy measurements



- Difficulty:
  - along T^c: solution is biased towards zero
  - along T: no cost and only data constraint – solution can be biased away from zero
  - the misses' set $\Delta_t \subset T^c$, while the extras' set, $\Delta_{e,t} \subset T$
    - need α small to add Δ_t, need α large to delete $\Delta_{e,t}$

(recall: $\Delta_t := N_t \setminus T = N_t \setminus \hat{N}_{t-1}$, $\Delta_{e,t} := T \setminus N_t = \hat{N}_{t-1} \setminus N_t$)

Possible Solutions

- Solution 1: improved support estimation (Add-LS-Del)
- Solution 2: use "slow signal value change" to constrain $(\beta)_T$

  [Lu, Vaswani, Trans.SP, Jan'12], [Raisali, Vaswani, CISS'11]

$$\arg\min_{\beta} \|(\beta)_{T^c}\|_1 \text{ s.t. } \|y_t - A\beta\|_2 \leq \epsilon, \quad \|\beta_T - \mu_T\|_2 \leq \gamma$$

  - with $\mu := \hat{x}_{t-1}$, $T := \hat{N}_{t-1}$ (Reg-Mod-CS – ongoing work)
  - useful if signal value change is "slow enough"

Modified-CS with Add-LS-Del (improved support support estimation)[9]

- Modified-CS: set $T = \hat{N}_{t-1}$ and compute $\hat{x}_t$ as the solution of

$$\min_{\beta} \|\beta_{T^c}\|_1 \text{ s.t. } \|y_t - A\beta\|_2 \leq \epsilon$$

- Support Add using a small threshold
  - use α_{add} just large enough s.t. well-conditioned $(A)_{T_{\text{add}}}$
- Compute LS estimate on T_{add}, call it $\hat{x}_{t,\text{add}}$
  - reduces bias and error if $T_{\text{add}} \approx N_t$ [Candes,Tao'06]
- Support Delete by thresholding on $\hat{x}_{t,\text{add}}$ w/ a larger threshold
  - $\hat{x}_{t,\text{add}}$ more accurate $\Rightarrow \alpha_{\text{del}}$ can be larger

[9] introduced in [Vaswani,ICIP'08,T-SP'10] & also in [Dai,Milenkovic'09], [Needell,Tropp'09] for static case

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  - use α_{add} just large enough s.t. well-conditioned $(A)_{T_{\text{add}}}$
  - $T_{\text{add}} = T \cup \{i : |(\hat{x}_t)_i| > \alpha_{\text{add}}\}$
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  - $\hat{N}_t = T_{\text{add}} \setminus \{i : |(\hat{x}_{t,\text{add}})_i| \leq \alpha_{\text{del}}\}$

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Stability over time [Vaswani,T-SP, Aug'10] [10], [Vaswani,Allerton'10][11]

- Easy to bound the reconstruction error at a given time, t
  - the result depends on the support errors $|\Delta_t|$, $|\Delta_{e,t}|$
    (recall: $\Delta_t := N_t \setminus \hat{N}_{t-1}$, $\Delta_{e,t} := \hat{N}_{t-1} \setminus N_t$)
- **Key question for a recursive algorithm: when can we get a time-invariant and small bound on the error?**

[10] N. Vaswani, "LS-CS-residual (LS-CS): Compressive Sensing on the Least Squares Residual", IEEE Trans. Sig. Proc., Aug. 2010

[11] N. Vaswani, Stability (over time) of Modified-CS for Recursive Causal Sparse Reconstruction, Allerton 2010

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- **Key question for a recursive algorithm: when can we get a time-invariant and small bound on the error?**
- Solution approach: first obtain conditions under which time-invariant bounds on $|\Delta_t|$, $|\Delta_{e,t}|$ hold
  - direct corollary: time-invariant bound on the recon error

[10] N. Vaswani, "LS-CS-residual (LS-CS): Compressive Sensing on the Least Squares Residual", IEEE Trans. Sig. Proc., Aug. 2010

[11] N. Vaswani, Stability (over time) of Modified-CS for Recursive Causal Sparse Reconstruction, Allerton 2010

Signal change model, measurement model and our result

Signal change model:

- S_a additions and removals from the support **at each time**
- support size constant at S_0
- new elements added at a small value, r; magnitude increases at rate r per unit time, until it reaches a maximum magnitude dr
  - similarly for decrease before removal

Measurement model:

$$y_t = Ax_t + w_t, \quad \|w_t\|_2 \leq \epsilon$$

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Measurement model:

$$y_t = Ax_t + w_t, \quad \|w_t\|_2 \leq \epsilon$$

Our result: "stability" holds if

1. S_a and S_0 are small enough (for a given A),
   - ensures the error bound holds at all times
2. r is large enough
   - ensures newly added elements detected within a finite delay

Theorem (Modified-CS stability [Vaswani, Allerton'10])

If

1. *support estimation threshold,* $\alpha = 8.79\epsilon$
2. *support size, support change size* S_0, S_a *satisfy*
   - $\delta_{S_0+3S_a} < (\sqrt{2}-1)/2$ *(for a given A)*
3. *new element initial value and increase rate,* $r \geq 8.79\epsilon$,
4. *at initial time,* $t = 0$, n_0 *large enough s.t.* $\delta_{2S_0} < (\sqrt{2}-1)/2$

then, at all times, t,

- *final support errors,* $|\tilde{\Delta}_t| \leq 2S_a$ *and* $|\tilde{\Delta}_{e,t}| = 0$
- *initial support errors,* $|\Delta_t| \leq 2S_a$ *and* $|\Delta_{e,t}| \leq S_a$
- *and so recon error satisfies* $\|x_t - \hat{x}_{t,modcs}\|_2 \leq 8.79\epsilon$

recall: $\Delta_t := N_t \setminus \hat{N}_{t-1}$, $\Delta_{e,t} := \hat{N}_{t-1} \setminus N_t$, $\tilde{\Delta}_t := N_t \setminus \hat{N}_t$, $\tilde{\Delta}_{e,t} := \hat{N}_t \setminus N_t$

Proof Outline: use induction

Here, "bounded" ⇔ bounded by a time-invariant value

- Induction assumption:
  - final support errors (misses and extras) at $t-1$ bounded
- $+$ signal model $\Rightarrow$ predicted support errors at t bounded
- $+$ n large enough (or S_0 small enough) $\Rightarrow$ Mod-CS error bounded
- $+$ α large enough $\Rightarrow$ no extras
- $+$ r large enough $\Rightarrow$ all elements with mag $> 2r$ detected (bounded misses)
- $\Rightarrow$ final support errors (misses and extras) at t bounded

Theorem (Modified-CS-with-Add-LS-Del stability [Vaswani,Allerton'10])

Let $e := (x - \hat{x}_{add})_{T_{add}}$. *If*

$$\|e\|_\infty \leq (1/\sqrt{S_a})\ \|e\|_2,$$

1. *(addition and deletion thresholds)*
    - α_{add} *is large enough s.t. at most* S_a *false adds per unit time,*
    - $\alpha_{del} = \sqrt{\frac{2}{S_a}}\epsilon + 2\theta_{S_0+2S_a,S_a} r$,
2. *(support size, support change size)* S_0, S_a *satisfy*
    - $\delta_{S_0+3S_a} < (\sqrt{2}-1)/2$ *and* $\theta_{S_0+2S_a,S_a} < \frac{1}{4}$ *(for a given A),*
3. *(new coeff. increase rate)* $r \geq \max(G_1, G_2)$, *where*

$$G_1 := \frac{\alpha_{add} + 8.79\epsilon}{2},\ \ G_2 := \frac{\sqrt{2}\epsilon}{\sqrt{S_a}(1 - 2\theta_{S_0+2S_a,S_a})}$$

then, all the same conclusions hold.

Proof Outline – 1 [Vaswani,Allerton'10] [12]

- Goal: ensure that within a finite delay d_0, all newly added elements get detected and all zeroed (removed) elements get deleted
  - simpler case: fix $d_0 = 2$
- Starting point
  - conditions and bound for Modified-CS error at t
    - simple modification of Candes' approach for CS
  - conditions and bound for LS step error at t – also easy
- Key lemmas: sufficient conditions to ensure that, at a given t,
  1. an undetected large-enough element *gets added*
  2. an existing large-enough element *does not get falsely deleted*
  3. a falsely detected zero element *does get deleted*

[12] N. Vaswani, "Stability (over time) of Modified-CS for Recursive Causal Sparse Reconstruction", Allerton 2010, submitted to IEEE Trans. Info. Th.

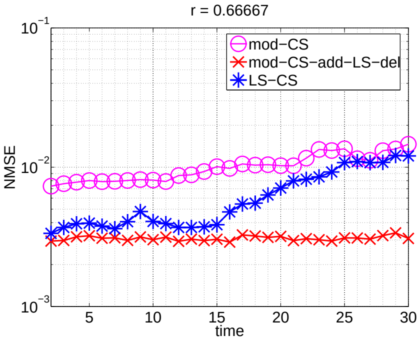
Proof Outline – 2: Induction step idea

- Assume $|\tilde{\Delta}_{t-1}| \le 2S_a$, $|\tilde{\Delta}_{e,t-1}| = 0$ (induction assumption)
- Above + signal model $\Rightarrow |\Delta_t|, |\Delta_{e,t}|$ bounded
  (recall: $\Delta_t := N_t \setminus \hat{N}_{t-1}$, $\Delta_{e,t} := \hat{N}_{t-1} \setminus N_t$, $\tilde{\Delta}_t := N_t \setminus \hat{N}_t$, $\tilde{\Delta}_{e,t} := \hat{N}_t \setminus N_t$)
- Above + S_0, S_a small enough $\Rightarrow$ Mod-CS error bounded at t
- Add step
  - above + signal model + r large enough $\Rightarrow$ elements with mag. $\ge 2r$ definitely get detected,
  - need α_{add} large enough s.t. few and bounded false adds
  - above two ensure support errors bounded after the add step
- LS and Delete step
  - above + S_0, S_a small enough $\Rightarrow$ LS step error bounded
  - above + signal model + r large enough $\Rightarrow$ only elements $< 2r$ may get falsely deleted ($|\tilde{\Delta}_t| \le 2S_a$)
  - above + α_{del} large enough $\Rightarrow$ all extras deleted ($|\tilde{\Delta}_{e,t}| = 0$)

Discussion

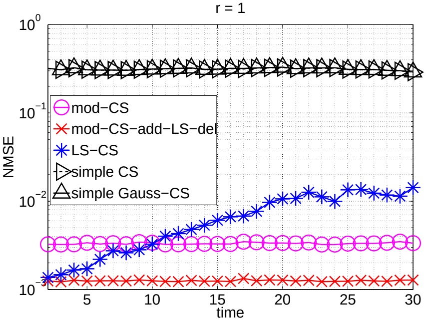
- Slow supp change $\Rightarrow S_a \ll S_0 \Rightarrow$ supp error bound, $2S_a$, is small compared to the supp size, S_0 (meaningful result)
- Modified-CS stability result – only needs $\delta_{S_0+2S_a} < (\sqrt{2}-1)/2$
  - needs weaker conditions on A than simple CS
    - Simple CS needs $\delta_{2S_0} < (\sqrt{2}-1)/2$ (for same error bound)
- Modified-CS-Add-LS-del stability result – needs
  - weaker conditions on A than CS (for same error bound)
  - weaker conditions on r than modified-CS
    - it needs $r \geq (\alpha_{\text{add}} + 8.79\epsilon)/2$ but modified-CS needs $r \geq 8.79\epsilon$
  - weaker conditions on both A and r compared to LS-CS result

Normalized mean squared error (NMSE) v/s time

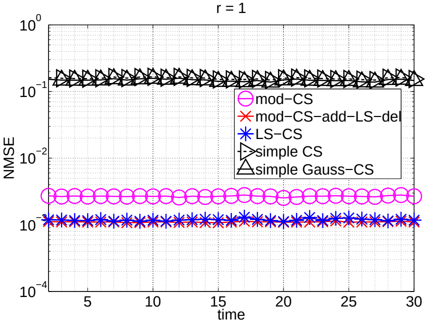


- *A*: random-Gaussian, $n \times m$, $n = 29.5\%$; noise: unif(-0.13,0.13);
- new elem's added at mag. $r = 0.67$; incr. at rate r, until reach $M = 2$
- $m = 200$, support size, $S_0 = 0.1m$, support change size, $S_a = 0.1S_0$
- ModCS-Add-LS-del stable, others are not

Normalized mean squared error (NMSE) v/s time



$r = 1, \ n = 29.5\%$



$r = 1, \ n = 32.5\%$

- $m = 200, \ S_0 = 0.1m, \ S_a = 0.1S_0, \ d = 3.$
- ModCS needs larger r, LS-CS needs larger r and larger n
- Simple-CS has large error even with $n = 32.5\%$

Video: Background subtraction

image sequence, $M_t = L_t + S_t$

background sequence, L_t: low rank, changing subspace

foreground sequence, F_t: sparse w/ correlated support changes

$N_t = \text{support}(F_t)$, $(S_t)_{N_t} = (F_t - L_t)_{N_t}$, $(S_t)_{N_t^c} = 0$

The Problem

- Measurement: $M_t := L_t + S_t$
  - S_t: sparse vector, with correlated support change over time
  - L_t: low dimensional vector (matrix $L := [L_{t-\tau}, \dots L_t]$ is low rank)
    - subspace in which L_t lies changes gradually over time
    - matrix P_t: its columns span the subspace in which L_t lies
- Given P_0, recursively recover S_t, L_t and the matrix P_t

The Problem

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- Given P_0, recursively recover S_t, L_t and the matrix P_t
- Recursive Robust PCA:
  - S_t: corruption (outlier), L_t: signal, P_t: its PC matrix
- RecSparsRec in Large but Low-dimensional Noise:
  - S_t: sparse signal, L_t: corruption (low dimensional noise)
  - our solutions apply even if $M_t = \Psi S_t + L_t$, Ψ: fat matrix

Motivation and Applications

- Existing work [Candes,Wright,Ma,Li], [DeLaTorre,Black], ...
  - simple thresholding (recovers only S_t)
  - detect outliers and either downweight them, e.g. RSL, or fill in using heuristics
  - PCP - recover L, S from $M = L + S$ (S: sparse but not low rank, L: low rank but not sparse)
- Need an approach that can
  - handle correlated S_t's (PCP cannot)
  - can handle fairly large support-sized S_t's (RSL, PCP cannot)
  - recover small magnitude S_t's (RSL, thresh cannot)
  - work in real-time
- Applications: recover sparse signals (most natural signals) in large but spatially correlated noise (most natural noise sources)
  - video/audio denoising, fMRI based active region detection, sensor nets, ...

ReProCS: Recursive Projected CS [Qiu,Vaswani,Allerton'10, Allerton'11][13]

$$M_t = S_t + L_t, \quad L_t = P_t a_t$$

- Update $\hat{P}_t$ every-so-often: recursive PCA
- Project M_t into space perp to $\hat{P}_t$: get y_t
- Recover S_t from y_t: noisy sparse recovery
- Compute $\hat{L}_t := M_t - \hat{S}_t$

[13] C. Qiu and N. Vaswani, Real-time Robust Principal Components' Pursuit, Allerton, 2010
C. Qiu and N. Vaswani, Recursive Sparse Recovery in Large but Correlated Noise, Allerton 2011

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$$M_t = S_t + L_t, \quad L_t = P_t a_t$$

- Update $\hat{P}_t$ every-so-often: recursive PCA
  - $\hat{P}_t = \text{recursive-PCA}(\hat{P}_{t-1}, [\hat{L}_{t-\tau}, \dots \hat{L}_{t-1}])$
- Project M_t into space perp to $\hat{P}_t$: get y_t
  - $y_t := (\hat{P}_{t,\perp})' M_t = (\hat{P}_{t,\perp})' S_t + \beta_t, \quad \beta_t$: small noise
- Recover S_t from y_t: noisy sparse recovery
  - $\hat{S}_t = \arg\min_b \|b\|_1 \text{ s.t. } \|y_t - (\hat{P}_{t,\perp})' b\|_2 \le \epsilon$
- Compute $\hat{L}_t := M_t - \hat{S}_t$

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Support-predicted Modified-CS in ReProCS [Qiu,Vaswani,ISIT'11][14]

- If $r := rank(\hat{P}_t)$ small enough for a given $s := |\text{support}(S_t)|$
  - $\frac{s}{n-r}$ large enough for CS to work

[14] C. Qiu and N. Vaswani, Support-Predicted Modified-CS for Recursive Robust Principal Components' Pursuit, ISIT, 2011

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- If s too large or r too large: need Modified-CS
- Video: support changes over time much more
  - e.g. 10x10 block: one pixel motion – supp change of 10
  - $T = \hat{N}_{t-1}$ is not a good approx to N_t
- Support-predicted Modified-CS idea:
  - use $T = \text{model-predict}(\hat{N}_{t-1})$ in Mod-CS
  - use $\hat{N}_t$ to update correlation model parameters

[14] C. Qiu and N. Vaswani, Support-Predicted Modified-CS for Recursive Robust Principal Components' Pursuit, ISIT, 2011

Experiments

- Chenlu Qiu's webpage
- ReProCS magic (S_t invisible in video, its support large, is correlated)
- ReProCS overlay (real bgnd, foregnd somewhat visible but overlay)
- ReProCS (modCS) overlay (very large support of S_t: ReProCS fails)

Acknowledgements

- This talk is mostly based on joint work with my Ph.D. students Wei Lu and Chenlu Qiu
- Research support: NSF grants
  - CCF-1117125 (Recursive Robust PCA)
  - CCF-0917015 (Recursive Reconstruction of Sparse Signal Sequences)
  - ECCS-0725849 (Change Detection in Nonlinear Systems and Applications in Shape Analysis)
- The fMRI work is in collaboration with Dr. Ian Atkinson at UIC Center for MR Research

Ongoing and Future Work

- RecSparsRec in Large but Correlated Noise – Rec Robust PCA
- Regularized ModCS and Kalman filtered ModCS (KalMoCS)
  - open q – when is KalMoCS stable w.r.t. a genie-aided KF?

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- Regularized ModCS and Kalman filtered ModCS (KalMoCS)
  - open q – when is KalMoCS stable w.r.t. a genie-aided KF?
- Functional MRI (fMRI) applications [Lu, Li, Atkinson, Vaswani, ICIP'11]
- Computer Vision
  - ReProCS and applications in video [Qiu, Vaswani, Allerton'10, ISIT'11]
  - Large dimensional visual tracking – use ideas from RecSparsRec