Recursive Causal Reconstruction of Sparse Signal Sequences

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Our Goal

- *Causally & recursively* reconstruct a time seq. of sparse signals
- with slowly changing sparsity patterns
- from a *small number* of linear projections at each time

- *"recursive": use only current measurements vector and previous reconstruction to get current reconstruction*

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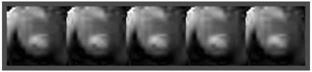
- Applications
  - real-time dynamic MRI reconstruction
    - interventional radiology apps, e.g. MRI-guided surgery
    - fMRI-based study of neural activation patterns
  - single-pixel video imaging with a real-time video display, . . .

- Why causal?
  - needed for real-time applications
- Why causal & recursive?
  - much faster than causal & batch
    - $O(m^3)$ v/s $O(t^3\ m^3)$ at time t (m: signal length)
  - also much faster than offline & batch
- Why reduce the number of measurements needed?
  - data acquisition in MRI or single-pixel camera is sequential: fewer meas $\Rightarrow$ faster acquisition (needed for real-time)

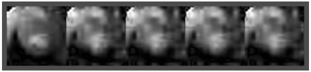
- Most existing work is either
  - for static sparse reconstruction or
  - or is offline & batch [Wakin et al'06(video)],[Gamper et al'08, Jung et al'09 (MRI)]
- Fails if applied to online problem with few measurements

Example: dynamic MRI recon. of a cardiac sequence

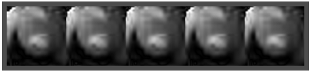
Original sequence



CS−reconstructed sequence



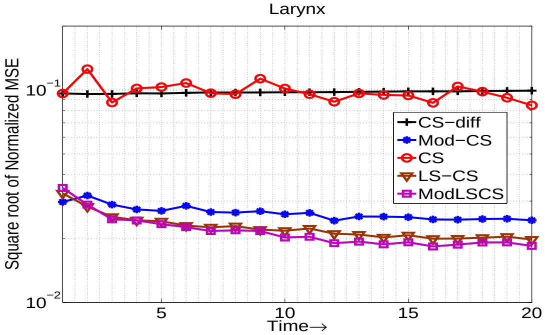
Modified−CS reconstructed sequence



using only 16% Fourier measurements at $t > 0$ (50% at $t = 0$), existing work (CS) gives large reconstruction error (10-12%), proposed approach (modified-CS) is very accurate

Example: dynamic MRI recon of a vocal tract sequence

videos: `http://www.ece.iastate.edu/~luwei/modcs/`



using only 19% Fourier measurements at all times,
existing work (CS, CS-diff) has large error

What is sparse reconstruction?

- Reconstruct a sparse signal x from $y := Ax$ (noiseless) or $y := Ax + w$ (noisy),
  - when A is a fat matrix
- Solved if one can find the sparsest vector satisfying $y = Ax$
  - and $spark(A) > 2|support(x)|$
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  - convex relaxation approaches, e.g. BP, BPDN, DS, ...
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- Compressed Sensing (CS) literature provides the missing theoretical guarantees for the practical approaches

Notation [Candes,Romberg,Tao'05]

- Notation:
  - $|T|$: cardinality of set T
  - $T^c = [1, 2, \ldots m] \setminus T$: complement of set T
  - $\|\beta\|_k$: ℓ_k norm of vector β, $\|\beta\|$: ℓ_2 norm
  - $\|A\|$: spectral matrix norm (induced 2-norm) of matrix A
  - β_T: sub-vector containing elements of β with indices in set T
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$$|c_1' {A_{T_1}}' A_{T_2} c_2| \le \theta_{S_1,S_2} \|c_1\|_2 \; \|c_2\|_2$$ [Candes,Romberg,Tao'05]

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  - *easy to see: $\|A_{T_1}{}' A_{T_2}\| \leq \theta_{|T_1|,|T_2|}$*

Compressive sensing [Candes,Romberg,Tao'05][Donoho'05]

- ℓ_1 min approaches
  - Basis pursuit (BP) [Chen,Donoho,Saunders'97]: $\min_\beta \|\beta\|_1 \ s.t. \ y = A\beta$
  - BP denoising (BPDN): $\min_\beta \|\beta\|_1 \ s.t. \ \|y - A\beta\|_2 \le \epsilon$
  - BPDN - unconst.: $\gamma \min_\beta \|\beta\|_1 + \|y - A\beta\|_2^2$
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- If x is S-sparse and $\delta_{2S} + \theta_{S,2S} < 1$,
  - noiseless measurements: BP gives exact reconstruction
  - noisy meas.: DS or BPDN error can be bounded

    [Candes,Tao'06][Tropp'05]

Problem definition

- Measure

$$y_t = Ax_t \text{ (noise-free) or } \; y_t = Ax_t + w_t \text{ (noisy)}$$

  - $A = H\Phi$, H: measurement matrix, Φ: sparsity basis matrix
  - y_t: measurements $(n \times 1)$
  - x_t: sparsity basis coefficients $(m \times 1)$, $m > n$
  - N_t: support of x_t (set of indices of nonzero elements of x_t)
- Goal: recursively reconstruct x_t from $y_0, y_1, \ldots y_t$,
  - i.e. use only $\hat{x}_{t-1}$ and y_t for reconstructing x_t

Assumptions

- Measurement basis is “incoherent” w.r.t. sparsity basis
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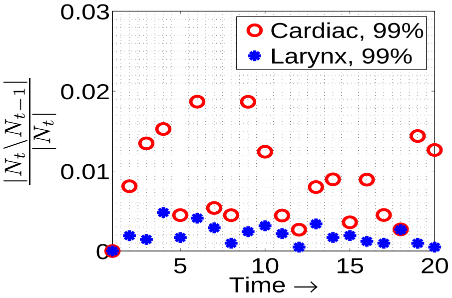
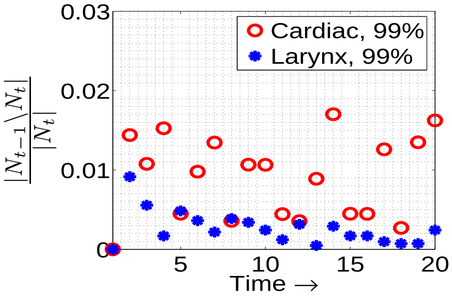
- *Usually nonzero elements of x_t also change slowly over time*

Slow support change in medical image sequences



(a) additions



(b) deletions

- N_t: 99%-energy support of the 2D-DWT of the image
- additions: $N_t \setminus N_{t-1}$, deletions: $N_{t-1} \setminus N_t$
- **maximum size of additions/deletions is less than** $0.02|N_t|$

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   - significant performance improvement with fewer measurements
   - but difficult to analyze

Key Contributions [Vaswani, ICIP'08, Trans. SP (to appear)] [Vaswani, Lu, ISIT'09,Trans. SP (to appear)]

Assume: "slow support change"

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4. Demonstrated all the above for recon'ing real image sequences (approx. sparse) from both partial Fourier (MRI) & Gaussian meas's

Related Work

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  - $CS(y_t - y_{t-1})$: designed to only recon $x_t - x_{t-1}$
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  - unstable if try to recon x_t from few measurements
- [von Borries et al,CAMSAP'07]: static CS with prior support knowledge
  - did not give any exact reconstruction guarantees or error bounds or experimental results

Parallel, later and not-so-related work

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  - [Angelosante,Giannakis, DSP'09]
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- **Our goals are very different from:**
  - homotopy methods e.g. [Asif,Romberg'09], [Rozell et al'07]
    - speed up optimization, do not reduce no. of meas's reqd.
  - reconstruct **one** signal recursively from seq. arriving meas's,
    - e.g. [Sequential CS,Maliotov et al,ICASSP'08], [Garrigues et al'08], [Asif,Romberg'08], [Angelosante,Giannakis,RLS-Lasso,ICASSP'09]
  - multiple measurements vector (MMV) problem

Outline

- Sparse reconstruction with partially known support
  - problem definition
  - LS-CS-residual and error bound
  - Modified-CS and exact reconstruction conditions
  - Stability over time
- Summary
- Ongoing work: KF-CS-residual, KF-mod-CS

Sparse reconstruction with partly known support

- Rewrite the support, N_t, as

$$N_t = T \cup \Delta \setminus \Delta_e$$

  - T: "known" part of the support at t, may have error
  - $\Delta_e := T \setminus N_t$: error in T, unknown
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  - at $t = 0$, T = empty or use prior knowledge

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  2. Fourier sparse signals/images: usually most low frequencies present
  3. fMRI brain activation tracking: use initial frame support as "known part"

LS-CS-residual (LS-CS) [Vaswani, ICIP'08, ICASSP'09, Trans. SP (to appear)]

- Our problem: reconstruct x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$ or from $y := Ax + w$, *when T is known*
- CS-residual idea:
  - compute an initial LS estimate assuming T is correct support

$$(\hat{x}_{\text{init}})_T = {A_T}^{\dagger} y$$
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  - CS on observation residual, add back $\hat{x}_{\text{init}}$

$$\hat{x}_{\text{CSres}} = CS(\tilde{y}_{\text{res}}) + \hat{x}_{\text{init}}$$

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  - compute the observation residual

$$\tilde{y}_{\text{res}} = y - A\hat{x}_{\text{init}}$$

  - CS on observation residual, add back $\hat{x}_{\text{init}}$

$$\hat{x}_{\text{CSres}} = CS(\tilde{y}_{\text{res}}) + \hat{x}_{\text{init}}$$

- Notice that $\tilde{y}_{\text{res}} = A\beta + w$, $\beta = x - \hat{x}_{\text{init}}$

Why CS-residual works better?

- Notice that $\tilde{y}_{\text{res}} = A\beta + w$, where $\beta = x - \hat{x}_{\text{init}}$ with

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Reconstruction error bound [Vaswani, Trans. SP (to appear)]

- **Bound reconstruction error as a function of $|T|, |\Delta|$**
  - L1: obtain error bound for CS on sparse-compressible vectors
  - $(\beta)_T$ is "compressible" part of $\beta := x - \hat{x}_{\text{init}}$
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  - under these sufficient conditions,
    - possible to obtain another CS error bound
    - can argue: our bound is smaller

recall: T: "known" support, Δ: unknown part of support, Δ_e: error in known part

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	$n = 59$, $\sigma = 0.04$	$n = 59$, $\sigma = 0.09$	$n = 59$, $\sigma = 0.44$	$n = 100$, $\sigma = 0.04$
DS, $\lambda = 4\sigma$	0.6545	0.6759	0.9607	0.2622
DS, $\lambda = 0.4\sigma$	0.5375	0.5479	1.0525	**0.0209**
CS-residual, $\lambda = 4\sigma$	**0.0866**	**0.1069**	**0.1800**	0.0402
CS-residual w/ add-then-del	0.0044	0.0205	0.1793	0.0032

$n = 59$: **CS-residual error much smaller than CS error**

Support estimation [Vaswani, ICIP'08, Trans. SP (to appear)]

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$$\begin{aligned} T_{\text{add}} &= T \cup \{i : |(\hat{x}_{CSres})_i| > \alpha_{\text{add}}\} \\ \hat{x}_{\text{add}} &= LS(T_{\text{add}}, y_t) \\ \hat{N} &= T_{add} \setminus \{i : |(\hat{x}_{add})_i| \leq \alpha_{\text{del}}\} \end{aligned}$$

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  - Similar idea also introduced in [Needell-Tropp,CoSaMP'08]

LS-CS-residual (LS-CS) algo [Vaswani, Trans. SP (to appear)]

At each time t,

- *Initial LS.*
  - compute $\hat{x}_{t,\text{init}} = \text{LS}(T, y_t)$
  - compute residual, $\tilde{y}_{t,\text{res}} = y_t - A\hat{x}_{t,\text{init}}$
- *CS-residual.*
  - compute $\hat{x}_{t,\text{CSres}} = \text{CS}(\tilde{y}_{t,\text{res}}) + \hat{x}_{t,\text{init}}$
- *Support Additions and LS.*
  - compute $\tilde{T}_{\text{add}} = T \cup \text{threshold}(\hat{x}_{t,\text{CSres}}, \alpha_{\text{add}})$
  - compute $\hat{x}_{t,\text{add}} = \text{LS}(\tilde{T}_{\text{add}}, y_t)$
- *Support Deletions and LS.*
  - compute $\hat{N}_t = \tilde{T}_{\text{add}} \setminus \text{threshold}(\hat{x}_{t,\text{add}}, \alpha_{\text{del}})$
  - compute $\hat{x}_t = \text{LS}(\hat{N}_t, y_t)$

Exact reconstruction from fewer noiseless measurements?

- Consider noise-free measurements, i.e. $y := Ax$.
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Exact reconstruction from fewer noiseless measurements?

- Consider noise-free measurements, i.e. $y := Ax$.
- Can CS-residual achieve exact reconstruction using fewer measurements?
- Answer: NO
  - No. of meas. needed for exact recon depends on support size
  - CS-residual reconstructs $\beta := x_t - \hat{x}_{t,\text{init}}$ from the LS residual
  - Support of β is $T \cup \Delta$ and $|T \cup \Delta| \geq |N|$ (support size of x)
- Need something else...

recall: T: "known" support, Δ: unknown part of support, Δ_e: error in known part

Modified-CS: noiseless measurements [Vaswani,Lu, ISIT'09, Trans. SP (to appear)]

- Our problem: reconstruct a sparse x with support $N = T \cup \Delta \setminus \Delta_e$ from $y := Ax$, *when T is known*

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- Replace ℓ_0 norm by ℓ_1 norm: get a convex problem:

$$\min_\beta \|(\beta)_{T^c}\|_1 \ s.t. \ y = A\beta \quad \textbf{(modified-CS)}$$

recall: T: known part of support, Δ: unknown part, Δ_e: error in known part

Exact recon with modified-CS [Vaswani,Lu, ISIT'09, Trans. SP (to appear)]

$$\min_{\beta} \|\beta_{T^c}\|_1 \; s.t. \; y = A\beta \quad \text{(modified-CS)}$$

Theorem

x is the unique minimizer of (modified-CS) if $\delta_{|T|+|\Delta|} < 1$ and

$$(\theta_{|\Delta|,|\Delta|} + \delta_{2|\Delta|} + \theta_{|\Delta|,2|\Delta|}) + (\delta_{|T|} + \theta^2_{|\Delta|,|T|} + 2\theta^2_{2|\Delta|,|T|}) < 1$$

recall: $T = N \cup \Delta_e \setminus \Delta$, T: known part of support, Δ: unknown part, Δ_e: error in known part

Comparing the sufficient conditions

- *Modified-CS needs* $\delta_{|T|+|\Delta|} < 1$ and

$$Mcond := (\delta_{2|\Delta|} + \theta_{|\Delta|,|\Delta|} + \theta_{|\Delta|,2|\Delta|}) + (\delta_{|T|} + \theta^2_{|\Delta|,|T|} + 2\theta^2_{2|\Delta|,|T|}) < 1$$

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- If $|\Delta| \approx |\Delta_e| \ll |N|$ (typical for medical image seq's),

$$Mcond < Ccond$$

  - *the difference* $(Ccond - Mcond)$ *is larger when* n *is smaller*
  - *e.g. if* $n < 2|N|$, $Ccond > 1$, *but* $Mcond < 1$ *can hold*

recall: n is the number of measurements, T: known part of support, Δ: unknown part, Δ_e: error in known part

Comparison with the best sufficient conditions for CS

- CS gives exact recon if

$$\delta_{2|N|} < \sqrt{2} - 1 \quad \text{or} \quad \delta_{2|N|} + \delta_{3|N|} < 1$$ [Candes'08, Candes-Tao'06]

- Modified-CS gives exact recon if

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|N|+|\Delta_e|-|\Delta|} + \delta^2_{|N|+|\Delta_e|} + 2\delta^2_{|N|+|\Delta_e|+|\Delta|} < 1$$

- use $\delta_{ck} \leq c\delta_{2k}$ [CoSaMP'08]
- If $|\Delta| = |\Delta_e| = 0.02|N|$ (typical in medical sequences),
  - **sufficient condition for CS:**

$$\delta_{2|\Delta|} < 1/241.5$$

  - **sufficient condition for modified-CS:**

$$\delta_{2|\Delta|} < 1/132.5$$

Simulations: probability of exact reconstruction

Simulation setup:

- signal length, $m = 256$, support size $s = |N| = 0.1m$
- use random-Gaussian A, varied n, $|\Delta|$ and $|\Delta_e|$
- for each choice, Monte Carlo averaged over N, $(x)_N$, Δ, Δ_e
- we say "works" (gives exact recon) if $\|x - \hat{x}\|_2 < 10^{-5}\|x\|_2$

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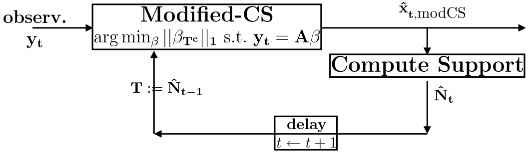
n	mod-CS $\lvert\Delta\rvert, \lvert\Delta_e\rvert \le 0.08\lvert N\rvert$	mod-CS $\lvert\Delta\rvert, \lvert\Delta_e\rvert \le 0.20\lvert N\rvert$	CS $(\Delta = N, \Delta_e = \emptyset)$
19%	**0.998**	0.68	**0**
25%	1	0.99	0.002
40%	1	1	0.98

recall: n is number of measurements, Δ: unknown part of support, Δ_e: error in known part

Modified-CS for time sequences



Initial time ($t = 0$):

- use T_0 from prior knowledge, e.g. wavelet approx. coeff's
- typically need more measurements at $t = 0$

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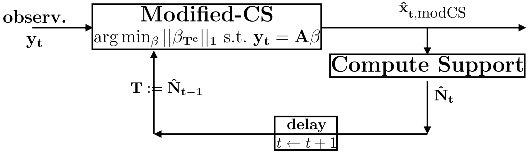

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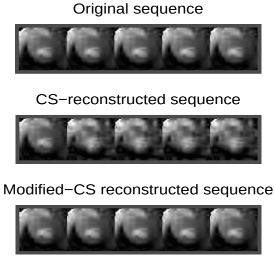
- typically need more measurements at $t = 0$

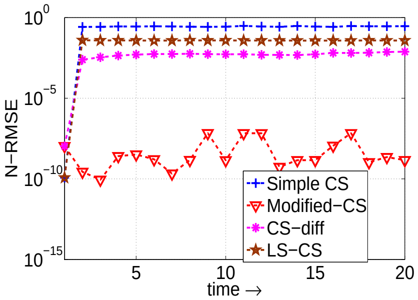
Stability: (trivial in the noise-free case)

- error stable at zero if $Mcond < 1$ at $t = 0$ and at all $t > 0$

Exact recon of a sparsified cardiac sequence



(c) $n_0 = 50\%$, $n = 16\%$



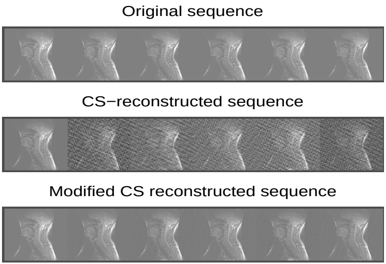
(d) $n_0 = 50\%$, $n = 16\%$

support size $\sim 10\%$

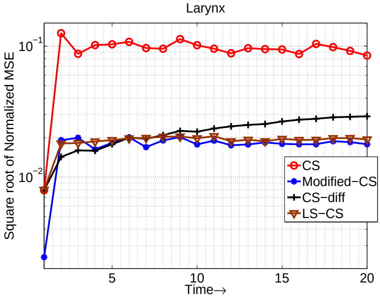
using $n = 16\%$ MRI measurements at $t > 0$, $n_0 = 50\%$ at $t = 0$.

modified-CS gives exact recon (NRMSE $\sim 10^{-8}$), others do not

Small error recon of a true larynx sequence



(e) $n_0 = 50\%,\ n = 19\%$



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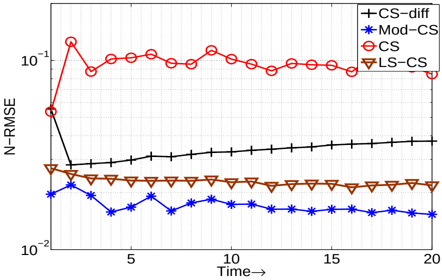
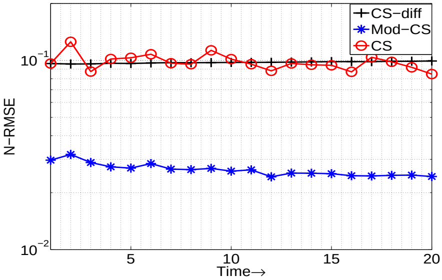


(g) $n_0 = 20\%,\ n = 19\%$



(h) $n_0 = 19\%,\ n = 19\%$

reducing n_0 (no. of measurements at $t = 0$)

Noisy measurements

- Mod-CS(noisy): relax the data constraint, e.g.

$$\min_{\beta} \|\beta_{T^c}\|_1 \text{ s.t. } \|y_t - A\beta\|_2 \leq \epsilon$$

  - use add-then-delete for support estimation

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- Easy to bound error as a function of $|T|, |\Delta|$ [Lu,Vaswani,ICASSP'10], [Jacques,Arxiv'09]
  - **but $|T|, |\Delta|$ depend on accuracy of previous recon**
  - **bound may keep increasing over time – limited use**

Stability [Vaswani, Trans. SP (to appear)]

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  - **i.e. need conditions for "stability" over time**
- **Approach:**
  - *obtain a time-invariant bound on the support errors (extras & misses)*
  - *argue: bound small compared to support size*
  - directly implies time-invariant and small bound on recon error

Stability of modified-CS and LS-CS: setting

- Bounded measurement noise.
  - Why? -
    - Gaussian noise: error bounds at t hold with “large” probability
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- In case of 1.: perfect support estimation possible after a small delay

Stability of modified-CS and LS-CS: summary [Vaswani, Trans. SP (to appear)]

- For a given n (no. of meas.) and noise level,
  1. if use enough measurements for accurate recon at $t = 0$
  2. if
     - the support is small enough, and
     - the support changes slowly enough,
  3. if the nonzero coefficients increase/decrease fast enough, and
  4. if addition & deletion thresholds are appropriately set,

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  - support errors (no. of extras and misses) are bounded
- **Can argue: our sufficient conditions allow larger support sizes, for a given n, than CS results**

Result 1: support changes every-so-often [Vaswani, Trans. SP (to appear)]

- Signal model:
  - S_a additions (removals) to (from) support every d frames
  - support size is always either S_0 or $S_0 - S_a$
  - the magnitude of the i^{th} new coeff increases at rate a_i for d time units and then becomes constant
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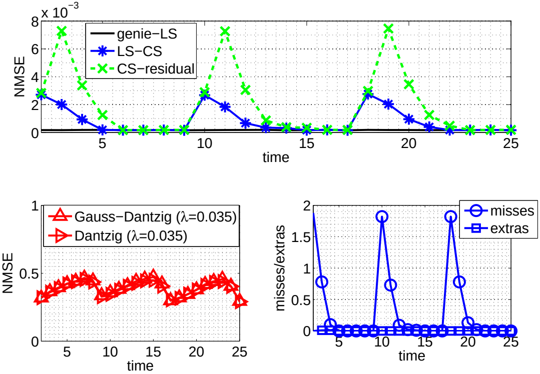
- If "conditions" hold, then
  - **at all times, misses and extras are bounded:**

$$|N_t \setminus \hat{N}_t| \leq S_a, \text{ and } |\hat{N}_t \setminus N_t| \leq 2S_a + 4$$

  - **within a short delay, $S_a + 2$, after a new addition, $\hat{N}_t = N_t$**
  - **remains this way until next addition time**

substituting $d_0 = 2$ in [Vaswani, LS-CS-residual, TSP (to appear)]

Simulations: verifying stability



- $m = 200$, support size, $S_0 = 20$
- $S_a = 2$ additions/removals every $d = 8$ frames
- 29.5% measurements at $t > 0$, noise $\sim unif(-c, c)$, $c = 0.05$

Proof strategy: key lemmas [Vaswani, Trans. SP (to appear)]

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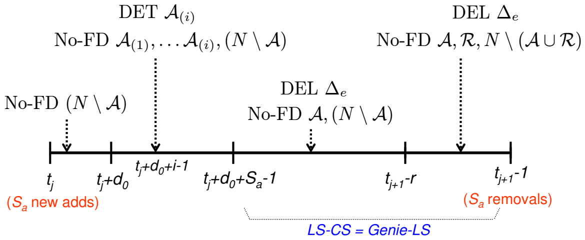
- *(true deletion)* all extras in support estimate (zero coeff's) **do get deleted**

Proof strategy: induction step [Vaswani, Trans. SP (to appear)]



DET: detection No-FD: no false deletion DEL: true deletion

- t_j : j^{th} support addition time, $t_{j+1} - 1$: support removal time
- $\mathcal{A}$: set added at t_j (increasing coeff's),
  - $\mathcal{A}_{(i)}$: i^{th} largest element of $\mathcal{A}$
- $N \setminus \mathcal{A}$: previously added set (constant coeff's)

Result 2: support changes at every time

- Signal model 2:
  - S_a additions and S_a removals **at each time**
  - support size constant at S_0
  - every new coeff's magnitude increases at rate r until it reaches a max value M
  - similar model for coeff decrease
- Noise, $\|w\| \leq \epsilon$
- If
  1. accurate recon at initial time,
  2. $\delta_{S_0+4S_a} < 0.414$ and $\theta_{S_0+2S_a,S_a} < 1/\sqrt{18S_a}$.
     - if LS error bound equal in all directions: only need $\theta_{S_0+2S_a,S_a} < \sqrt{S_0/18S_a}$
  3. $r > f_{incr}(S_0, S_a, \epsilon, \alpha_{add}, \alpha_{del})$,
  4. α_{add}, α_{del} large enough,

then $|N_t \setminus \hat{N}_t| \leq 2S_a$ **and** $|\hat{N}_t \setminus N_t| = 0$

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- Compare: CS needs $\delta_{2S_0} < 0.414$

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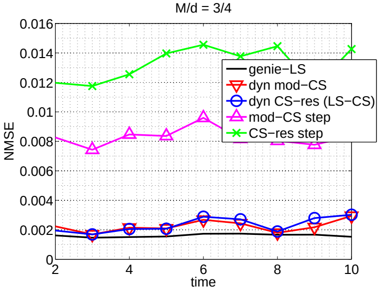

(i) $r = 0.75$

- $m = 200$, $S_0 = 20$, additions/removals, $S_a = 2$ at each time
- 29.5% measurements at $t > 0$, noise $\sim unif(-c, c)$, $c = 0.1266$
- *CS error 22-30% in all cases*

Summary

$T := \hat{N}_{t-1}, \quad \mu_T := (\hat{x}_{t-1})_T$

- CS on observation residual
  - initial estimate: compute using $LS(y_t, T)$ or use previous recon or use KF'ed estimate

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- Support estimation in either case
  - add-LS-delete is better than simple thresholding

- If support changes slowly enough,
- under much weaker sufficient conditions than CS,
  - modified-CS gives exact reconstruction
    - its stability proof is trivial
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- For dynamic MRI and video reconstruction,
  - significant improvement over CS, Gauss-CS and CS-diff
  - only slightly worse than batch methods (batch-CS, k-t-focuss)

Key References

- **Namrata Vaswani, LS-CS-residual (LS-CS): Compressive Sensing on the Least Squares Residual, To Appear in IEEE Trans. Signal Processing**
  - Namrata Vaswani, *Kalman Filtered Compressed Sensing*, IEEE Intl. Conf. Image Proc. (ICIP), 2008
  - Namrata Vaswani, *Analyzing Least Squares and Kalman Filtered Compressed Sensing*, IEEE Intl. Conf. Acous. Speech. Sig. Proc. (ICASSP), 2009
- **Namrata Vaswani and Wei Lu, Modified-CS: Modifying Compressive Sensing for Problems with Partially Known Support, To Appear in IEEE Trans. Signal Processing**
  - Namrata Vaswani and Wei Lu, *Modified-CS: Modifying Compressive Sensing for Problems with Partially Known Support*, IEEE Intl. Symp. Info. Theory (ISIT), 2009

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  - Samarjit Das
  - Fardad Raisali

Slow support and signal value change [Vaswani, ICIP'08, ICASSP'09]

- $\Leftrightarrow$ **Track a time sequence of signals with slowly changing "principal" directions (in a given sparsity basis) and slowly changing principal coefficient values**
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- *KF-CS error is stable (so 1. and 2. hold): under strong assumptions*

Proposed solution 2: Regularized Modified-CS

- Most practical apps: the (significantly) nonzero elements of x_t also change slowly
- To also use this fact, we can solve

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- Above computes a *causal MAP estimate* if
  - posterior at $t-1$ approx by a Dirac delta at μ
  - $(x_t)_T$ are i.i.d. Gaussian with mean μ_T and variance σ^2,
  - $(x_t)_{T^c}$ are i.i.d. Laplacian with mean zero and scale b,
  - *and we set* $\gamma = b/2\sigma^2$

Regularized Modified CS (RegModCS): simulations

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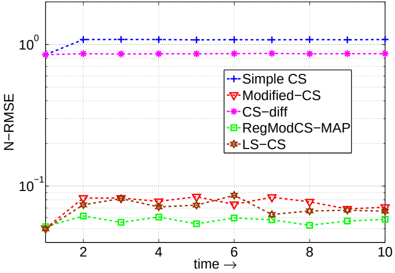
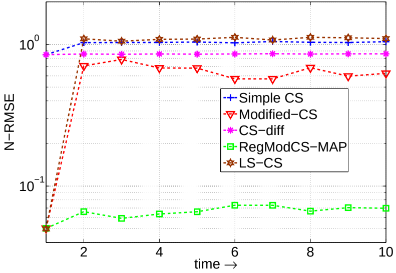


(l) $n_0 = n = 19\%$



(m) $n_0 = 19\%, \ n = 6\%$

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Most current work: non-Bayesian, only recursive approaches

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- **Real dynamic MRI problems** (ongoing: fMRI)