Kalman Filtering

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Notes are based on Vincent Poor's book.

1 Kalman Filter as a causal MMSE estimator

Consider the following state space model (signal and observation model).

$$\begin{aligned} Y_t &= H_t X_t + W_t, \ W_t \sim \mathcal{N}(0, R) && (1) \\ X_t &= F_t X_{t-1} + U_t, \ U_t \sim \mathcal{N}(0, Q) && (2) \end{aligned}$$

where $X_0, \{U_t, t = 1, \ldots \infty\}, \{W_t, t = 0, \ldots \infty\}$ are mutually independent and $X_0 \sim \mathcal{N}(0, \Sigma_0)$.

Recall that

$$Cov[X|Y] \triangleq \mathbb{E}[(X - \mathbb{E}[X|Y])(X - \mathbb{E}[X|Y])^T|Y].$$

Define

$$\begin{aligned} \hat{X}_{t|s} &\triangleq \mathbb{E}[X_t|Y_{0:s}] \\ \Sigma_{t|s} &\triangleq Cov[X_t|Y_{0:s}] = \mathbb{E}[(X_t - \hat{X}_{t|s})(\cdot)^T|Y_{0:s}] && (3) \end{aligned}$$

Thus $\hat{X}_{t|s}$ is the MMSE estimator of X_t from observations $Y_{0:s}$ and $\Sigma_{t|s}$ is its error covariance conditioned in $Y_{0:s}$.

We claim that that $\hat{X}_{t|t}$ and $\hat{X}_{t|t-1}$ satisfy the following recursion **(Kalman filter)**.

$$\begin{aligned} \hat{X}_{t|t-1} &= F_t \hat{X}_{t-1|t-1} \\ \Sigma_{t|t-1} &= F_t \Sigma_{t-1|t-1} F_t^T + Q, \\ K_t &= \Sigma_{t|t-1} H_t^T (H_t \Sigma_{t|t-1} H_t^T + R)^{-1} \\ \hat{X}_{t|t} &= \hat{X}_{t|t-1} + K_t (Y_t - H_t \hat{X}_{t|t-1}) \\ \Sigma_{t|t} &= [I - K_t H_t] \Sigma_{t|t-1} && (4) \end{aligned}$$

with initialization, $\hat{X}_{0|-1} = 0$, $\Sigma_{0|-1} = \Sigma_0$.

1.1 Conditional Gaussian Distribution

In the proof we will need the following result for jointly Gaussian random variables. If X, Y have the joint PDF

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathcal{N}(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \Sigma_Y & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_X \end{bmatrix})$$

then

$$\begin{aligned} \mathbb{E}[X|Y] &= \mu_X + \Sigma_{XY}\Sigma_Y^{-1}(Y - \mu_Y) \\ Cov[X|Y] &= \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX}, \end{aligned} \tag{5}$$

Proof: One way to prove this is to write out the expression for the conditional PDF and use the block matrix inversion lemma. A shorter and nicer proof is as follows. The idea is to define a r.v. Z that is a linear function of X and Y and is such that $Cov(Z, Y) = 0$. Because it is a linear function, Z and Y are also jointly Gaussian and hence $cov = 0$ will imply independence. Then if one writes of X as a linear function of Z and Y, getting the above quantities becomes very easy because $\mathbb{E}[f(Z)|Y] = \mathbb{E}[f(Z)]$.

1. Let $Z = X + BY$. We want $Cov(Z, Y) = 0$. But notice that $Cov(Z, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)^T] + B\mathbb{E}[(Y - \mu_Y)(Y - \mu_Y)^T] = \Sigma_{XY} + B\Sigma_Y$. Thus if we let $B = -\Sigma_{XY}\Sigma_Y^{-1}$ we will get $Cov(Z, Y) = 0$ and so using joint-Guassianity Z and Y are independent.

2. Thus, $Z = X - \Sigma_{XY}\Sigma_Y^{-1}Y$ and so $X = Z + \Sigma_{XY}\Sigma_Y^{-1}Y$. Also, Z and Y are independent. Thus,

$$\mathbb{E}[X|Y] = \mathbb{E}[Z|Y] + \Sigma_{XY}\Sigma_Y^{-1}Y = \mu_X - \Sigma_{XY}\Sigma_Y^{-1}\mu_Y + \Sigma_{XY}\Sigma_Y^{-1}Y \tag{6}$$

and since $Cov(Z|Y) = Cov(Z)$, $Cov(Y|Y) = 0$ and $Cov(Z, BY|Y) = \mathbb{E}[(Z - \mu_Z)|Y](Y - \mu_Y)^T = \mathbb{E}[(Z - \mu_Z)](Y - \mu_Y)^T = 0$, we get

$$\begin{aligned} Cov[X|Y] &= Cov(Z - BY, Z - BY|Y) \text{ where } B := -\Sigma_{XY}\Sigma_Y^{-1} \\ &= Cov(Z|Y) + Cov(BY|Y) - Cov(Z, BY|Y) - Cov(BY, Z|Y) \\ &= Cov(Z) \\ &= \Sigma_X + \Sigma_{XY}\Sigma_Y^{-1}\Sigma_Y\Sigma_Y^{-1}\Sigma_{XY} - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{XY} - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{XY} \\ &= \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX} \end{aligned} \tag{7}$$

1.2 Proof of KF as MMSE estimator

1. Use induction. Base case for $t = 0$ follows directly from the signal model. Assume that (4) holds at $t - 1$.

2. To compute $\hat{X}_{t|t-1}$, take $\mathbb{E}[\cdot|Y_{0:t-1}]$ on the signal model, (2). Then use the fact that U_t is independent of $Y_{0:t-1} = f(X_0, U_{0:t-1}, W_{0:t-1})$ to show that $\mathbb{E}[U_t|Y_{0:t-1}] = \mathbb{E}[U_t] = 0$ (since U_t is zero mean).

3. To compute $\Sigma_{t|t-1}$, take $Cov[\cdot|Y_{0:t-1}]$ on the signal model, (2). Show that the cross-terms which contain $\mathbb{E}[(X_{t-1} - \hat{X}_{t-1|t-1})U_t^T|Y_{0:t-1}]$ or its transpose are zero using the following approach.

   (a) Let $Z \triangleq X_{t-1} - \hat{X}_{t-1|t-1}$.

   (b) It is easy to see that both Z and $Y_{0:t-1}$ are functions of $X_0, U_{0:t-1}, W_{0:t-1}$, i.e. $\{Z, Y_{0:t-1}\} = f((X_0, U_{0:t-1}, W_{0:t-1})$. Thus, U_t is independent of $\{Z, Y_{0:t-1}\}$ (using the model assumption).

   (c) U independent of $\{Z, Y\}$ implies (i) U independent of Y; (ii) U independent of Z given Y (conditionally independent).

   - Proof: U independent of $\{Z, Y\}$ implies that $f(y, u) = \int f(z, y, u)dz = \int f(z, y)f(u)dz = f(y)f(u)$. Thus (i) follows. To show (ii), notice that $f(z, u|y) = f(z|y)f(u|z, y)$ by chain rule. U independent of $\{Z, Y\}$ implies that $f(u|z, y) = f(u)$. Also, (i) implies that $f(u) = f(u|y)$. Thus, $f(z, u|y) = f(z|y)f(u|y)$ and thus (ii) holds.

   (d) Thus U_t, Z are conditionally independent given $Y_{0:t-1}$.

   (e) Thus the cross term, $\mathbb{E}[ZU_t^T|Y_{0:t-1}] = \mathbb{E}[Z|Y_{0:t-1}]\,\mathbb{E}[U_t|Y_{0:t-1}]^T = \mathbb{E}[Z|Y_{0:t-1}]\mathbb{E}[U_t]^T = 0$ and the same holds for its transpose. The second equality follows because U_t is independent of $Y_{0:t-1}$ while the third follows because U_t is zero mean.

4. For the update step, first derive the expression for the joint pdf of X_t, Y_t conditioned on $Y_{0:t-1}$, $f(x_t, y_t|y_{0:t-1})$, and then use the conditional mean and covariance formula for joint Gaussians given in (5), to obtain the expression for $\hat{X}_{t|t}$ and $\Sigma_{t|t}$. To obtain the joint pdf expression, use the following approach.

   (a) Since the three random variables are jointly Gaussian, $f(x_t, y_t|y_{0:t-1})$ will also be Gaussian. Thus we only need expressions for its mean and covariance. Clearly,

$$f(x_t, y_t|y_{0:t-1}) \quad = \quad \mathcal{N}(\begin{bmatrix} x_t \\ y_t \end{bmatrix}; \begin{bmatrix} \hat{X}_{t|t-1} \\ H\hat{X}_{t|t-1} \end{bmatrix}, \begin{bmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY}^T & \Sigma_X \end{bmatrix}), \text{ where}$$

$$\begin{aligned}
\Sigma_X &= \Sigma_{t|t-1}, \\
\Sigma_{XY} &= Cov[X_t, Y_t|Y_{0:t-1}] = \mathbb{E}[(X_t - \hat{X}_{t|t-1})(H(X_t - \hat{X}_{t|t-1}) + W_t)^T|Y_{0:t-1}] \\
&= \Sigma_{t|t-1}H^T + cross_1 = \Sigma_{t|t-1}H^T + 0, \\
\Sigma_Y &= Cov[Y_t|Y_{0:t-1}] = H\Sigma_{t|t-1}H^T + R + cross_2 + cross_2^T = H\Sigma_{t|t-1}H^T + R + 0 + 0
\end{aligned}$$

(b) To show that $cross_1 = 0$ and $cross_2 = 0$, use the fact that $(X_t - \hat{X}_{t|t-1})$ and W_t are conditionally independent given $Y_{0:t-1}$; W_t is independent of $Y_{0:t-1}$ and W_t is zero mean. These can be shown in a fashion analogous to step 3.

This finishes the proof of the induction step and thus of the result.

A few important things to notice are as follows.

1. Since the expressions for $\Sigma_{t|t-1}$ and $\Sigma_{t|t}$ do not depend on $Y_{0:t-1}$, thus these are also equal to the unconditional error covariances, i.e. $\Sigma_{t|t-1} \triangleq \mathbb{E}[(X_t - \hat{X}_{t|t-1})(\cdot)^T | Y_{0:t-1}] = \mathbb{E}[(X_t - \hat{X}_{t|t-1})(\cdot)^T]$ and similarly, $\Sigma_{t|t} \triangleq \mathbb{E}[(X_t - \hat{X}_{t|t})(\cdot)^T | Y_{0:t}] = \mathbb{E}[(X_t - \hat{X}_{t|t})(\cdot)^T]$

2. It can be shown that the innovations, $Z_t \triangleq Y_t - H\hat{X}_{t|t-1}$, are pairwise uncorrelated. Since they are jointly Gaussian, this means they are mutually independent.

   (a) To show uncorrelated-ness one needs to use $\mathbb{E}[\mathbb{E}[Z|Z_1, Z_2]|Z_1] = \mathbb{E}[Z|Z_1]$ (law of iterated expectations applied to $\tilde{Z} = Z|Z_1$).

   (b) Consider an $s < t$,

$$
\begin{aligned}
\mathbb{E}[Z_t Z_s^T] &= \mathbb{E}[\mathbb{E}[Z_t Z_s^T | Y_{0:s}]] \\
&= \mathbb{E}[\mathbb{E}[Z_t | Y_{0:s}] Z_s^T] \\
&= \mathbb{E}[\mathbb{E}[(Y_t - H\hat{X}_{t|t-1}) | Y_{0:s}] Z_s^T] \\
&= \mathbb{E}[(\mathbb{E}[Y_t | Y_{0:s}] - \mathbb{E}[H\hat{X}_{t|t-1} | Y_{0:s}]) Z_s^T] \\
&= \mathbb{E}[(H\hat{X}_{t|s} - \mathbb{E}[H\hat{X}_{t|t-1} | Y_{0:s}]) Z_s^T] \\
&= \mathbb{E}[(H\hat{X}_{t|s} - H\hat{X}_{t|s}) Z_s^T] \\
&= 0 \qquad (8)
\end{aligned}
$$

   We used $\mathbb{E}[\mathbb{E}[Z|Z_1, Z_2]|Z_1] = \mathbb{E}[Z|Z_1]$ to show that $\mathbb{E}[H\hat{X}_{t|t-1}|Y_{0:s}] = H\hat{X}_{t|s}$. Here $Z \equiv X_t$, $Z_1 \equiv Y_{0:s}$, $Z_2 \equiv Y_{s+1:t}$.

3. Another expression for K_t:

1.3 Kalman filter with control input

Consider a state space model of the form

$$
\begin{aligned}
Y_t &= HX_t + r(Y_1, Y_2, \ldots Y_{t-1}) + W_t, \quad W_t \sim \mathcal{N}(0, R) \\
X_t &= FX_{t-1} + q(Y_1, Y_2, \ldots Y_{t-1}) + GU_t, \quad U_t \sim \mathcal{N}(0, Q)
\end{aligned}
$$

with $X_0, \{U_t\}_{t=1}^{\infty}, \{W_t\}_{t=0}^{\infty}$ being mutually independent and $X_0 \sim \mathcal{N}(0, \Sigma_0)$.

The above is a state space model, but with a nonzero "feedback control" input in both equations. To derive the Kalman recursion for this model, use the exact same procedure outlined above. Since at time t, the control inputs are functions of $Y_{0:t-1}$, when we take $\mathbb{E}[\cdot|Y_{0:t-1}]$ of either the signal model or the observation model, r and q just get pulled out as constants. Thus the expressions for the error covariances does not change at all.

2 Kalman filter as a causal linear MMSE estimator

Consider the state space model of (1), (2), but with the difference that X_0, U_t, W_t's are no longer Gaussian, but are just some zero mean random variables with the given covariances. Also, instead of being mutually independent, they are only pairwise uncorrelated.

For this model, the Kalman filter of (4) is the causal linear MMSE estimator, i.e. $\hat{X}_{t|t-1}$ is the linear MMSE of X_t from $Y_{0:t-1}$, $\hat{X}_{t|t}$ is the linear MMSE of X_t from $Y_{0:t}$, and the unconditional error covariances, $\mathbb{E}[(X_t - \hat{X}_{t|t-1})(.)^T] = \Sigma_{t|t-1}$ and $\mathbb{E}[(X_t - \hat{X}_{t|t})(.)^T] = \Sigma_{t|t}$.

We show the above using the orthogonality principle: $\hat{X}$ is the linear MMSE of X from observations $Y_{0:n-1}$ if and only if

$$\mathbb{E}[(X - \hat{X})] = 0 \text{ and } \mathbb{E}[(X - \hat{X})Y_l^T] = 0, \ \forall \ l = 0, 1, \dots n-1, \tag{9}$$

Outline of Proof.

1. Use induction. The base case is easy, since you are using no observations. Assume that our result holds for $t-1$, i.e. $\hat{X}_{t-1|t-1}$ is linear MMSE of X_{t-1} from $Y_{0:t-1}$ and $\mathbb{E}[(X_{t-1} - \hat{X}_{t-1|t-1})(.)^T] = \Sigma_{t-1|t-1}$.

2. The expression for $\hat{X}_{t|t-1}$ is given in (4). Thus,

$$\mathbb{E}[(X_t - \hat{X}_{t|t-1})Y_l^T] = F\mathbb{E}[(X_{t-1} - \hat{X}_{t-1|t-1})Y_l^T] + F\mathbb{E}[U_t Y_l^T] = 0 + 0, \ \forall l = 0, 1, \dots t-1 \tag{10}$$

   The first term is zero because of the induction hypothesis and the orthogonality principle. The second term is zero because of uncorrelated-ness of U_t and Y_l (follows because Y_l is a linear function of $X_0, U_{1:l}, W_{0:l}$). Also,

$$\mathbb{E}[(X_t - \hat{X}_{t|t-1})] = F\mathbb{E}[X_{t-1} - \hat{X}_{t-1|t-1}] + \mathbb{E}[U_t] = 0 \tag{11}$$

   This follows by induction assumption and orthogonality principle and since U_t is zero mean.

   Thus by the orthogonality principle, $\hat{X}_{t|t-1}$ is the L-MMSE of X_t from $Y_{0:t-1}$.

3. It is easy to show that $\mathbb{E}[(X_t - \hat{X}_{t|t-1})(.)^T] = F\Sigma_{t-1|t-1}F^T + Q + 0 + 0 = \Sigma_{t|t-1}$ by showing that the cross-terms are zero. The cross-terms are of the form $\mathbb{E}[(X_{t-1} - \hat{X}_{t-1|t-1})U_t^T]$ (or its transpose). They are zero since $(X_{t-1} - \hat{X}_{t-1|t-1})$ is a linear function of $X_0, U_{1:t-1}, W_{0:t-1}$, all of which are uncorrelated with U_t, and U_t is zero mean.

4. The expression for $\hat{X}_{t|t}$ is given in (4). Using the expression for Y_t,

$$X_t - \hat{X}_{t|t} = X_t - [(I - K_t H)\hat{X}_{t|t-1} + K_t Y_t] = (I - K_t H)(X_t - \hat{X}_{t|t-1}) + K_t W_t \quad (12)$$

Thus,

$$\mathbb{E}[(X_t - \hat{X}_{t|t})Y_l^T] = (I - K_t H)\mathbb{E}[(X_t - \hat{X}_{t|t-1})Y_l^T] + K_t \mathbb{E}[W_t Y_l^T] \quad (13)$$

For $l = 0, 1, \ldots t-1$, one can use the previous step; uncorrelated-ness of W_t and $Y_{0:t-1}$ and W_t being zero mean, to show that the above is zero. Consider $l = t$.

$$\begin{aligned}
\mathbb{E}[(X_t - \hat{X}_{t|t})Y_t^T] &= (I - K_t H)\mathbb{E}[(X_t - \hat{X}_{t|t-1})Y_t^T] - K_t \mathbb{E}[W_t Y_t^T] \\
&= (I - K_t H)\mathbb{E}[(X_t - \hat{X}_{t|t-1})(HX_t + W_t)^T - K_t \mathbb{E}[W_t W_t^T] \\
&= (I - K_t H)\Sigma_{t|t-1} H^T - K_t R \\
&= 0 \qquad (14)
\end{aligned}$$

(a) The last equality follows from the expression for K_t.

(b) The second-last one follows because

i. $\mathbb{E}[(X_t - \hat{X}_{t|t-1})X_t^T] = \mathbb{E}[(X_t - \hat{X}_{t|t-1})(X_t - \hat{X}_{t|t-1} + \hat{X}_{t|t-1})^T] = \Sigma_{t|t-1} + \mathbb{E}[(X_t - \hat{X}_{t|t-1})\hat{X}_{t|t-1}^T]$.

ii. $\mathbb{E}[(X_t - \hat{X}_{t|t-1})\hat{X}_{t|t-1}^T] = \mathbb{E}[\mathbb{E}[(X_t - \hat{X}_{t|t-1})|Y_{0:t-1}]\hat{X}_{t|t-1}^T] = \mathbb{E}[0.\hat{X}_{t|t-1}^T] = 0$

iii. W_t and $(X_t - \hat{X}_{t|t-1})$ are uncorrelated and W_t is zero mean.

(c) The third-last one follows using (1) and the fact that W_t and X_t are uncorrelated and W_t is zero mean.

Also, using (12),

$$\mathbb{E}[(X_t - \hat{X}_{t|t})] = (I - K_t H)\mathbb{E}[X_t - \hat{X}_{t|t-1}] + K_t \mathbb{E}[W_t] = 0 \quad (15)$$

The first term is zero by prediction step claim and orthogonality principle; the second term is zero since W_t is zero mean.

Thus by orthogonality principle, $\hat{X}_{t|t}$ is L-MMSE of X_t from $Y_{0:t}$.

5. To get the expression for $\mathbb{E}[(X_t - \hat{X}_{t|t})(.)^T]$, expand it out using the expression for $\hat{X}_{t|t}$ from (4) and then simplify it using (1) and the expression for K_t.

(a) $\mathbb{E}[(X_t - \hat{X}_{t|t})(.)^T] = (I - K_t H)\mathbb{E}[(X_t - \hat{X}_{t|t-1})(.)^T](I - K_t H)^T + K_t \mathbb{E}[W_t W_t^T]K_t^T + cross_3 + cross_3^T = (I - K_t H)\Sigma_{t|t-1}(I - K_t H)^T + K_t R K_t^T + 0 + 0$.

(b) The cross term, $cross_3$, contains $\mathbb{E}[(X_t - \hat{X}_{t|t-1})W_t^T]$ which can be shown to be zero since W_t and $(X_t - \hat{X}_{t|t-1})$ are uncorrelated and W_t is zero mean.

(c) Use the expression for K_t and simplify to show that $(I - K_t H)\Sigma_{t|t-1}(I - K_t H)^T + K_t R K_t^T = (I - K_t H)\Sigma_{t|t-1}$: homework problem.