

# Compressed Sensing

April  
29  
Continued

③

1)  $y = Ax$ .  $A = \text{fat}$   $x: N \times 1$ .

$A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{K \times N}$   $K < N$ .

$N = 256$   
 $K = 72$ .

$\hat{x} = ?$   
 $x$  solutions.

→ a min regularization of some kind,

Option 1: Minimum energy: ~~and~~  $\min \|x\|_2$  st.  $y = Ax$ .

$\hat{x} = A^T (AA^T)^{-1} y$ .

~~this~~

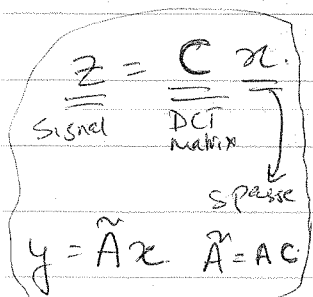
2) Option 2: ~~use~~ I tell you  $x$  is a sparse vector.  
only  $s = \ll N$  elements of  $x$  are non zero.

→ Min energy soln is NOT sparse.

→ Why sparsity

↓  
reason why JPEG works

Compressibility (most natural signals are sparse approximately)



JPEG: DCT of signal; zeros out small coeffs.  
↓  
gives a lot of compression at small loss of quality ⇒  
this assumption holds

3) When can I do: find  $\min \|x\|_0$  st  $y = Ax$ . &  
hope to get  $\hat{x} = x_{\text{true}}$ .

When  $K > 2s$  & any  $2s$  <sup>or smaller columns</sup> ~~columns~~  $s$ -m of  $A$  is full rank.

i.e.  $\text{rank}(A_T^T A_T) = |T|$ .  $A_T = [a_{T_1} \ a_{T_2} \ \dots \ a_{T_s}]$   
 $T = \{T_1, T_2, \dots, T_s\}$ .

why this is enough?

say this is not sufficient then

$$\exists x_1 \neq x_2 \text{ s.t. } x_1, x_2 \text{ on } T, |T| \leq S,$$
$$y = Ax_1 = Ax_2 \quad x_2 \text{ on } T_2, |T_2| \leq S.$$

$$\text{f. } y = Ax_1 = Ax_2.$$

$$\Rightarrow A(x_1 - x_2) = 0.$$

then

$$x_1 - x_2 \text{ n.z. on } T_1 \cup T_2.$$

$$\Rightarrow A_{T_1 \cup T_2} (x_1 - x_2)_{T_1 \cup T_2} = 0.$$

pick row corresp. to elements in  $T_1 \cup T_2$ .

$$\nexists A_{T_1 \cup T_2} (x_1 - x_2)_{T_1 \cup T_2} = 0.$$

$\Rightarrow A_{T_1 \cup T_2}$  is not full rank.

this is a contradiction.

4) So far inf.  $\min \|x\|_1$  s.t.  $y = Ax$  works if

$K > 2S$  &  $A$  satisfies RIP at  $\text{RIP}(2S)$   
Full rank  $(2S)$ . RIP(2S) slightly stronger

RIP( $S_0$ ) : for any set  $T$  with  $|T| \leq S_0$ , ~~let~~ let  $\delta_{S_0}$  is smallest quantity s.t.

$$\text{s.t. } 1 - \delta_{S_0} \leq \lambda(A_T^T A_T) \leq 1 + \delta_{S_0} \quad \forall |T| \leq S_0$$

~~$\delta_{S_0} < 1$~~  if  $\delta_{S_0} < 1$  then

$A$  satisfies RIP( $S_0$ ).

5) Above is computationally complex : combinatorial

Relax :  $\min \|x\|_1$  s.t.  $y = Ax$ .

The soln of (P1) is equal to that

of (P0), i.e. P1 gives the sparsest

correct solution if

$$\left. \begin{array}{l} \rightarrow k > 3s \\ \rightarrow A \text{ satisfies RIP}(3s) \\ \rightarrow \delta_s + \delta_{2s} + \delta_{3s} < 1 \end{array} \right\} \#CS.$$

$$(P1) \quad \hat{x} = \underset{x}{\text{argmin}} \|x\|_1 \quad \hat{x} = x_{\text{true}} \\ \text{st. } y = Ax$$

6) Why P1 easy: convert to LP.

$$\min_{x, u} \sum_{i=1}^N u_i \quad \text{st. } \begin{cases} -u_i \leq x_i \leq u_i & \forall i=1, \dots, N \\ y_j - (Ax)_j \leq 0 & j=1, \dots, K \end{cases}$$

7) Practical aspects: where do we get such A's?

Thm 1: If A is random Fourier,  $k > 3s$ , then

$k = O(s \log N)$  then

A satisfies #CS. whp.

Random Fourier: we pick  $k$  rows of  $F_{N \times N}$ .

iid uniformly..

row 1 ~  $\text{unif}(\frac{1}{\sqrt{N}} \text{ to } \frac{1}{\sqrt{N}})$

row 2 - same way.

where we use MRI, CT (cross-sectional imaging)

capture FT of cross-section one at a time.

$$y = A x$$

$x = \text{sparse} \therefore$  ~~image of an image~~ :

$\downarrow$   
if only edges are of interest.

angiogram & blood vessel image.

b) or  $x = W \theta$        $W = \text{DWT}$

$\theta = \text{sparse}$ .

$$y = A W \theta$$

$A W$  also satisfies Thm 1.

c) Thm 2     $A = \text{random Bernoulli}$  or random  $G$ .

Applic<sup>n</sup>  $\rightarrow$  Sensor net.

channel error decoding

$\rightarrow$  Compressive imaging

Thm 3  $\rightarrow$   $\Delta$  if  $A = \text{random } B$ .

then  $A \not\perp$  any orthon basis

$B$  is also random Bernoulli.