Compressed Sensing

April 29 continued (3)

1) $y = Ax$. A: fat $x: N \times 1$.

$A = [\quad]_{K \times N}$ $K < N$. $N = 256$. $K = 72$.

$\hat{x} = ?$
∞ solutions.

→ ~~min.~~ Regularizⁿ of some kind,

Option 1: Minimum energy: $\min \|x\|_2$ st. $y = Ax$.

$$\hat{x} = A^T (AA^T)^{-1} y.$$

2) Option 2: ~~case~~ I tell you x is a sparse vector.
only $s = 8 << N$. element of x are nonzero.
→ min energy soln ~~is~~ not sparse.
→ why sparsity
↓
reason why JPEG works
↓
Compressibility (most natural signals are sparse approximately

$\underline{z} = \underline{C}\, x$.
Signal DCT matrix → sparse
$y = \tilde{A}x$ $\tilde{A} = AC$

JPEG: DCT of signal, zeros out small coeffs.
↓
gives a lot of compression at small loss of quality ⇒
this assumption holds

3) What can I do: find $\min \|x\|_0$ st $y = Ax$. &

hope to get $\hat{x} = x_{true}$.

When $K > 2S$ & any $2S$, or smaller column s-m of A is full rank.

i.e. $\text{rank}(A_T' A_T) = |T|$. $A_T = [a_{T_1}\ a_{T_2} \ldots a_{T_j}]$

$T = \{T_1, T_2, \ldots T_j\}$.

why this is enough?

say this is not sufficient then

$\exists\ x_1$ & x_2 s.t. : x_1 n.z. on T_1, $|T_1| \le S$,

~~$y = Ax_1$~~ x_2 n.z. on T_2. $|T_2| \le S$.

& $y = Ax_1 = Ax_2$. $\Rightarrow |T_1 \cup T_2| \le 2S$

then $\Rightarrow A(x_1 - x_2) = 0$.

$x_1 - x_2$ n.z. on $T_1 \cup T_2$.

$$\Rightarrow A_{T_1 \cup T_2} \underbrace{(x_1 - x_2)_{(T_1 \cup T_2)}}_{\text{pick rows corresp. to elements in } T_1 \cup T_2} = 0$$

$$\Rightarrow A_{T_1 \cup T_2}^{\dagger} A_{T_1 \cup T_2} \underbrace{(x_1 - x_2)_{(T_1 \cup T_2)}}_{\neq 0} = 0.$$

$\Rightarrow$ $A_{T_1 \cup T_2}$ is <u>not</u> full rank.

this is a contradiction $\therefore$

4) <u>So far</u> ~~mAs~~ $\min \|x\|_0$ st $y = Ax$ works if

$K > 2S$ & A satisfies ~~RNR~~ or ~~RIP(2S)~~

Full rank(2S). $\boxed{\text{RIP}(2S) \text{ slightly stronger}}$

RIP(S_0) : ~~A~~ for any set T with $|T| \le S_0$, ~~$\exists$~~ let δ_{S_0} is smallest quantity s.t.

s.t. $1 - \delta_{S_0} \le \lambda.(A_T' A_T) \le 1 + \delta_{S_0}$ $\forall\ |T| \le S_0$

~~δ_S ...~~ if $\delta_{S_0} < 1$ then

A satisfies RIP(S_0).

5) Above is computationally complex $\therefore$ combinatorial.

<u>Relax</u> $\therefore$ $\min \|x\|_1$ s.t. $y = Ax$.

The soln of (P1) is equal to that of (P0), i.e. P1 gives the ~~sparsest~~ correct solution if

- $\rightarrow$ $K > 3S$
- & $\rightarrow$ A satisfies RIP(3S)
- & $\rightarrow$ $\delta_S + \delta_{2S} + \delta_{3S} < 1$.

} #CS.

(P1) $\hat{x} = \underset{\substack{x \\ \text{st } y = Ax}}{\arg\min} \|x\|_1$ $\quad \hat{x} = x_{true}$.

6) Why P1 easy : convert to LP.

$$\min_{x, u} \sum_{i=1}^{N} u_i \quad \text{s.t.} \begin{cases} -u_i \le x_i \le u_i & \forall\, i = 1 \ldots N \\ y_j - (Ax)_j \le 0 & j = 1, \ldots K. \end{cases}$$

7) Practical aspects : where do we get such A's ?

Thm 1 : If A is random Fourier, ~~& K > 3S~~; & $K = O(S \log N)$ then A satisfies #CS. whp.

random Fourier : pick K rows of $F_{N \times N}$ iid uniformly ..

row 1 ~ ~~unif(...)~~

row 2 — same way.

Where use MRI, CT (cross-sectional imaging)

capture FT of cross-section one at a time

$y = A \underset{\sim}{x}$

x = sparse : ~~image of an organ~~ :

if only edges are of interest.

angiograms - blood vessel image.

b) or $x = W\theta$ $\quad W$ = DWT

θ = sparse.

$y = \underset{\sim}{AW}\,\theta$

AW also satisfies Thm 1

c) Thm 2 $\quad A$ = random Bernoulli or random G.

Applic^n → Sensor nets. $\qquad$ channel error decoding

→ Compressive imaging

Thm 3 ~~B~~. If A = random B.

then $A\Phi$ (Φ any ortho basis)

is also random Bernoulli