EE 424 #2: Time-domain Representation of Discrete-time Signals

January 18, 2011

Contents

READING: § 1.1–§ 1.4 in the textbook[1].

[1] A. V. Oppenheim and A. S. Willsky. *Signals & Systems*. Prentice Hall, Upper Saddle River, NJ, second edition, 1997

Discrete-time (DT) Sinusoids

RECALL THE DEFINITION OF PERIODIC DT SIGNALS:

Definition (Periodic DT signals)**.** *If a signal $x[n]$ satisfies*

$$x[n+N] = x[n] \quad \textit{for all } n$$

then it is periodic with period N. The smallest positive N that satisfies the above equation is called the fundamental period.

Start with a continuous-time (CT) sinusoid:

$$x(t) = A\cos(\omega t + \theta) \quad t \in (-\infty, +\infty)$$

where ω is the analog frequency in radians per second. This CT sinusoid is periodic with fundamental period

$$T_{\text{per}} = 2\pi/\omega$$

for any ω.

Sample with sampling interval $T = 2\pi/\omega_0$ to obtain a DT sinusoid:

$$x[n] = x(t)\big|_{t=nT} = A\,\cos(\omega\, n\, T + \theta).$$

The sampling frequency is ω_0. Define *discrete-time frequency* $\Omega = \omega T = 2\pi \frac{\omega}{\omega_0} = 2\pi \frac{T}{T_{\text{per}}}$ in radians (rad); then,

$$x[n] = A\cos(\Omega n + \theta). \tag{1}$$

This $x[n]$ is *not always* periodic. Define

$$\Omega' = \omega T.$$

Then, $x[n]$ in (1) is periodic with period N if and only if[2]

$$\Omega'(n+N) = \Omega' n + 2\pi m \quad \forall n \in \mathbb{N}$$

for some $m, N \in \mathbb{N}$. Solving this equation leads to

$$\Omega' N = 2\pi m$$

and

$$\Omega' = \omega T = \frac{2\pi m}{N} \quad m, N \in \mathbb{N}$$

i.e. $\Omega' = \omega T$ must be a rational multiple of 2π. The fundamental period of the DT sinusoid in (1) is the smallest positive N satisfying the above condition. To find the fundamental period, express

$$\Omega' = \frac{2\pi m}{N} \quad m, N \in \mathbb{N}$$

using the smallest positive N. Clearly, the discrete-time frequency Ω corresponds to a collection of *rational multiples of* 2π.

[2] Here, $\mathbb{N}$ denotes the set of all integers.

Examples

WE NOW PRESENT EXAMPLES OF SAMPLING CT SINUSOIDS.

Example 1.

$$x_1(t) = \cos(\frac{\pi}{2} t).$$

The frequency of this sinusoid is $\omega = 0.5\pi$ *rad/s and its fundamental period is* $T_{\text{per}} = 2\pi/\omega = 4$ *s.*

Sample $x_1(t)$ *with sampling interval* $T = 1$ *s:*

$$x_1[n] = x_1(t)\big|_{t=nT} = \cos(\frac{\pi}{2} n).$$

The frequency of this DT sinusoid is $\Omega = 0.5\pi$ *rad, which is a rational multiple of* 2π*; hence,* $x_1[n]$ *is periodic.*

To find the fundamental period, express the discrete-time frequency as

$$0.5\pi = \frac{2\pi m}{N} = \frac{2\pi}{4} \quad m, N \text{ integers}$$

using the smallest positive N. The fundamental period is

$$N = 4$$

see Fig. 1.

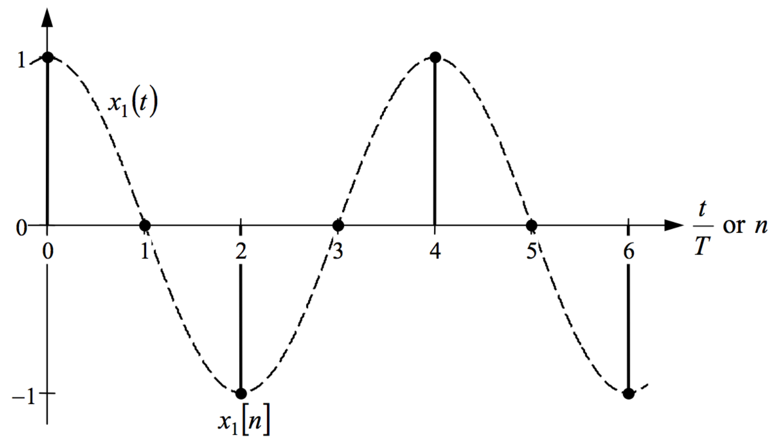


Figure 1: $x_1(t)$ and $x_1[n]$ in Example 1.

Example 2.

$$x_2(t) = \cos(2\,t).$$

The frequency of this sinusoid is $\omega = 2$ *rad/s and its fundamental period is* $T_{\text{per}} = 2\,\pi/\omega = \pi$ *s.*

Sample $x_2(t)$ *with sampling interval* $T = 1$ *s:*

$$x_2[n] = x_2(t)\big|_{t=n\,T} = \cos(2\,n).$$

The frequency of this DT sinusoid is $\Omega = 2$ *rad, which is not a rational multiple of* $2\,\pi$*; hence,* $x_2[n]$ *is not periodic, see Fig. 2.*

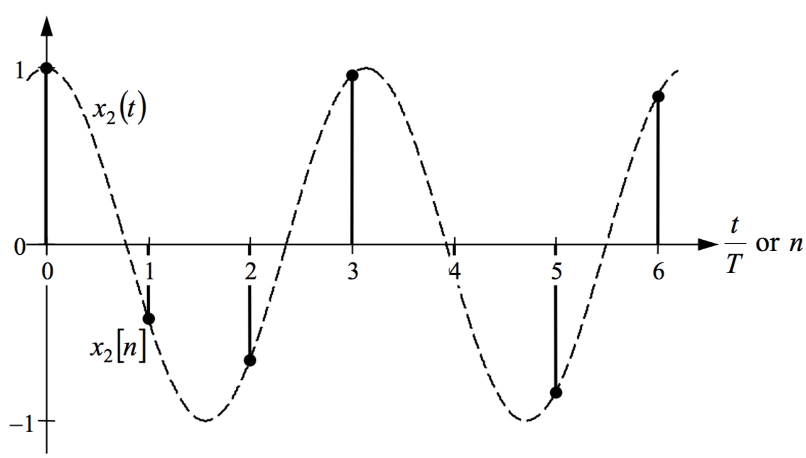


Figure 2: $x_2(t)$ and $x_2[n]$ in Example 2.

Frequency ambiguity

Comments:

- A given DT sinusoid corresponds to samples of CT sinusoids of many different frequencies.

  EXAMPLE:

  $$\left.\begin{aligned} x_1(t) &= \cos(\pi t), \quad \omega = \pi, \quad T_{\text{per}} = 2\pi/\pi = 2 \\ x_2(t) &= \cos(3\pi t), \quad \omega = 3\pi, \quad T_{\text{per}} = 2\pi/(3\pi) = 2/3 \end{aligned}\right] \text{different CT signals}$$

  Sample with sampling interval $T = 1$ s:

  $$\left.\begin{aligned} x_1[n] &= \cos(\pi n), \quad \Omega' = \pi = \tfrac{1}{2} 2\pi, \quad N = 2 \\ x_2[n] &= \cos(3\pi n), \quad \Omega' = 3\pi = \tfrac{3}{2} 2\pi, \quad N = 2 \end{aligned}\right] \text{identical DT signals}$$

  Note that

  $$\Omega = \pi \text{ rad} = 3\pi \text{ rad}.$$

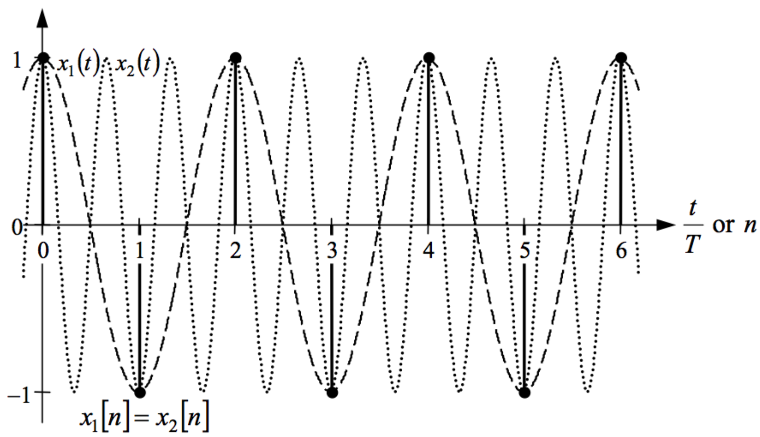


Figure 3: $x_3(t)$ and $x_3[n]$.

- This frequency ambiguity is the origin of aliasing.
- Consider a family of CT sinusoids at frequencies $\omega + k\,\omega_0$ $k \in \mathbb{N}$:

  $$x_k(t) = A \cos\big((\omega + k\,\omega_0)\,t + \theta\big).$$

  where

  $$\omega_0 = 2\pi/T$$

  is the sampling frequency that we will use to sample these sinusoids.

- $x_k(t)$ are distinct signals for different values of k.
- Sample $x_k(t)$ using sampling interval T to obtain a family of DT sinusoids:

  $$\begin{aligned} x_k[n] &= A \cos\big((\omega + k\,\omega_0)\,n\,T + \theta\big) = A \cos(\Omega' n + 2\pi k n + \theta) \\ &= A \cos(\Omega' n + \theta). \end{aligned}$$

  $\Omega' = \omega T$

- CONCLUSION: All CT sinusoids at frequencies $\omega + k\,\omega_0$ $k \in \mathbb{N}$ yield the same DT signal (i.e. same samples) when sampled at the sampling frequency
$$\omega_0 = \frac{2\,\pi}{T}\ \text{rad/s}.$$
- All DT sinusoids at frequencies $\Omega' + k\,2\,\pi$ $k \in \mathbb{N}$ have the same samples. Indeed,
$$\Omega'\ \text{rad} = \Omega' + 2\,\pi\ \text{rad} = \Omega' - 2\,\pi\ \text{rad} = \Omega' + 2 \cdot 2\,\pi\ \text{rad}\ \cdots$$
are all a single DT frequency in radians.
- Based on the above conclusion, two analog frequencies ω_1 and ω_2 are ambiguous after sampling if
$$|\omega_1 - \omega_2| = k\,\omega_0, \quad k\ \text{integer.}$$
Consider a lowpass signal $x(t)$ with spectrum given in Fig. 4. Taking $\omega_1 = -\omega_{\text{m}}$ and $\omega_2 = \omega_{\text{m}}$, we obtain that there is no ambiguity if this $x(t)$ is sampled at
$$\omega_0 > 2\,\omega_{\text{m}}.$$

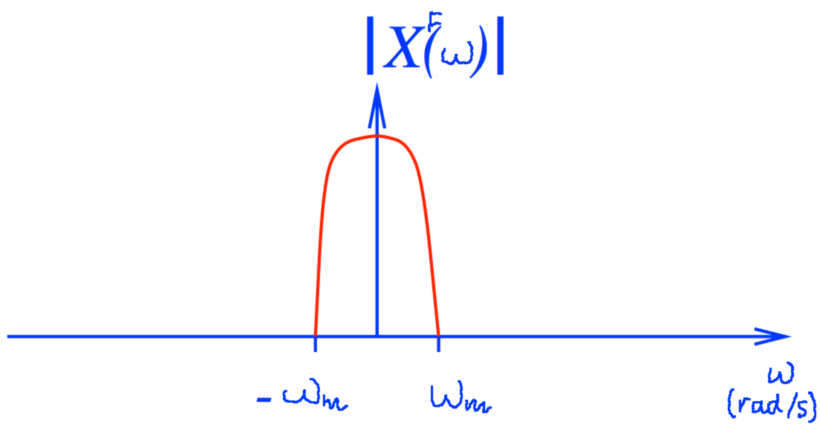


Figure 4: Spectrum of a lowpass CT signal $x(t)$.

- To eliminate this ambiguity, we often restrict ourselves to
  CT frequencies: $-\omega_0/2 < \omega < \omega_0/2$
  DT frequencies: $-\pi < \Omega < \pi$.

Periodicity of a sum of DT sinusoids

THE DT SIGNAL

$$x[n] = A_1 \cos(\Omega_1 n + \theta_1) + A_2 \cos(\Omega_2 n + \theta_2)$$

is periodic provided that both frequencies Ω_1 and Ω_2 are rational multiples of 2π:

$$\begin{array}{l} \Omega_1 = \frac{m_1}{N_1} 2\pi \\ \Omega_2 = \frac{m_2}{N_2} 2\pi \end{array} \quad \text{smallest } N_1 \text{ and } N_2 \Rightarrow \begin{array}{l} N_1 \\ N_2 \end{array} \text{ are fundamental periods.}$$

Then, the fundamental period N of $x[n]$ is the *least common multiple (lcm)* of N_1 and N_2:

$$N = \text{lcm}(N_1, N_2).$$

EXAMPLE:

$$\begin{array}{ll} \Omega_1 = \frac{4}{5} 2\pi & N_1 = 5 \\ \Omega_2 = \frac{7}{2} 2\pi & N_2 = 2 \end{array} \qquad N = \text{lcm}(2, 5) = 10.$$

DT Exponential Functions

DT COMPLEX SINUSOID:

$$x[n] = e^{j(\Omega n + \theta)} = \cos(\Omega n + \theta) + j \sin(\Omega n + \theta).$$

DT REAL-VALUED EXPONENTIAL FUNCTION:

$$x[n] = C\alpha^n \quad \forall n \in \mathbb{N} \quad \alpha \in \mathbb{R}.$$

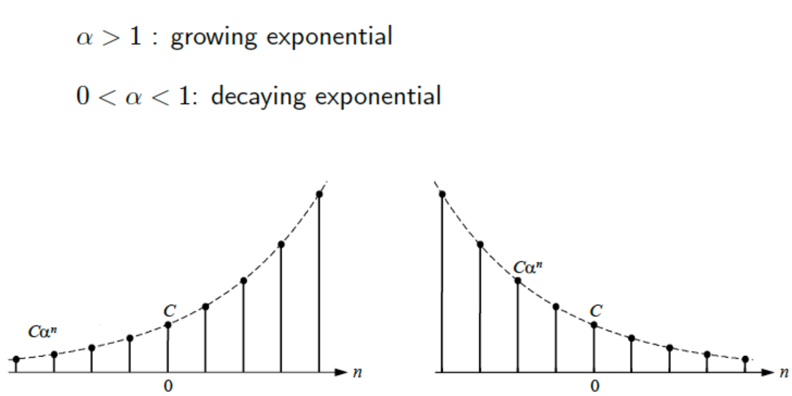


Figure 5: DT exponential functions for $\alpha > 0$.

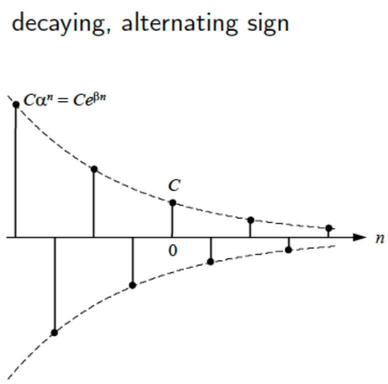


Figure 6: DT exponential functions for $-1 < \alpha < 0$.

DT Singularity Functions

DT Unit Step:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{2}$$

where $n \in \mathbb{N}$, see Fig. 7.

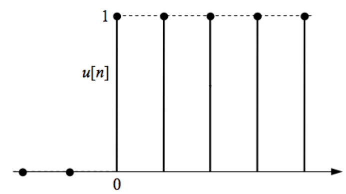


Figure 7: DT unit step. Corresponds to sampled $u(t)$, CT unit step.

DT Unit Ramp:

$$r[n] = n\, u[n] = \begin{cases} 0, & n < 0 \\ n, & n \geq 0 \end{cases} \tag{3}$$

where $n \in \mathbb{N}$, see Fig. 8.

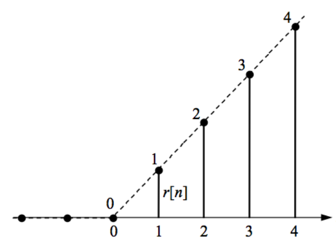


Figure 8: DT unit ramp. Corresponds to sampled $r(t)$, CT unit ramp.

DT Unit Impulse:

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

where $n \in \mathbb{N}$, see Fig. 9.

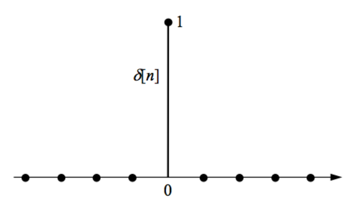


Figure 9: DT unit impulse. *Does not* correspond to sampled $\delta(t)$, CT unit impulse.

Unit impulse is *first difference* of unit step:

$$\delta[n] = u[n] - u[n-1].$$

Unit step is *running sum* of unit impulse:[3]

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$

[3] $\sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 0, & n < 0 \\ 1, & n \geqslant 0 \end{cases}$.

Energy and Power of DT Signals

Energy of a DT signal over all time: is written as[4]

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{4}$$

[4] (4) is computed as follows:

$$E_\infty = \lim_{N \nearrow +\infty} \sum_{n=-N}^{N} |x[n]|^2.$$

AVERAGE POWER OF A DT SIGNAL OVER ALL TIME:

$$P_\infty = \lim_{N \nearrow +\infty} \left(\frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \right).$$

DT Energy Signal: $0 < E_\infty < +\infty, \quad P_\infty = 0$

DT Power Signal: $0 < P_\infty < +\infty, \ E_\infty = +\infty$

There exist signals that are neither energy nor power signals.

PERIODIC DT SIGNALS. Suppose

$$x[n+N] = x[n] \quad \forall n \in \mathbb{N}.$$

Then, for nonzero $x[n]$, $E_\infty = +\infty$ and we can compute the power over one period N as

$$P_N = \frac{1}{N} \underbrace{\sum_{n=\langle N \rangle}}_{\text{any } N \text{ consecutive samples (one period)}} |x[n]|^2.$$

Examples of Energy and Power Calculations

EXAMPLE 1. Find the energy and power of $x[n] = 0.5^{|n|}$ for all n, see Fig. 10.

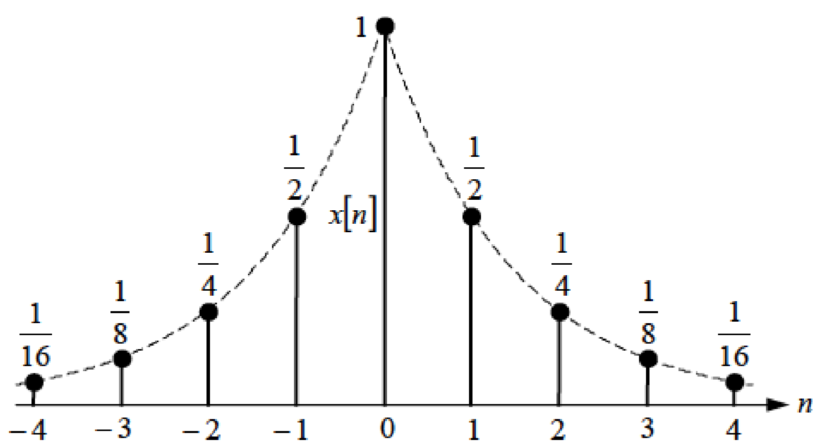


Figure 10: $x[n] = 0.5^{|n|}$ as a function of n.

Energy

$$
\begin{aligned}
E_\infty &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{2|n|} \\
&= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-2n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \\
&= \sum_{n=-\infty}^{-1} \left(\frac{1}{4}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \\
&= E_1 + E_2
\end{aligned}
$$

- E_2: standard geometric series $E_2 = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$
- E_1: graphically we can see $E_1 = E_2 - (1)^2 = \frac{1}{3}$
- To compute E_1 by brute force: put in terms of standard geometric series:

Let $k = -1 - n$

$$E_1 = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k+1} = \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} = \frac{1}{4} \frac{1}{1-\frac{1}{4}} = \frac{1}{3}$$

$$E_\infty = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$$

Power

$$P_\infty = 0.$$

EXAMPLE 2. Find the energy and power of $x[n] = \cos(\Omega_0 n + \theta)$.

Assume $\Omega_0 = \frac{m}{N} \cdot 2\pi$, the period is N, and $0 \leq \Omega_0 \leq \pi$

Energy

$$E_\infty = \infty$$

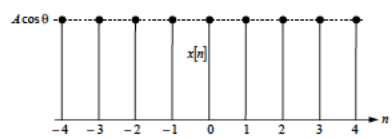
Power

$$P_N = \frac{1}{N} \sum_{n=0}^{N-1} A^2 \cos^2(\Omega_0 n + \theta)$$

$\Omega_0 = 0$

$N = 1 \qquad x[n] = A\cos\theta \quad \forall n$

$$P_N = \frac{1}{1} \sum_{n=0}^{0} A^2 \cos^2\theta = A^2 \cos^2\theta$$

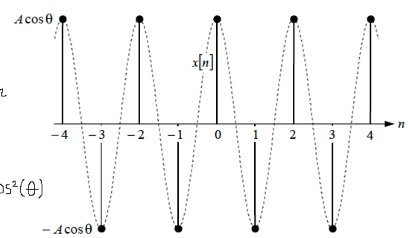


$\Omega_0 = \pi$

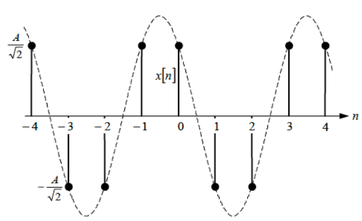
$N = 2$

$x[n] = A\cos(\pi n + \theta) = A\cos\theta(-1)^n$

$$P_N = \frac{1}{2} \sum_{n=0}^{1} A^2 \cos^2\theta(-1)^{2n} = A^2 \cos^2(\theta)$$



$0 < \Omega_0 < \pi$ e.g. $\Omega_0 = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$, $N = 4$



Compute P_N for general N. Recall inverse Euler's formula:

$$\cos(\Omega_0 n + \theta) = \tfrac{1}{2}\left(e^{j(\Omega_0 n + \theta)} + e^{-j(\Omega_0 n + \theta)}\right)$$

and note that we now focus on periodic $x[n]$ with

$$\Omega_0 = 2\pi \frac{m}{N} \quad m = 1, 2, \ldots, N-1. \tag{5}$$

Then

$$\begin{aligned} P_N &= \frac{1}{N}\sum_{n=0}^{N-1} A^2\cos^2(\Omega_0 n+\theta) = \frac{A^2}{4N}\sum_{n=0}^{N-1}(e^{j(\Omega_0 n+\theta)}+e^{-j(\Omega_0 n+\theta)})^2 \\ &= \frac{A^2}{4N}\Big(\sum_{n=0}^{N-1} e^{2j(\Omega_0 n+\theta)}\Big) + \frac{A^2}{4N}\Big(\sum_{n=0}^{N-1} e^{-2j(\Omega_0 n+\theta)}\Big) + \frac{A^2}{4N}2N \\ &= \frac{A^2}{2} + e^{2j\theta}S_1 + e^{-2j\theta}S_2 = \frac{A^2}{2} \end{aligned}$$

where

$$\begin{aligned} S_1 &= \sum_{n=0}^{N-1} e^{2j\Omega_0 n} \overset{z_1=e^{2jm/N}}{=} \sum_{n=0}^{N-1} z_1^n = \frac{z_1^N-1}{z_1-1} = 0 \\ S_2 &= \sum_{n=0}^{N-1} e^{-2j\Omega_0 n} \overset{z_2=e^{-2jm/N}}{=} \sum_{n=0}^{N-1} z_2^n = \frac{z_2^N-1}{z_2-1} = 0 \end{aligned}$$

since $z_1^N = 1$ and $z_2^N = 1$.

To summarize, for $N > 2$, the power of the periodic DT sinusoid

$$x[n] = A\cos\big(2\pi\frac{m}{N}n+\theta\big) \quad m = 0,1,\ldots,N$$

is

$$P_N = \frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \begin{cases} A^2\cos^2\theta, & m=0, N=1 \\ A^2\cos^2\theta, & m=1, N=2 \\ \frac{A^2}{2}, & m=1,2,\ldots,N-1, N>2 \end{cases}.$$

Hence, the power of a periodic DT sinusoid [at frequency Ω_0 in (5)] with period $N > 2$ is

$$P_N = \frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \frac{A^2}{2}$$

which is *not a function of θ, m, or N*.

References

A. V. Oppenheim and A. S. Willsky. *Signals & Systems*. Prentice Hall, Upper Saddle River, NJ, second edition, 1997.