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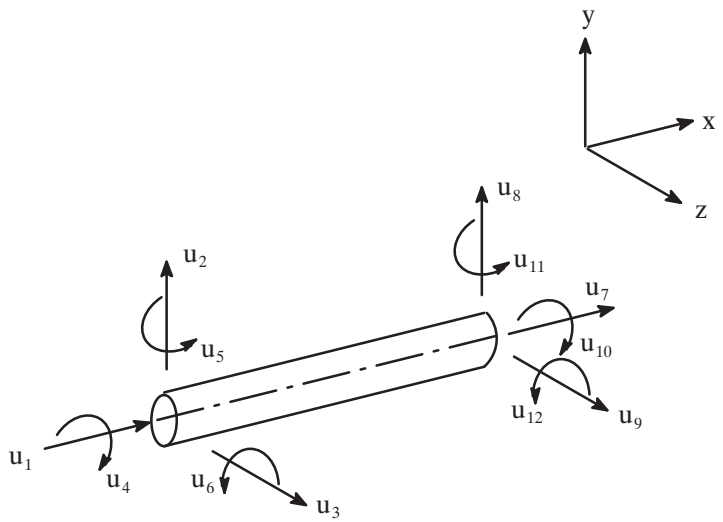
FINITE ELEMENT REVIEW

The finite element equilibrium equation and the values of the various stiffness coefficients developed using virtual work and energy methods in Aero E 321 can also be determined directly using simple beam theory and strength of materials concepts. We will use the nomenclature developed in Aero E 321 for beam elements, with each generalized displacement degree of freedom and each corresponding generalized force being numbered sequentially as illustrated below for displacements for the 12 degree of freedom cases.



In simple beam theory, we assume

1. sufficiently small deflections that the material may be considered as Hookean, (stress proportional to strain, or generalized force proportional to generalized displacement)
2. sufficiently small angular deflections that e.g., $\sin\theta \simeq \theta$.
3. plane cross section remain plane under angular deflections.

These assumptions allow us to linearize the governing equations and assure that there are no cross–interactions (e.g. pure bending does not cause torsion, etc.).

Equilibrium Equation

Consider a simple beam oriented in 3–dimensional space with n (=12) degrees of freedom. The degree of freedom i has a generalized displacement u_i , and a generalized force F_i .

In general the deflections in all n degrees of freedom will contribute to F_i,

$$\text{or} \quad F_i = F_i(u_1, u_2, \ldots u_n) \qquad 1.$$

Using the small deflection assumptions, i.e., assuming a Hookean material, we can use the first term only, of a Taylor series expansion, to write the force change ΔF_i resulting from all deflections Δu_j as:

$$\Delta F_i = \frac{\partial F_i}{\partial u_1}\Delta u_1 + \frac{\partial F_i}{\partial u_2}\Delta u_2 + \frac{\partial F_i}{\partial u_3}\Delta u_3 + \cdots + \frac{\partial F_i}{\partial u_n}\Delta u_n \qquad 2.$$

where the partial derivatives are all constant and evaluated at the undeflected state.

Taking $\Delta F_i \equiv F_i$ and $\Delta u_i \equiv u_i$

the equation can be written:

$$F_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial u_j} u_j$$

If we define $k_{ij} \equiv \frac{\partial F_i}{\partial u_j} = \frac{F_i}{u_j}$ 3.

Then $F_i = \sum_{j=1}^{n} k_{ij} u_j$

If we now consider all generalized forces F_i as a vector $\{F\}$

and all generalized displacements u_i as a vector $\{u\}$

our equation can be written

$$\{F\} = [K]\{u\} \qquad 4.$$

where $[K]$ is the stiffness matrix

Determination of the Stiffness Matrix

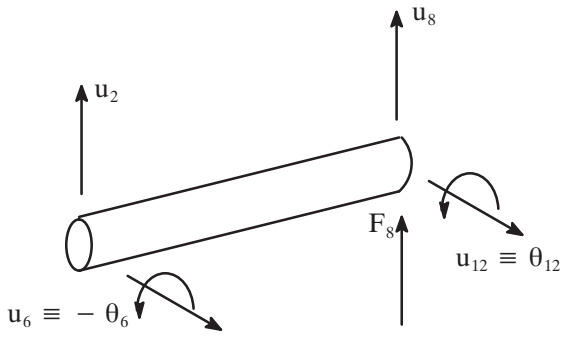
Each of the stiffness terms k_{ij} can be considered to be the generalized force corresponding to degree of freedom i, resulting from a unit displacement of degree of freedom j, with all other degrees of freedom fixed at a value of zero. In other words the generalized force F_8, (vertical force at right end of a beam in the Figure) divided by the generalized displacement u_6, (the angular rotation in the vertical plane at the left end of the beam) with all other deflections held rigid by the appropriate forces and moments, gives us the stiffness k_{86} . This allows us to

choose an appropriate free body diagram and analyse it to determine the stiffness using Equation 3.

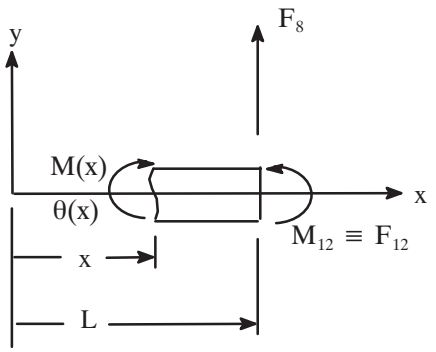
The sign convention used in finite element analysis is different from that used in strength of materials analysis. For the former case, the standard Cartesian convention is used for linear and angular dispacements and for corresponding forces and moments. These are then established by your choice of (x,y,z). For the latter case, the convention is the standard one for positive faces (normal to the surface lies in the positive coordinate direction), but is reversed for negative faces. An important exception is in equations for deflections where x,y, and z are always in the direction of the coordinate axes.

Let us consider two examples:

1. Determination of k_{86}



Finite Element (external forces)



Free–body Diagram (surface traction forces)

$$k_{86} = \frac{F_8}{u_6}$$

Strength of materials analysis tells us that the linear deflection y and the angular deflection $\frac{dy}{dx}$ of a simple beam can be determined from:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad 5.$$

The boundary conditions we need (there are many others) for this problem are:

$$x = 0 \text{ , } y(0) = 0 \quad = u_2 \qquad 6.$$

$$x = L \text{ , } y(L) = 0 \quad = u_8 \qquad 7.$$

$$\left\{\frac{dy}{dx}\right\}_{x=L} \equiv u_{12} = \theta_{12}(L) = 0 \qquad 8.$$

We wish to find $\left\{\frac{dy}{dx}\right\}_{x=0} \equiv u_6 = -\theta_6(0)$ 9.

From the free–body diagram, $M(x) = M_{12} + F_8(L - x)$ (Note: F_8 is +'ve in both systems)

Substituting this into Equation 5 and integrating,

$$EI\frac{dy}{dx} = M_{12}x + F_8\left(Lx - \frac{x^2}{2}\right) + C_1$$

Using B.C. 8, $$0 = M_{12}L + F_8\left(\frac{L^2}{2}\right) + C_1$$

or $$EI\frac{dy}{dx} = M_{12}(x - L) + F_8\left(Lx - \frac{x^2}{2} - \frac{L^2}{2}\right) \quad 10.$$

and Equation 9 becomes $$EIu_6 = -M_{12}L - F_8\frac{L^2}{2} = -EI\theta_6 \quad 11.$$

We now need a relationship between F_6 and M_{12}

and we obtain this by integrating Equation 10 once again,

$$EIy = M_{12}\left(\frac{x^2}{2} - Lx\right) + F_8\left(L\frac{x^2}{2} - \frac{x^3}{6} - \frac{L^2x}{2}\right) + C_2$$

Applying B.C. 6 $$0 = C_2$$

Applying B.C. 7 $$M_{12} = -F_8\frac{L}{3} \quad 12.$$

Substituting Equation 12 into Equation 11, we obtain finally,

$$\frac{F_8}{u_6} = \frac{F_8}{-\theta_6} = \boxed{-\frac{6EI}{L^2} = k_{86}} \quad 13.$$

2. Determination of $k_{12\,6}$

$$k_{12\,6} = \frac{F_{12}}{u_6} = \frac{M_{12}}{-\theta_6} \quad 14.$$

Substitute for F_8 from Equation 12 into Equation 11, and obtain

$$EI\theta_6 = -\frac{L}{2}M_{12}$$

Substituting into Equation 14 then gives

$$\boxed{k_{12\,6} = \frac{2EI}{L}}$$

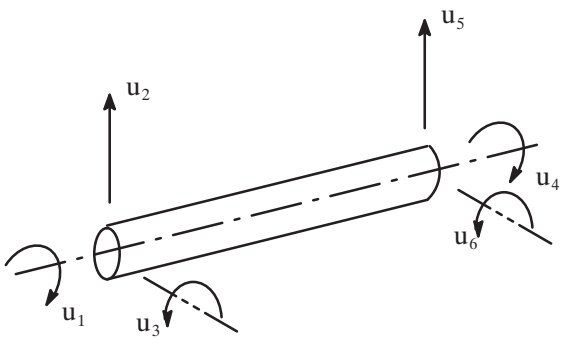
The stiffness matrix for the 12 degree of freedom case is shown on the next page. The student

must be prepared to derive each one using the strength of materials approach as in the examples above.

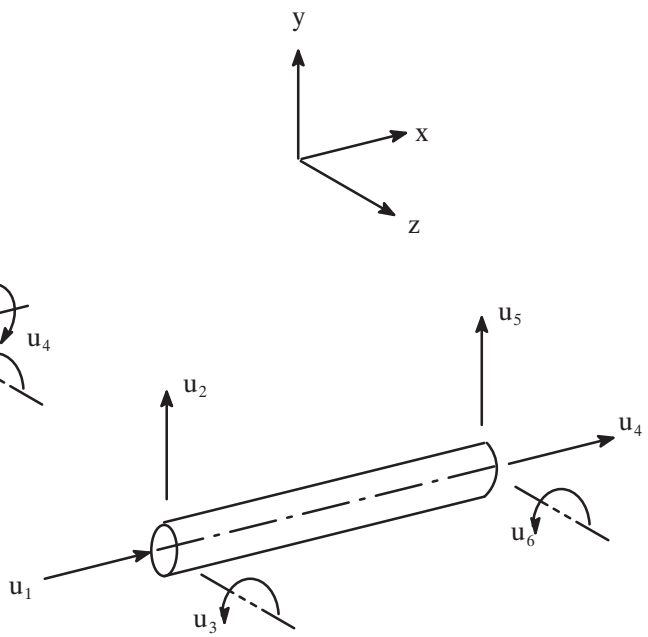
	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{EA}{L}$						$\frac{-EA}{L}$					
2		$\frac{12EI_z}{L^3}$				$\frac{6EI_z}{L^2}$		$\frac{-12EI_z}{L^3}$				$\frac{6EI_z}{L^2}$
3			$\frac{12EI_y}{L^3}$		$\frac{-6EI_y}{L^2}$				$\frac{-12EI_y}{L^3}$		$\frac{-6EI_y}{L^2}$	
4				$\frac{GJ}{L}$						$\frac{-GJ}{L}$		
5			$\frac{-6EI_y}{L^2}$		$\frac{4EI_y}{L}$				$\frac{6EI_y}{L^2}$		$\frac{2EI_y}{L}$	
6		$\frac{6EI_z}{L^2}$				$\frac{4EI_z}{L}$		$\frac{-6EI_z}{L^2}$				$\frac{2EI_z}{L}$
7	$\frac{-EA}{L}$						$\frac{EA}{L}$					
8		$\frac{-12EI_z}{L^3}$				$\frac{-6EI_z}{L^2}$		$\frac{12EI_z}{L^3}$				$\frac{-6EI_z}{L^2}$
9			$\frac{-12EI_y}{L^3}$		$\frac{6EI_y}{L^2}$				$\frac{12EI_y}{L^3}$		$\frac{6EI_y}{L^2}$	
10				$\frac{-GJ}{L}$						$\frac{GJ}{L}$		
11			$\frac{-6EI_y}{L^2}$		$\frac{2EI_y}{L}$				$\frac{6EI_y}{L^2}$		$\frac{4EI_y}{L}$	
12		$\frac{6EI_z}{L^2}$				$\frac{2EI_z}{L}$		$\frac{-6EI_z}{L^2}$				$\frac{4EI_z}{L}$

Stiffness Matrix

Special In–plane 6 degree of Freedom Cases



a) torsion, no axial load



b) axial, no torsion load

$\frac{GJ}{L}$			$\frac{-GJ}{L}$		
	$\frac{12EI_z}{L^3}$	$\frac{6EI_z}{L^2}$		$\frac{-12EI_z}{L^3}$	$\frac{6EI_z}{L^2}$
	$\frac{6EI_z}{L^2}$	$\frac{4EI_z}{L}$		$\frac{-6EI_z}{L^2}$	$\frac{2EI_z}{L}$
$\frac{-GJ}{L}$			$\frac{GJ}{L}$		
	$\frac{-12EI_z}{L^3}$	$\frac{-6EI_z}{L^2}$		$\frac{12EI_z}{L^3}$	$\frac{-6EI_z}{L^2}$
	$\frac{6EI_z}{L^2}$	$\frac{2EI_z}{L}$		$\frac{-6EI_z}{L^2}$	$\frac{4EI_z}{L}$

$\frac{EA}{L}$			$\frac{-EA}{L}$		
	$\frac{12EI_z}{L^3}$	$\frac{6EI_z}{L^2}$		$\frac{-12EI_z}{L^3}$	$\frac{6EI_z}{L^2}$
	$\frac{6EI_z}{L^2}$	$\frac{4EI_z}{L}$		$\frac{-6EI_z}{L^2}$	$\frac{2EI_z}{L}$
$\frac{-EA}{L}$			$\frac{EA}{L}$		
	$\frac{-12EI_z}{L^3}$	$\frac{-6EI_z}{L^2}$		$\frac{12EI_z}{L^3}$	$\frac{-6EI_z}{L^2}$
	$\frac{6EI_z}{L^2}$	$\frac{2EI_z}{L}$		$\frac{-6EI_z}{L^2}$	$\frac{4EI_z}{L}$