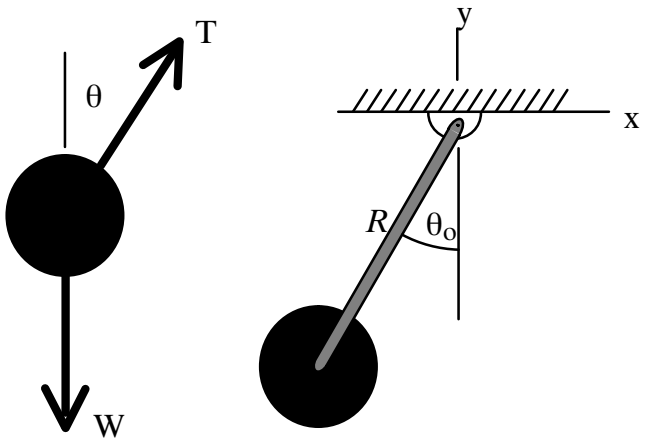
Aero 361 Problem 2

A spherical mass of .25 slugs is hanging still at sea level 3 feet below a frictionless and rigidly supported bearing. The mass is perturbed from this position by rotating it through an angle, θ_o, and then releasing it. We wish to determine the motion of the mass center after release.

We will:

1. develop the equation governing this motion
2. linearize this equation
3. develop the exact solution for θ(t) for the linear equation
4. develop a numerical solution scheme applicable to the linear and nonlinear problems



1. Develop the equation governing this motion

From the above free body diagram, we write:

$$m\ddot{x} = T\sin\theta$$
$$m\ddot{y} = T\cos\theta - W$$

also:

$$x = -R\sin\theta$$
$$y = -R\cos\theta$$

Since W=mg, elimination of T from these two equations yields:

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

2. Linearize the equation

For small values of θ, sinθ $\simeq$ θ, so the equation becomes:

$$\ddot{\theta} \;=\; -\frac{g}{R}\theta$$

3. Exact solution to the linearized equation

The exact solution to the linearized equation is given as:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{R}}t\right)$$

where θ_o is the initial release angle.

4. Develop of a numerical solution scheme

Consider the equation:

$$\dot{z} \quad = f(z) \;\;,\;\; z(0) = z_0$$

A first order integration scheme for this equation is:

$$z^{n+1} = z^n + dt\, f(z^n)$$

A second order scheme is given by the two step scheme:

$$z^* = z^n + dt\, f(z^n)$$
$$z^{n+1} = \frac{1}{2}\left[z^n + z^* + dt\, f(z^*)\right]$$

A fourth order scheme (Runge–Kutta) is given by the four step scheme:

$$z^* = z^n + \frac{dt}{2}\, f(z^n)$$
$$z^{**} = z^n + \frac{dt}{2}\, f(z^*)$$
$$z^{***} = z^n + dt\, f(z^{**})$$
$$z^{n+1} = z^n + \frac{dt}{6}\left[f(z^n) + 2f(z^*) + 2f(z^{**}) + f(z^{***})\right]$$

4.1. Application to a second order equation

We reformulate both our second order linear and nonlinear equations above as systems of first order equations. Consider:

$$\vec{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$$

$$z_1 = \theta$$

$$z_2 = \dot{\theta}$$

$$\dot{\vec{z}} = \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} z_2 \\ -\frac{g}{R} z_1 \end{Bmatrix}_{linear} = \begin{Bmatrix} z_2 \\ -\frac{g}{R} sin(z_1) \end{Bmatrix}_{nonlinear}$$

Now we simply apply one of the above numerical integration schemes to a vector of z values instead of a scalar value.