Numerical Integration

Let's consider the simplest of the differential equations discussed earlier, namely

$$\frac{dx}{dt} = f(t) \quad , \qquad f(t) \text{ known} \quad .$$

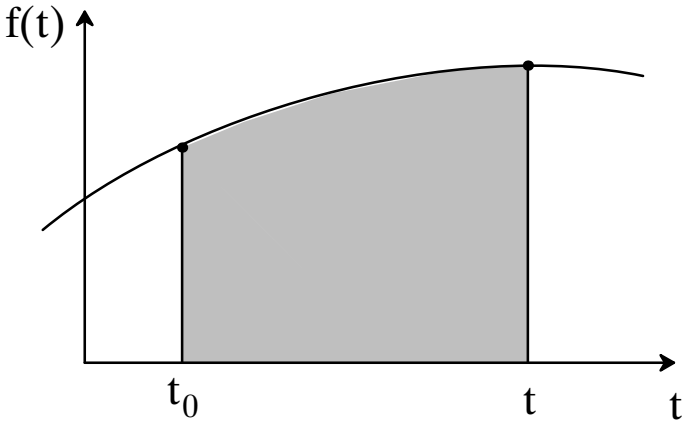
The exact solution for x(t) is given by

$$x(t) - x_0 = \int_{t_0}^{t} f(t)dt \quad ,$$

with the initial condition

$$x(t_0) = x_0 \quad .$$

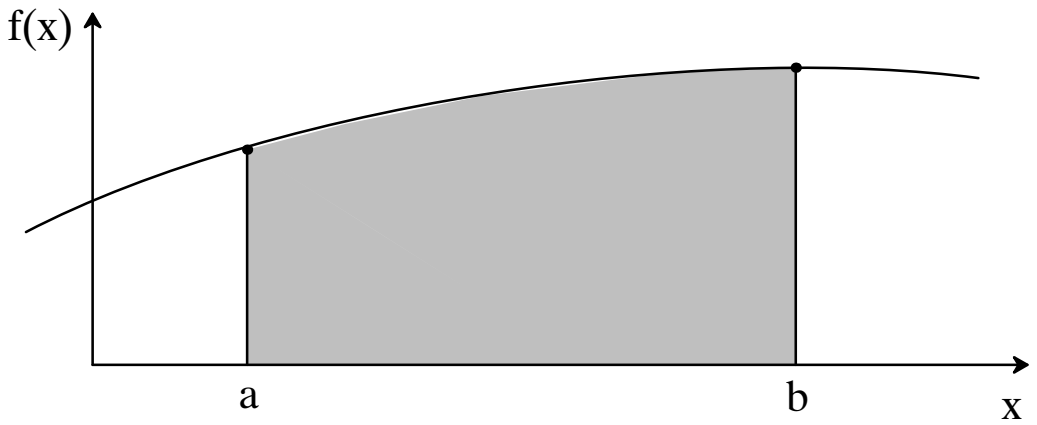
The basic problem here is to compute the integral of some known function f(t) over a fixed interval:



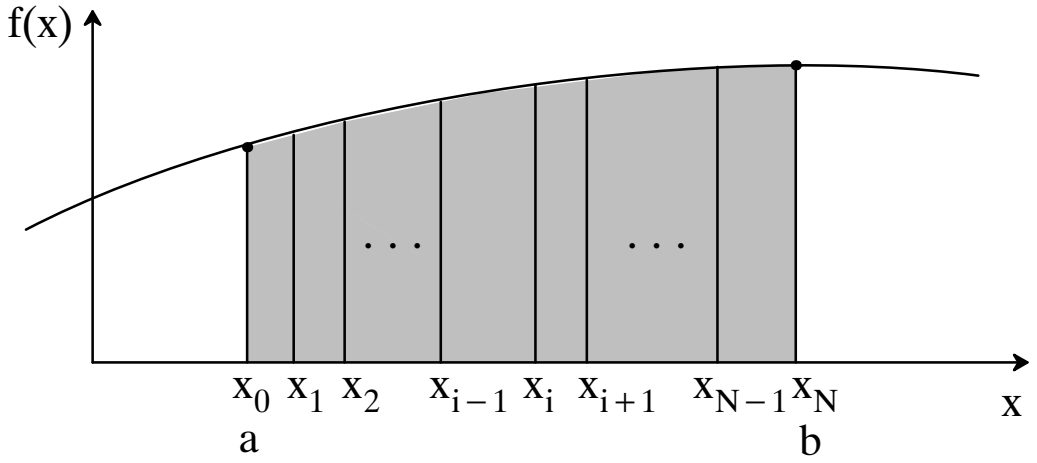
Equivalently we could change the independent variable to x and write the integral in the more familiar form

$$\int_{a}^{b} f(x)dx \quad ,$$

in which case the integration region is



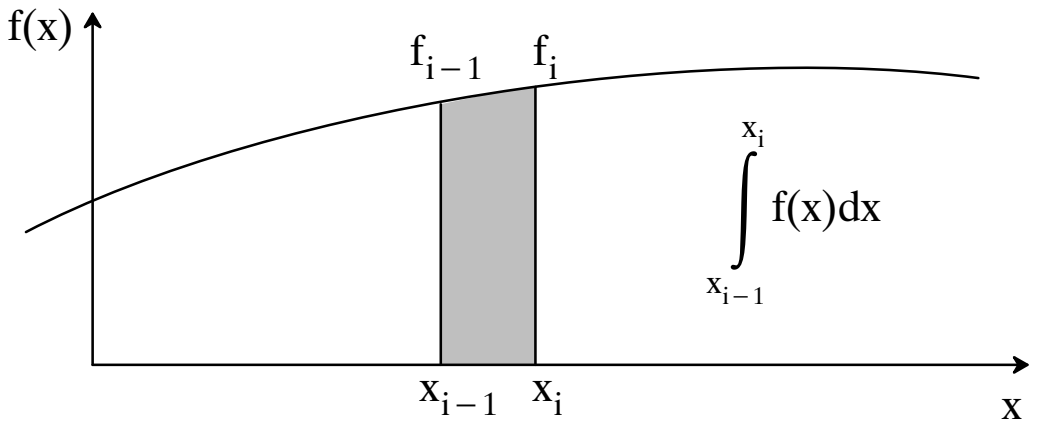
The basic strategy is to divide the interval [a,b] into a number of sub−intervals, or *elements*, and then approximate the function on each element. The interval division is



Notice that the grid points may be equally or unequally spaced. We will do both cases. Given the above grid, and an exact evaluation of the function, the original integral may be written as

$$\int_a^b f(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)dx \quad .$$

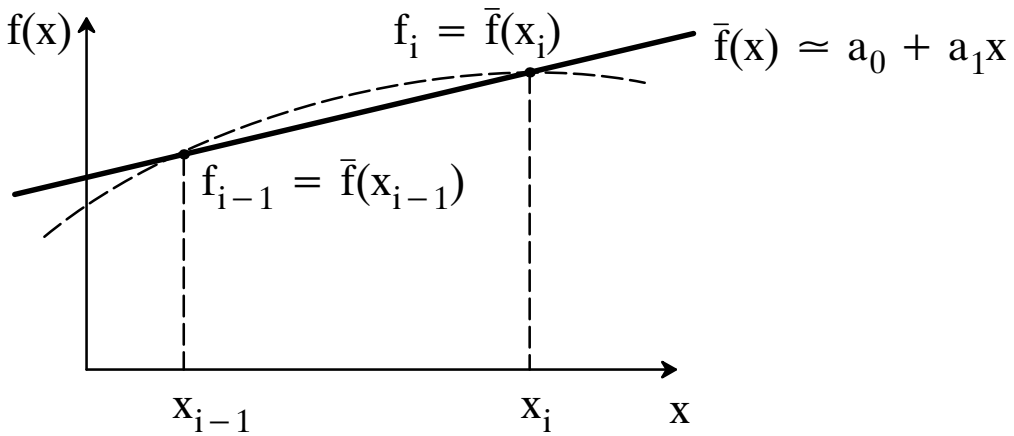
Now consider the integral on an interior element (i.e. one of the terms in the above summation):



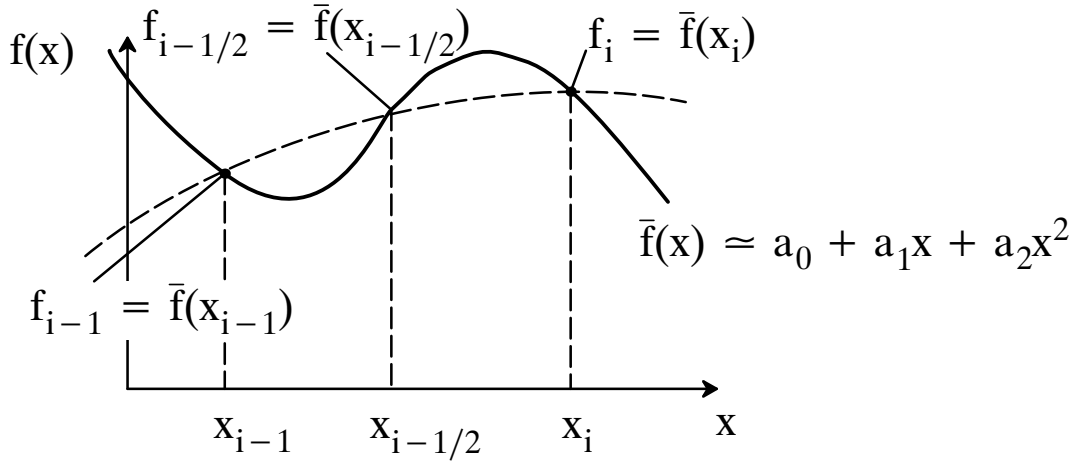
The problem here is that we may not know the function f(x) at all points (often−times we only know f(x) at the element endpoints). In fact, even if we do know the function at every point, the computer can only handle discrete data and we still need to define the function only at a limited number of points. The Newton−Cotes integration formulae approximate f(x) by a polynomial on the element interior, i.e.

$$f(x) \simeq \bar{f}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \ .$$

The a's are found by evaluating the function and/or its derivatives at points where they are assumed to be known. For example, the trapezoidal rule uses a linear interpolating polynomial and evaluates the constants by requiring that the interpolating polynomial take on the function values at the element endpoints



Simpson's 1/3 rule, on the other hand, assumes a cubic interpolating polynomial which matches the function value at three points (due to the fact that there are three unknown constants in a cubic polynomial):



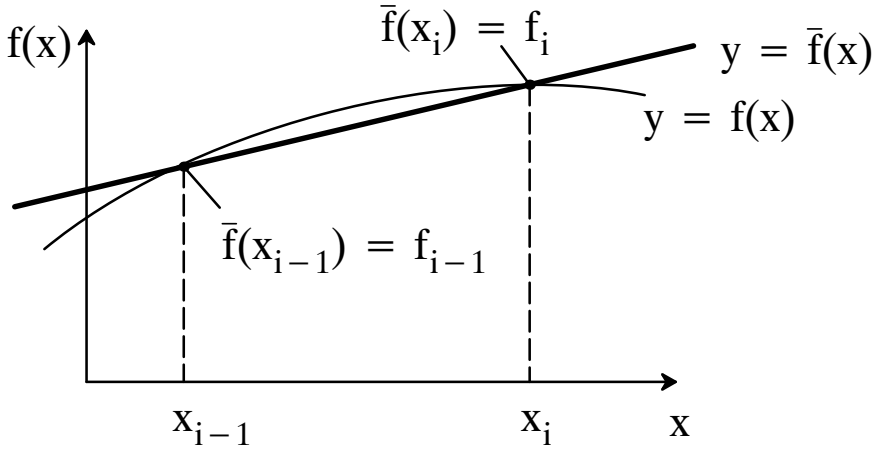
Now let's look at the details of the trapezoidal rule. We will define the approximating function to be

$$\bar{f}(x) = a_0 + a_1x$$

and require that the approximating function take on the known function values at the element endpoints, i.e.

$$\bar{f}(x_i) = a_0 + a_1x_i = f(x_i) = f_i \ ,$$

$$\bar{f}(x_{i-1}) = a_0 + a_1x_{i-1} = f(x_{i-1}) = f_{i-1} \ .$$



This gives two equations for the two unknown a's:

$$a_0 + a_1 x_i = f_i \qquad a_0 + a_1 x_{i-1} = f_{i-1} \quad .$$

Solve the right equation for a_0

$$a_0 = f_{i-1} - a_1 x_{i-1}$$

and plug into the left equation to obtain

$$f_{i-1} - a_1 x_{i-1} + a_1 x_i = f_i \quad .$$

Solving for a_1 gives

$$a_1 = \frac{f_i - f_{i-1}}{x_i - x_{i-1}} \quad .$$

Substituting and solving for a_0 gives

$$\begin{aligned} a_0 = f_{i-1} - a_1 x_{i-1} &= f_{i-1} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}} x_{i-1} \\ &= \frac{f_{i-1}\left(x_i - x_{i-1}\right) - \left(f_i - f_{i-1}\right) x_{i-1}}{x_i - x_{i-1}} \end{aligned}$$

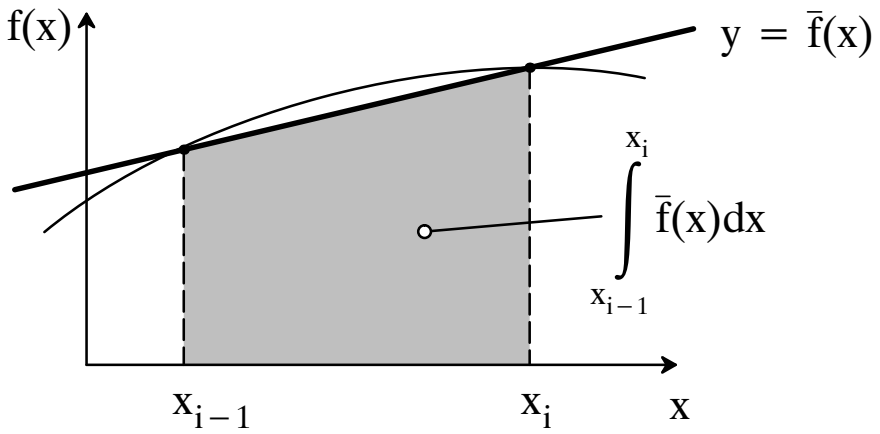
or

$$a_0 = \frac{f_{i-1} x_i - f_i x_{i-1}}{x_i - x_{i-1}} \quad .$$

The interpolating polynomial then becomes

$$\bar{f}(x) = a_0 + a_1 x = \frac{f_{i-1} x_i - f_i x_{i-1}}{x_i - x_{i-1}} + \frac{f_i - f_{i-1}}{x_i - x_{i-1}} x \quad .$$

We can now obtain an approximation to the integral on the element by integrating this approximating function over the element, i.e.



in which case:

$$\int_{x_{i-1}}^{x_i} f(x)dx \simeq \int_{x_{i-1}}^{x_i} \bar{f}(x)dx = \int_{x_{i-1}}^{x_i} \left\{ a_0 + a_1 x = \frac{f_{i-1}x_i - f_i x_{i-1}}{x_i - x_{i-1}} + \frac{f_i - f_{i-1}}{x_i - x_{i-1}} x \right\} dx$$

$$= \left[\left(\frac{f_{i-1}x_i - f_i x_{i-1}}{x_i - x_{i-1}} \right) x + \left(\frac{f_i - f_{i-1}}{x_i - x_{i-1}} \right) \frac{x^2}{2} \right]_{x_{i-1}}^{x_i} = \frac{f_{i-1}x_i - f_i x_{i-1}}{x_i - x_{i-1}} \left(x_i - x_{i-1} \right)$$

$$+ \frac{f_i - f_{i-1}}{x_i - x_{i-1}} \left[\frac{x_i^2 - x_{i-1}^2}{2} \right] = f_{i-1}x_i - f_i x_{i-1} + \frac{f_i - f_{i-1}}{x_i - x_{i-1}} \left[\frac{\left(x_i - x_{i-1} \right)\left(x_i + x_{i-1} \right)}{2} \right]$$

$$= f_{i-1}x_i - f_i x_{i-1} + \frac{\left(f_i - f_{i-1} \right)\left(x_i + x_{i-1} \right)}{2} = \frac{f_{i-1}x_i - f_i x_{i-1} + f_i x_i - f_{i-1}x_{i-1}}{2}$$

The last expression may be factored to give

$$\int_{x_{i-1}}^{x_i} f(x)dx \simeq \frac{\left(f_i + f_{i-1} \right)}{2} \left(x_i - x_{i-1} \right) \quad . \qquad \text{(A)}$$

This equation says that the final approximation for the integral on the element is simply the average of the function values at the element endpoints times the interval length. This is referred to as an *element expression* for the trapezoidal rule.

The Trapezoidal Rule for Unequally Spaced Points:

The total integral is simply a sum of all the integrals over the individual elements, or

$$\int_a^b f(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)dx \quad .$$

Substituting the element expressions for the integrals from the trapezoidal rule formula (A) gives

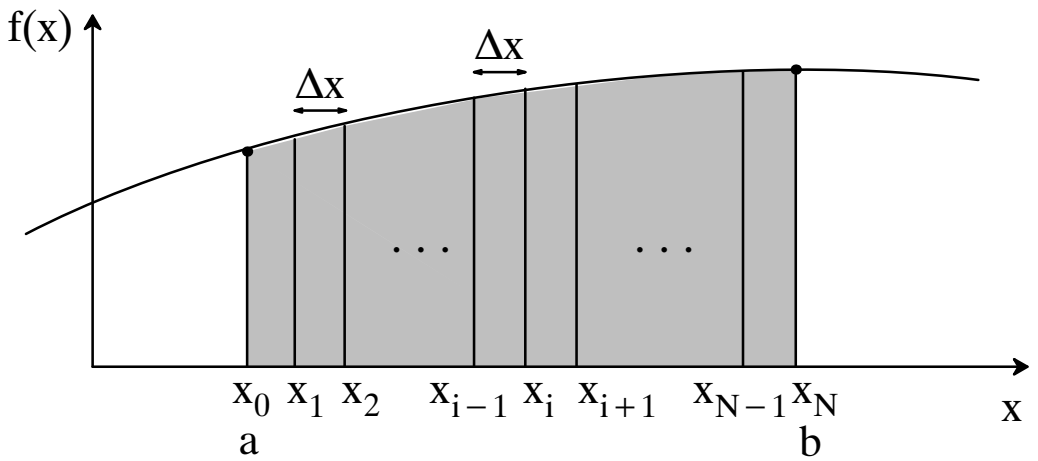
$$\boxed{\int_a^b f(x)dx = \frac{1}{2}\sum_{i=1}^{N}\left(f_i + f_{i-1}\right)\left(x_i - x_{i-1}\right)} \quad .$$

The Trapezoidal Rule for Equally Spaced Points:

Now let's assume that the element endpoints are equally spaced with

$$x_i - x_{i-1} = \Delta x \quad .$$

That is, the grid looks like:



The element expression then becomes

$$\int_{x_{i-1}}^{x_i} f(x)dx \simeq \frac{(f_i + f_{i-1})}{2}(x_i - x_{i-1}) = \frac{(f_i + f_{i-1})}{2}\Delta x \ ,$$

where Δx is constant. The total integral is just a sum of these element expressions, or

$$\int_a^b f(x)dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)dx = \sum_{i=1}^{N} \frac{(f_i + f_{i-1})}{2}\Delta x = \frac{\Delta x}{2}\sum_{i=1}^{N}(f_i + f_{i-1}) \ .$$

Expanding this summation gives

$$\int_a^b f(x)dx = \frac{\Delta x}{2}(f_0 + \underline{f_1 + f_1} + \underline{f_2 + f_2} + f_3 + \cdots + f_{N-2} + \underline{f_{N-1} + f_{N-1}} + f_N)$$

Notice that every interior point appears twice (i.e. the underscored terms), and so the above equation becomes

$$\int_a^b f(x)dx = \frac{\Delta x}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{N-2} + 2f_{N-1} + f_N)$$

or

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left[f_0 + f_N + 2\sum_{i=1}^{N-1} f_i\right]$$

or

$$\boxed{\int_a^b f(x)dx = \Delta x\left[\frac{f_0 + f_N}{2} + \sum_{i=1}^{N-1} f_i\right]} \ .$$