6. THREE−DIMENSIONAL FLOW: Finite Wing, Lift and Induced Drag

We have shown that flow around a wing can be considered to be the result of the free stream velocity, V_∞ and the velocities induced by a horseshoe vortex system composed of a bound vortex and two trailing vortices. The influence of the bound vortex will be much the same as it was for the two−dimensional case, with the Kutta condition being satisfied along the trailing edge. The trailing vortices give us a new effect, however, since they give a net downward flow between the wing tips extending far downstream. This affects both the lift and drag of the wing.

6.1 Lift

Locally, the sectional lift characteristics are nearly the same for the two−dimensional as for the three−dimensional wing, if we neglect, as a first approximation, the effect of flow along the span. Such an assumption is obviously more valid near the wing root than it is near the wing tips. We can then determine the overall lift from the section lift characteristics, if we know the effect of the trailing vortices on the local flow direction. In general, the sectional lift varies along the span, depending upon distributions of wing twist $\alpha(y)$,chord c(y), and section characteristics $\alpha_{l_0}(y)$, $C_{l_\alpha}(y)$ so that

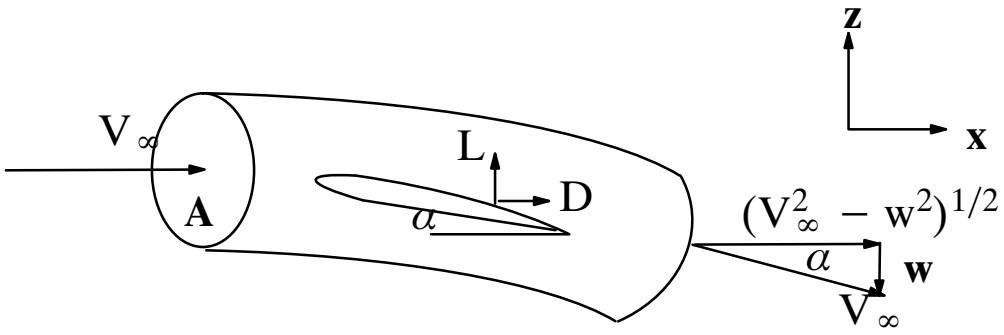
$l = l(y)$ where y is distance along the span

Then the total lift $L = \int_{-\frac{b}{2}}^{+\frac{b}{2}} l(y)dy$ where b is the wing span

or $C_L \equiv \frac{L}{qS} = \frac{1}{S}\int_{-\frac{b}{2}}^{+\frac{b}{2}} C_l c dy$ where S is the wing area

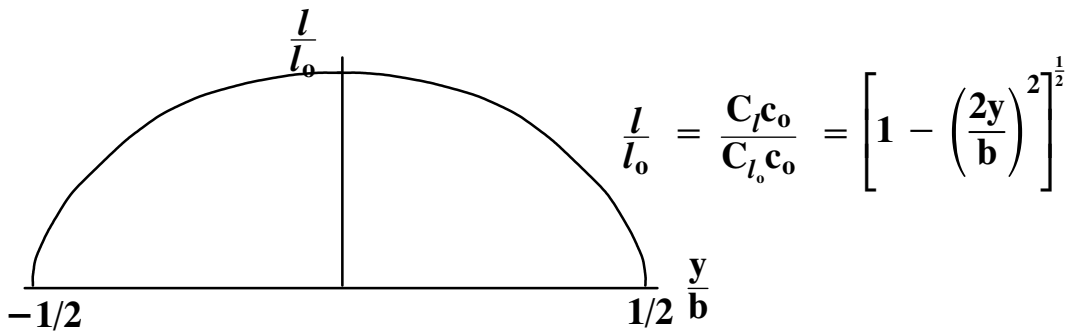
In order to find C_l we need to know how the trailing vortices influence the local flow near each airfoil section, and this we do by analyzing the CV shown below (the same as the one in Chapter 4 of these notes) using the one dimen-

sional momentum equation.



At very large distances upstream and downstream, the value of v_θ will approach zero more quickly than it does for the two−dimensional case, because now the bound vortex has a finite length of b, equal to the wing span and therefore has less influence. This means that if we choose our control volume as in the sketch, then v_θ can be neglected at the inlet and outlet of the CV, which are taken as being *very far* upstream and downstream ($x \to \pm\ \infty$). w is the velocity induced by the trailing vortices and is called "downwash". It is taken positive downwards as for v_θ and we will assume that it is constant along the span at the outlet.

This implies a very special lift distribution, called an elliptical distribution, as sketched below.



This C_l distribution can be obtained by combinations of twist, chord distribution and changing airfoil sections. A classical example where it was achieved through chord distribution is the Spitfire, with its universally−recognized elliptical planform.

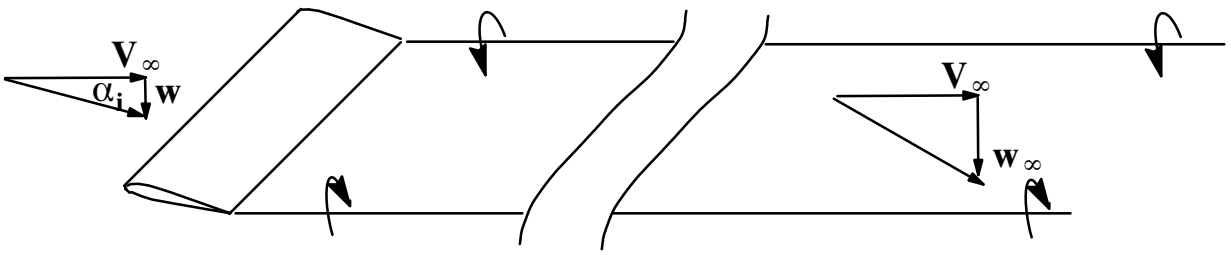
For the want of a better choice, we will take our CV diameter to be equal to the span of the wing so that its cross−sectional area will be

$$A = \pi\frac{b^2}{4}$$

Equation 4.2 then gives us

$$-L = \varrho V_\infty \frac{\pi b^2}{4}(-w_\infty)$$

giving $C_L = \pi AR\frac{w_\infty}{2V_\infty}$ **where the aspect ratio AR** $\equiv \frac{b^2}{S}$ **and S is the wing planform area**



As shown in the above sketch, w_∞ **is the (downwash) velocity at x =** ∞**, and results because of the downward flow that is induced by two trailing vortices that appear at that point in the flow to extend to** ∞ **in both directions. At the wing location, however, the vortices appear only semi−infinite in length and therefore the downwash induced at the wing is only 1/2 the downwash at** ∞**, or**

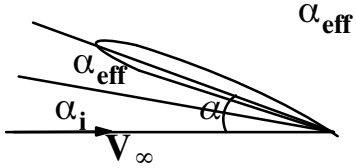
$$w = \frac{w_\infty}{2}$$

From the sketch we also note that

$$\frac{w}{V_\infty} = \alpha_i = \frac{w_\infty}{2V_\infty} = \frac{C_L}{\pi AR} \text{ for small } \alpha_i$$

where α_i **is called the induced angle of attack. The downwash, w, at the wing results in an induced angle of attack** α_i **so that the local airfoil section now "sees" a flow approaching it at an effective angle of attack** α_{eff} **such that**

$$\alpha_{eff} = \alpha - \alpha_i$$



Since we have taken w to be constant along the wing, α_i will also be constant along the wing span.

Recognizing that $\dfrac{dC_l}{d\alpha_{eff}} \equiv m_o = \dfrac{dC_L}{d\alpha_{eff}}$ **for α_i constant**

(m_o is the wing lift−curve slope)

and defining $\dfrac{dC_L}{d\alpha} \equiv m$

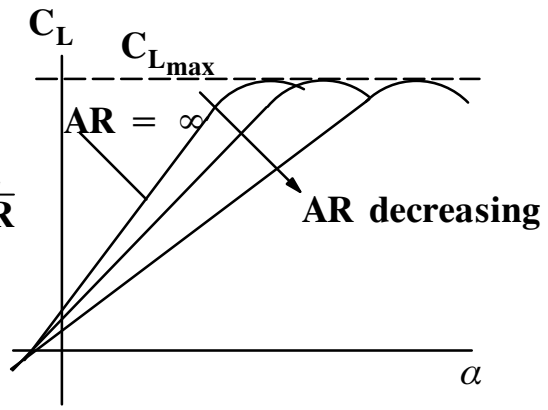
(m is the sectional lift−curve slope)

and using $\alpha_{eff} = \alpha - \alpha_i = \alpha - \dfrac{C_L}{\pi AR}$

we find by differentiating, that

$$\frac{d\alpha_{eff}}{dC_L} = \frac{d\alpha}{dC_L} - \frac{d\alpha_i}{dC_L}$$

or $\dfrac{1}{m_o} = \dfrac{1}{m} - \dfrac{1}{\pi AR}$



Finally $m = \dfrac{m_o}{1 + m_o/(\pi AR)}$

Remember, $m_o = 2\pi$ for a flat plate.

Curves for C_L vs α as a function of AR then appear as sketched above. The effect of the trailing vortices associated with a finite wing, is to reduce the lift−curve slope, and hence, the lift.

Summary of Lift Results:

For an elliptically loaded (constant w) wing:

$$C_L = \frac{1}{S}\int_{-\frac{b}{2}}^{+\frac{b}{2}} C_l c\,dy$$

$$\alpha_i = \frac{C_L}{\pi AR} = \frac{w}{V_\infty}$$

$$m = \frac{m_o}{1 + m_o/(\pi AR)}$$

6.2 Drag

Once we know the lift from the above derivations, we can determine the inviscid drag from three different approaches, which are equivalent:

1) momentum equation
2) induced angle of attack
3) downwash energy.

Since the flow is assumed inviscid, the only source of drag is pressure forces on the wing, and these are associated with the lift. Consequently, we refer to this drag as the lift−induced drag, D_i **.**

6.2.1 Momentum Equation Approach

Equation 4.1 can be written

$$D_i = -\varrho V^2 \frac{\pi b^2}{4}\left[\left(1 - \frac{w_\infty^2}{V_\infty^2}\right)^{\frac{1}{2}} - 1\right]$$

but for small $\frac{w_\infty}{V_\infty}$ **using only the first term of the binomial expansion**

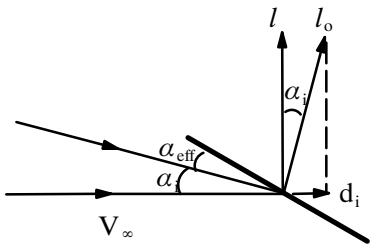
$$\left(1 - \frac{w_\infty^2}{V_\infty^2}\right)^{\frac{1}{2}} \simeq 1 - \frac{w_\infty^2}{2V_\infty^2}$$

$$\text{Then} \quad \frac{D_i}{\frac{\rho}{2} V_\infty^2 S} \simeq \pi AR \left(\frac{w_\infty}{2V_\infty} \right)^2$$

$$\text{But} \quad C_L = \pi AR \frac{w_\infty}{2V_\infty}$$

$$\text{Hence} \quad C_{D_i} = \frac{C_L^2}{\pi AR}$$

6.2.2 Induced Angle of Attack Approach



The free−stream flow direction at the wing is rotated downwards by the angle α_i, so that the local airfoil section "sees" an apparent angle of attack of $\alpha_{eff} = \alpha - \alpha_i$. The local section then generates a lift force l_o perpendicular to the apparent flow direction and a drag force, $d_o = 0$ along the apparent local flow direction. Lift force is of course defined as the force perpendicular to V_∞ and the drag lies in the direction of V_∞. Referring to the sketch above, then

$$l = l_o \cos \alpha_i \simeq l_o \quad \text{for small } \alpha_i$$

$$\text{and} \quad d_i = l_o \sin \alpha_i \simeq l_o \alpha_i \simeq l \alpha_i$$

$$\text{hence} \quad C_{d_i} = C_l \alpha_i$$

If we integrate over the whole wing with α_i = constant (elliptic)

$$C_{D_i} = C_L \alpha_i = C_L \frac{w}{V_\infty}$$

$$\text{giving} \quad C_{D_i} = \frac{C_L^2}{\pi AR} \quad \text{as before}$$

6.2.3 Downwash Energy Approach

After the wing has passed by, it leaves behind a wake which consists of a downward flow between the trailing vortices. The kinetic

energy associated with this downwash velocity w_∞, is lost energy which has been supplied by the work done by the induced drag force. Equating the work done by D_i per unit time, to the rate of deposit of kinetic energy into the wake, we obtain:

$$D_i V_\infty = \frac{1}{2}\dot{m}w_\infty^2 = \frac{1}{2}\varrho V_\infty \frac{\pi b^2}{4} w_\infty^2 = \frac{1}{2}\varrho V_\infty^3 \pi b^2 \left(\frac{w_\infty}{2V_\infty}\right)^2$$

$$\text{or} \qquad C_{D_i} = \frac{C_L^2}{\pi AR} \quad \text{once again}$$

The expression for the induced drag, as developed above is known as the Drag Polar. This drag is added to all other forms of drag to get the total drag of a wing. For example, if we include the effects of viscosity we whould include the friction and pressure drag of the airfoil sections or

$$C_{d_p} = C_{d_{pressure}} + C_{d_f} \qquad \text{where } C_{d_p} \text{ is the sectional profile drag}$$

and the total drag becomes

$$C_D = C_{D_p} + C_{D_i} \qquad C_{D_p} = \frac{1}{S}\int C_{d_p} c\, dy$$

6.3 Non−Elliptic Lift Distribution

The derivation above was applied to a wing with an elliptic lift distribution. In the case of a non−elliptic lift distribution, which in turn of course gives a downwash which varies along the span ($w = w(y)$ and $w_\infty = w_\infty(y)$), we must apply a correction to the results of the previous pages. This is done by introducing the Oswald span efficiency factors e and e_1:

$$C_{D_i} = \frac{C_L^2}{\pi eAR}$$

$$m = \frac{m_o}{1 + \dfrac{m_o}{\pi e_1 AR}}$$

A more exact analysis than we used here is required to find expressions for e and e_1:. Their values depend upon the details of the wing sections, twist, taper, etc.