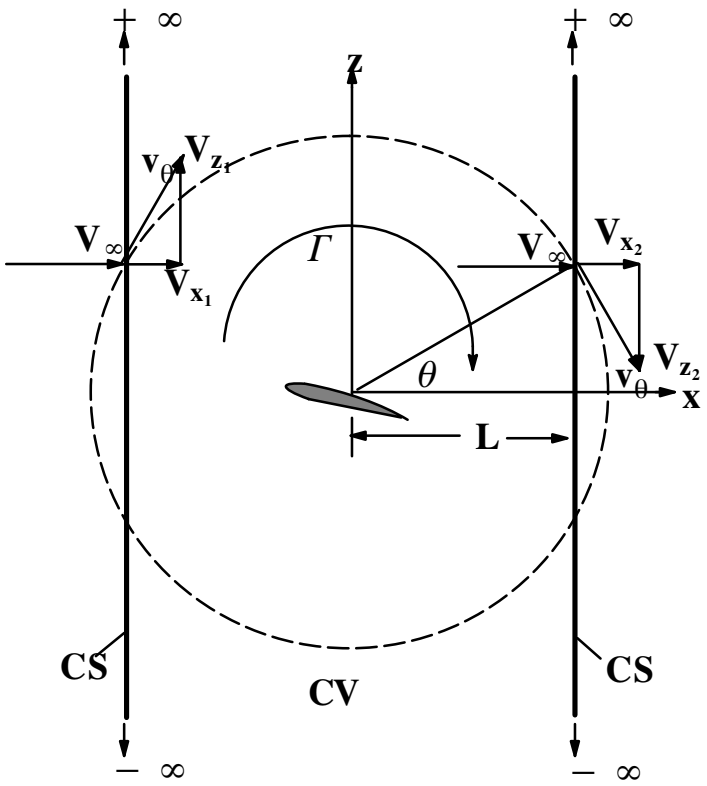
5. TWO−DIMENSIONAL FLOW: LIFT OF AIRFOILS

5.1 Kutta−Joukowski Theorem

Once steady flow is estblished on an airfoil in incompressible flow, we can assume we have inviscid conditions. At very large distances from the body, the influence of the lifting body will appear as a circulating flow (generated by the bound vortex for which $2\pi r v_\theta = \Gamma$, superimposed upon the free−stream (with velocity V_∞).



Since the influence of Γ is felt far from the body, we will assume a C.V. of unit depth as illustrated above. The upstream and downstream surfaces are sufficiently far from the body, that the pressure along them may be taken as the free−stream pressure p_∞. The surfaces are then taken to extend from $z = -\infty$ to $z = +\infty$. We may then write the momentum equation using equation (2.5) as:

$$-d = F_x = (\dot{m}V_x)_2 - (\dot{m}V_x)_1$$

$$= \int_{-\infty}^{+\infty} \varrho_\infty V_{x_2} V_{x_2} dz - \int_{-\infty}^{+\infty} \varrho_\infty V_{x_1} V_{x_1} dz$$

where terms in V_∞ cancel

From the sketch above, at z, $V_{x_1} = V_{x_2}$ by symmetry so that $d = 0$ (d is the drag/unit span) **5.1**

Similarily

$$-l = F_z = \int_{-\infty}^{+\infty} \varrho_\infty (V_{x_2} V_{z_2} - V_{x_1} V_{z_1}) dz$$

again, terms in V_∞ cancel

Now, returning to the sketch, we see that since

$V_{x_1} = V_{x_2} = V_\infty + v_\theta \sin\theta$ **and** $V_{z_2} = -V_{x_1} = -v_\theta \cos\theta$ **then**

$V_{x_2} V_{z_2} - V_{x_1} V_{z_1} = -2(V_\infty + v_\theta \sin\theta) v_\theta \cos\theta$

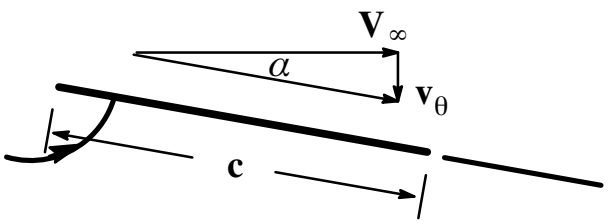
where $\cos\theta = \dfrac{L}{(L^2 + z^2)^{\frac{1}{2}}}$ **and** $\sin\theta = \dfrac{z}{(L^2 + z^2)^{\frac{1}{2}}}$

If we substitute these values into the expression for l and further take $rv_\theta = \dfrac{\Gamma}{2\pi}$ we will obtain

$$l = \varrho V_\infty \Gamma$$ ***(l* is the lift per unit span)** **5.2**

Equations 5.1 and 5.2 constitute the Kutta−Joukowski Theorem and directly relate the lift and drag of a two−dimensional body in incompressible flow to the circulation developed around it.

5.2 Estimating the Circulation, Γ



The circulation required to move the rear stagnation point down to the trailing edge can be estimated using the velocity triangle sketched above. Note that we have replaced the airfoil by a flat plate, so our results will now apply only to a very thin airfoil.

We assume that the velocity w at the trailing edge is just sufficient to deflect the free stream velocity through the angle α, i.e.,

$$v_\theta = V_\infty \sin\alpha \simeq V_\infty \alpha \qquad \text{for } \alpha \text{ small}$$

This movement occurs at the distance of about c/2 from the center of the airfoil and therefore from the center of the bound vortex. Hence, an angular momentum is given to the flow field of the amount

$$rv_\theta = \frac{c}{2} v_\theta = \frac{c\alpha V_\infty}{2}$$

Then $$\Gamma = 2\pi r v_\theta = \pi c V_\infty \alpha$$

Substituting into the Kutta−Joukowski equation,

$$l = \varrho V_\infty \Gamma = \varrho_\infty V_\infty^2 \pi c \alpha$$

$$C_l \equiv \frac{l}{qc} = 2\pi\alpha \quad \text{where } \alpha \text{ is in radians and } q = \frac{\varrho}{2} V^2$$

or $$\frac{dC_l}{d\alpha} \equiv C_{l_\alpha} \equiv m = 2\pi$$ **, the lift−curve slope**

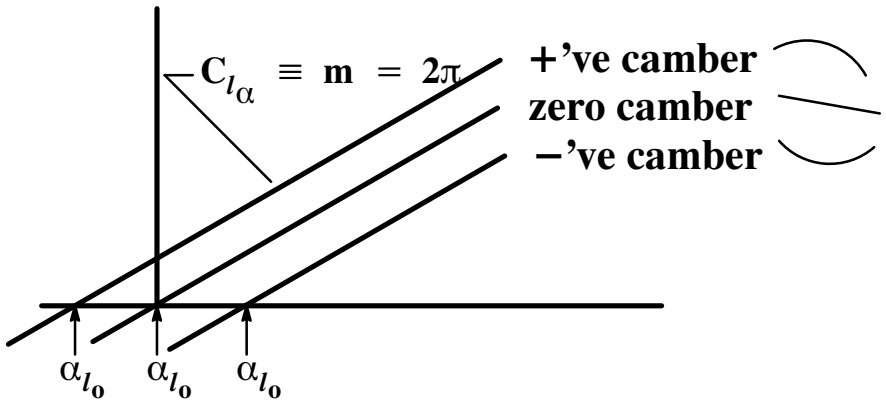
The lift curve slope for a flat plate, and for a very thin airfoil is 2π per radian.

If the plate is curved, then $C_l \neq 0$ **at** $\alpha = 0$
and we must write $C_l = 2\pi(\alpha - \alpha_{l_o})$
where $\alpha = \alpha_{l_o}$ **when** $C_l = 0$.

A curved plate is said to have positive camber if it is convex upwards and negative camber when it is convex downwards.

for positive camber $\alpha_{l_o} < 0$

for negative camber $\alpha_{l_o} > 0$

Does this make sense to you physically?

The results are summarized by the sketch below.