SUPPLEMENTARY NOTES IN AERODYNAMICS

for

AeroE 241

L.N. Wilson

Aerospace Engineering Department
Iowa State University, Ames, IA

Revised January 8, 1995

TABLE OF CONTENTS

	page
1. THE EQUATIONS OF MOTION	**1**
1.1 The Conservation Principles	**1**
1.2 Application to Fluids	**1**
1.3 Steady Flows	**2**
1.3.1 Mass Conservation	**2**
1.3.2 Momentum Conservation	**2**
1.3.3 Energy Conservation	**2**
1.4 Mathematical Form of Conservation Equations	**3**
1.4.1 Mass Conservation or Continuity Equation	**3**
1.4.2 Momentum Equation	**3**
1.4.3 Energy Equation	**4**
1.5 Incompressible Fluid, Constant Density Assumption	**4**
2. APPLICATIONS TO ONE–DIMENSIONAL FLOWS	**4**
2.1 One–Dimensional Flows	**4**
2.1.1 Streamlines and Streamtubes	**4**
2.1.2 One–Dimensional Flow Assumption	**5**
2.2 1–D Continuity Equation	**5**
2.3 1–D Momentum Equation	**6**
2.4 Aerodynamic Forces	**6**
2.5 Variable Conditions Across Inlet and Outlet	**8**
3. THE DIFFERENTIAL EQUATIONS OF MOTION	**8**
3.1 Continuity	**8**
3.2 Momentum – Eulers Equation, Bernoulli's Equation	**8**
3.2.1 Inviscid Flow Assumption	**8**
3.2.2 Streamline Equations	**9**

4. THE GENERATION OF LIFT **10**
4.1 The Origin of Lift Forces **10**
4.1.1 Finite Wing **11**
4.1.2 Airfoil **11**
4.2 Circulation **12**
4.2.1 The Kutta Condition **12**
4.2.2 Wing Vortex System **13**
4.3 Summary **14**

5. TWO–DIMENSIONAL FLOW: LIFT OF AIRFOILS **15**
5.1 The Kutta_Joukowski Theorem **15**
5.2 Estimating the Circulation **16**

6. THREE–DIMENSIONAL FLOW:
Finite Wing, Lift and Induced Drag **17**
6.1 Lift **17**
6.2 Drag **20**
6.2.1 Momentum Equation Approach **21**
6.2.2 Induced Angle of Attack Approach **22**
6.2.3 Downwash Energy Approach **22**
6.3 Non–Elliptic Lift Distribution **22**

7. PRESSURE FORCES AND MOMENTS ON AN AIRFOIL **23**
7.1 Forces **23**
7.2 Moments **24**

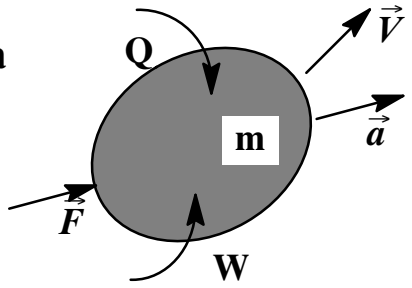
1. THE EQUATIONS OF MOTION

1.1 The Conservation Principles

The basic laws of motion are formulated with reference to a fixed quantity, or identity of matter.

Consider a free−body of mass m, moving at a velocity $\vec{V}$ and acted on by a force $\vec{F}$. This force results in an acceleration $\vec{a} = \frac{d\vec{V}}{dt}$

Suppose too, that an amount of work W, is done on the body, and that a quantity of heat Q, flows into the body.



The following three basic Laws, or Conservation Principles are then applied to determine the ensuing behavior of the body:

Name	Word Statement	Mathematical Statement
Mass Conservation	Mass cannot be created or destroyed	$m = \text{constant}$
Momentum Conservation (Newton's 2nd Law of Motion)	Force applied to a body increases its momentum at a rate proportional to the force	$\vec{F} = k\frac{d(m\vec{V})}{dt}$
Energy Conservation (First Law of Thermodynamics)	The internal (thermal) energy of a body is increased by the flow of energy (heat) across its surface and by the work done on it	$\Delta U = Q + \frac{W}{J}$

By using consistent sets of fundamental units, such as the SI or the English slug−foot−second system as we will throughout these notes, the constant of proportionality k, can be taken as unity. Joule's constant (J) is also not required if Q and W are in the same units.

Using mass conservation (m = constant), Newton's 2nd Law may be written in its more common form:

$$\vec{F} = m\frac{d\vec{V}}{dt} = m\vec{a}$$

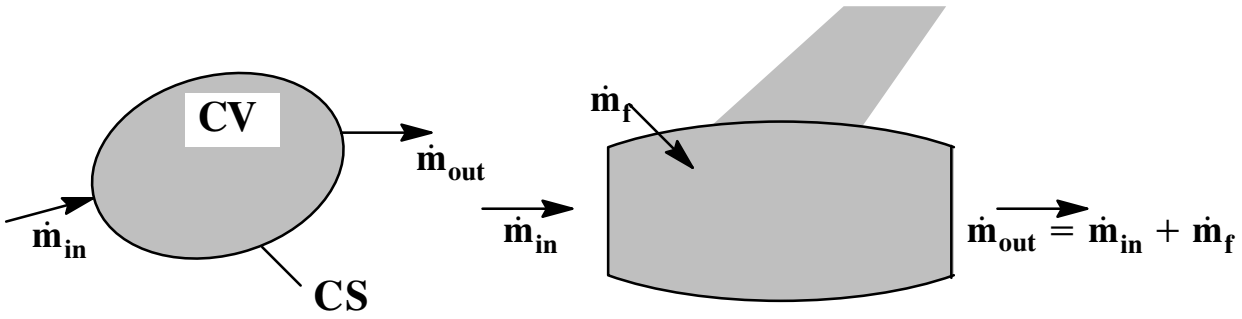
1.2 Application to Fluids

A solid material generally retains its integrity and does not break up into smaller pieces . If it does, the equations above still hold but it is more difficult to analyze such cases, because we must follow the motion of all of the pieces. This problem always exists for a fluid. As it moves, a single parcel of fluid not only distorts into very irregular shapes under the action of external forces, but it also breaks up into numerous, smaller parcels of fluid. We therefore find it mathematically more tractable, not to consider a fixed mass of fluid in our analyses, but rather consider a fixed volume in space, and analyze the flow of the fluid through it. Obviously, now the mass of fluid in the volume changes in time and we must re−word the conservation principles above.

It is still possible to analyze a fluid flow by considering a fixed mass of fluid, and following it, but it is generally more difficult for the reasons given above. The fixed mass approach is called the "Control Mass" approach, abbreviated as CM here. When we consider a fixed volume, we refer to the analysis as "Control Volume" approach, abbreviated CV here.

1.3 Steady Flow (s)

In Engineering, when we apply results of analysis to real problems we have to make certain assumptions in order to solve the problems. In each of the equations we develop here, we will indicate assumptions involved by using a code to indicate them directly following the equation. The first of these we will present is the steady flow assumption, and for it we use the code "s" as indicated in parentheses in the heading of this section.



1.3.1 Mass Conservation

Referring to the sketch above, we see that the mass within a CV is increased if there is a surplus of fluid flowing into it across its boundary, called the Control Surface, CS. If conditions exist such that no changes occur in time, within the CV or at any point on the CV, i.e., the influx of mass just balances the efflux, then we call the flow a "steady flow." Such conditions are very closely approximated in a large number of important Engineering problems. An example would be a jet engine operating on an aircraft flying straight and level with no accelerations. The rate at which mass flows from the exhaust equals the rate at which mass flows into the inlet plus the rate at which fuel mass is supplied to the engine.

We will assume steady flow for the remainder of these notes.

1.3.2 Momentum Conservation

For steady flow, momentum within the control volume will be constantin time too. The momentum that exists in the fluid within the CV will however be determined by the action of force $\vec{F}$. The momentum will be just the difference between the rate of momentum flow out of the CV and the rate of momentum flow into the CV, both flowing across the CS.

1.3.3 Energy Conservation

In the same way, energy of the fluid in the CV will not change in time for steady flow, but its value will be determined by the heat added and work done. Now, however, we must consider not only the internal (thermal) energy of the fluid, but also its kinetic energy and its potential energy. The energy for steady flow, however, will again be just the difference between the efflux of

energy and the influx of energy and will be balanced by the heat added and work done at the boundary of the CV.

Re−wording the Conservation Principles for steady flow through a control volume then, we can say:

Conserved Quantity	Word Statement
Mass	**All fluid flowing into the system flows out at the same rate.**
Momentum	**A force *on* a control volume *increases* the net rate of momentum flow out of the system relative to that flowing in. The rate of increase is proportional the force applied.**
Energy	**Heat added *to*, and work done *on*, a contol volume *increase* the net rate of energy flow out of the system, relative to that flowing in. The increase is proportional to the sum of the rate at which heat is added and work is done at the control surface.**

Note that the conservation laws for a CV are concerned with the flow *rate* of the conserved quantities. In other words we are dealing with the *rate* at which mass, momentum and energy are convected past a surface (the CS) fixed in space.

1.4 Mathematical Form of Conservation Equations

1.4.1 Mass Conservation or Continuity Equation

If a CV system has multiple inlets and outlets, then the mass conservation principle may be written

$$\Sigma(\dot{m})_{in} = \Sigma(\dot{m})_{out} \qquad 1.1 \quad (s)$$

where $\dot{m}$ is the mass flow rate, and the summations are carried out over all in(lets) and out(lets)

and $\dot{m} = \left\{\frac{dm}{dt}\right\}_{CS}$

1.4.2 Momentum Equation

Similarly,

$$\vec{F} = \Sigma\left\{\frac{d(m\vec{V})}{dt}\right\}_{out} - \Sigma\left\{\frac{d(m\vec{V})}{dt}\right\}_{in}$$

But $$\left\{\frac{d(m\vec{V})}{dt}\right\}_{CS} = m\frac{d\vec{V}}{dt} + \vec{V}\frac{dm}{dt}$$

$$= 0 + \vec{V}\dot{m}$$

since from the steady state assumption $\frac{d\vec{V}}{dt} = 0$ on the CS

$$\therefore \vec{F} = \Sigma(\dot{m}\vec{V})_{out} - \Sigma(\dot{m}\vec{V})_{in} \qquad 1.2 \quad (s)$$

Remember, $\vec{F}$ is the *total* force acting on the fluid in the CV.

1.4.3 Energy Equation

$$\dot{Q} + \dot{W} = \Sigma(\dot{E})_{out} - \Sigma(\dot{E})_{in} \qquad 1.3 \quad (s)$$

where $\dot{E} = \dot{U} + \dot{m}V^2/2$

= internal energy flow rate + kinetic energy flow rate

Note that the energy and continuity equations are scalar equations, whereas the momentum equation is a vector equation.

1.5 Incompressible Fluid, Constant Density Assumption (inc)

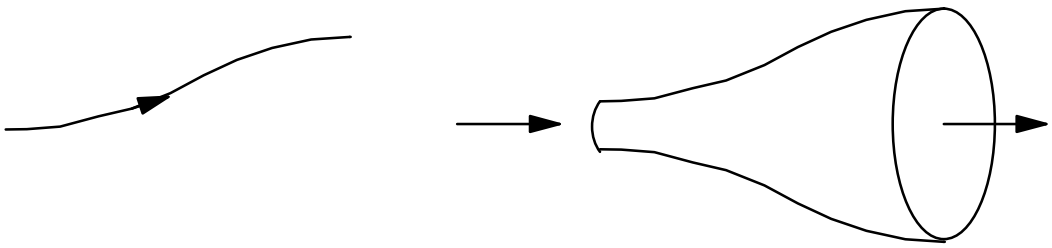
The density of a fluid under conditions of small or moderate temperature change, will itself change only because of compression due to pressure change. Liquids do not compress easily and for engineering purposes it is acceptable to assume that they are incompressible over a wide range of conditions. It is also usual to assume that they remain at constant density if temperature changes are small.

Gases are considerably more easily compressed than are liquids but we find that we can still assume they are incompressible as long as pressure changes are limited. They can also be assumed to be of constant density if temperature changes are small. Since pressure changes in a gas result from velocity changes, the constant density, or incompressible assumption can only be made for low speed gas flows. In air, a good rule of thumb is to limit the assumption to $M < 0.4$, where the Mach number M is the ratio of flow speed to speed of sound in air.

2. APPLICATIONS TO ONE−DIMENSIONAL FLOWS

2.1 One−Dimensional Flows

2.1.1 Streamlines and Streamtubes



streamline **streamtube**

A streamline is a line in a flow which is parallel to the local velocity vector. In other words, it is a line across which there is no flow. In steady flows, this is the same as the "particle path", the path followed by individual small parcels of fluid.

A streamtube is a tubular surface through which the fluid flows. Its surface is composed of streamlines, so that there is no flow across the stream

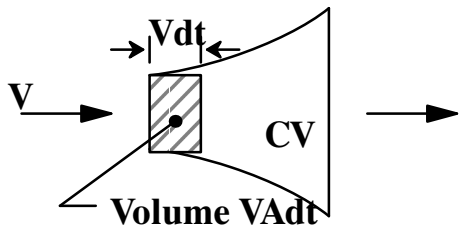
tube surface. It also serves as a convenient CV and CS for our purposes here.

2.1.2 One−Dimensional Flow Assumption (1D)

In many problems, we can assume either that conditions such as velocity, pressure, etc. are constant across the inlet and outlet of a streamtube, or that we are dealing with suitably averaged quantities. In this case, flow variables will be functions only of distance s, along the stream tube or streamline. We refer to this as a 1−D assumption and refer too, to 1−D flows of fluids. All the variables in the conservation equations for steady flows are then functions of only one distance variable s, and the equations are simplified considerably.

2.2 1−D Continuity Equation

We will apply the 1−D assumption to Equation 1.1 as follows:



Referring to the sketch above, in time dt, a volume of fluid VAdt crosses the inlet of area A. If the density of the fluid is ϱ , then the mass dm flows in, where

$$dm = \varrho VAdt$$

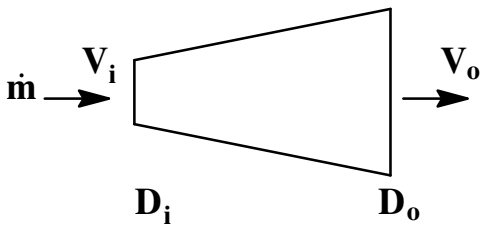
$$\text{or} \quad \frac{dm}{dt} = \varrho VA \equiv \dot{m} = \text{constant} \qquad \textbf{2.1 (s,1D)}$$

When the incompressible assumption is valid, this becomes

$$VA = \text{constant} \qquad \textbf{2.1a (s,1D,inc)}$$

Example:

Water at stp flows through a varying cross−section pipe at a rate of 20 lbm/s. What is the average flow speed at the pipe exit where the diameter is 1 inch?



Given: $\dot{m} = 20$ lbm/s $= 20/32.2$ slugs/s
$D_o = 1$ in $= 1/12$ ft
$\varrho_i = \varrho_{stp}$

Determine: V_o

Assume: steady, incompressible, 1−D flow

Equations & Constants:

Use: equation 2.1 $\dot{m} = \varrho_o V_o A_o$ or $V_o = \dfrac{\dot{m}}{\varrho_o A_o}$ (s,1D)

$$A_o = \frac{\pi D_o^2}{4} \quad \text{(i)}$$

$$\varrho_o = \varrho_i = 62.4/32.2 \text{ slugs/ft}^3$$

Solution: $V_o = \dfrac{\dot{m}}{\varrho_o A_o} = \dfrac{20/32.2}{(62.4/32.2)\pi(1/12)^2/4} = 58.8$ ft/s

2.3 1−D Momentum Equation

Using the 1−D assumption, Equation 1.2 becomes

$$\vec{F} = \dot{m}_{out}\vec{V}_{out} - \dot{m}_{in}\vec{V}_{in}$$

Using Equation 2.1, this becomes, for the case of a single inlet and single outlet

$$\vec{F} = \dot{m}(\vec{V}_{out} - \vec{V}_{in}) \qquad 2.2 \quad (s,1D)$$

Equation 2.2 is a vector equation which can be written in terms of its 3 Cartesian directions, (x,y,z).

$$\Sigma F_x = \dot{m}(V_{out_x} - V_{in_x}) \qquad 2.3a \quad (s,1D)$$

$$\Sigma F_y = \dot{m}(V_{out_y} - V_{in_y}) \qquad 2.3b \quad (s,1D)$$

$$\Sigma F_z = \dot{m}(V_{out_z} - V_{in_z}) \qquad 2.3c \quad (s,1D)$$

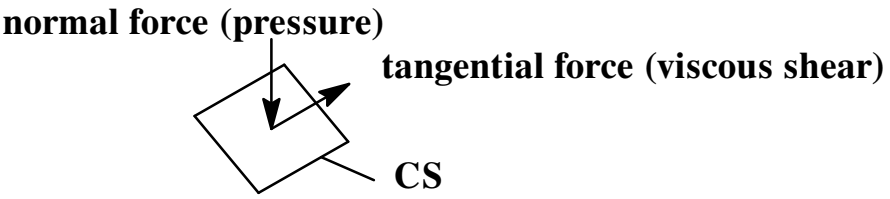
where ΣF_x , ΣF_y , and ΣF_z are the components of the total force acting on the CV

2.4 Aerodynamic Forces

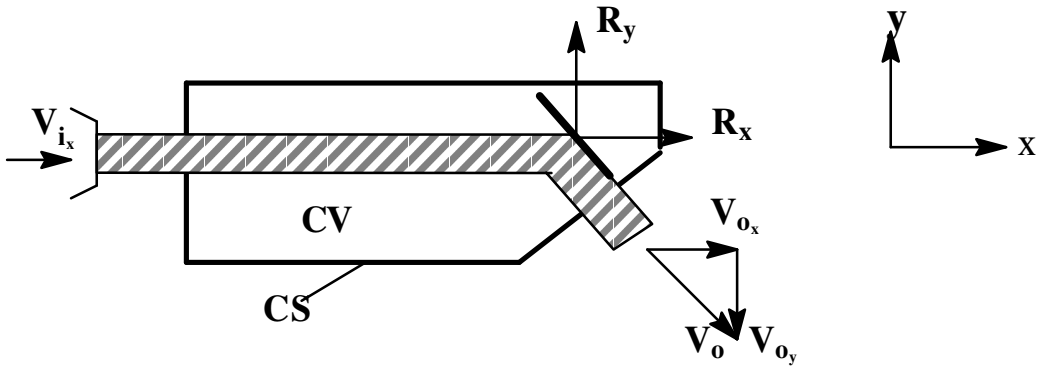
In order to use Equations 2.3, it is necessary to prescribe the total forces acting on the CV. In general such forces can be divided into two types, surface forces and body forces. As their names imply, surface forces act on the CS, whereas body forces act within or on the whole CV. A common example of body forces in fluids are those which result from gravity. It is these forces that give rise to the changes of pressure with altitude in the atmosphere, and are imporant in flows where there are large height changes. They are also important in liquids and are the source of buoyancy forces which keep boats afloat. Although we could include them in our analysis, we will exclude them for simplicity. They are not usually important in aerodynamics. They are the dominant forces however, in studies of the behavior of the atmosphere.

In these notes then, we are interested only in surface forces. In particular we are interested in what we classify as aerodynamic forces, and these are of two types, (i) pressure or normal forces and (ii) viscous or tangential shear forces.

Pressure forces are oriented normal to, and inwards towards the surface on which they act, while viscous forces act tangentially to the surface.



The key to solving problems involving the momentum equation is to choose the appropriate CV. If the surface of a CV is chosen such that it passes through a region where there are negligible velocity gradients, then we can neglect viscous forces there. This does not require that we neglect viscous effects inside the CV, but rather only on its surface. An example is a jet of water hitting an inclined plate as sketched below.

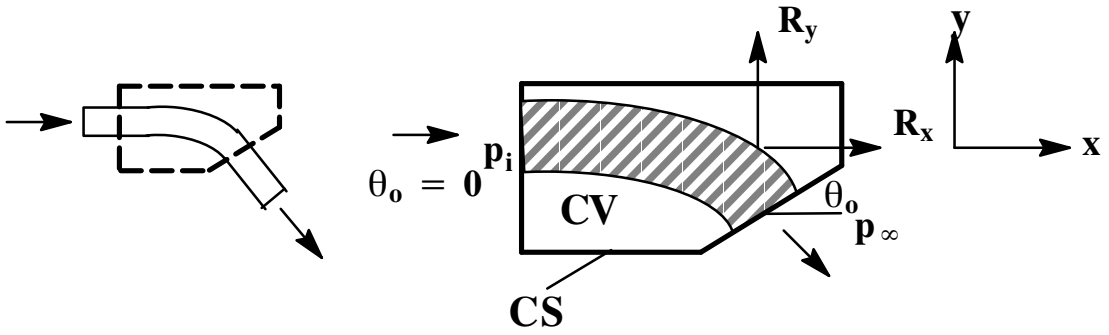


For the CV choice above, there are no appreciable viscous forces on *any* CS.

If we now consider the pressure forces, we can see that the pressure is the same on every surface as well. A free low−speed jet has no forces acting on it (when we neglect gravity), and therefore the pressure is constant through it and equal to the ambient pressure. Since the pressure is constant everywhere, there are no pressure forces acting on the surface of the CV. The only forces, then, are the viscous and pressure forces acting *on the fluid* by the plate. If we call this force $\vec{F}$, and the force *on the plate* $\vec{R}$, then equation 2.3a and 2.3b become:

$$F_x = \dot{m}(V_{o_x} - V_{i_x}) = -R_x$$
$$F_y = \dot{m}(-V_{o_y} - 0) = -R_y$$

Another example is the reaction force on a pipe bend.



In this case, the pressure on the inlet and outlet portion of the surfaces of the CV chosen, are not equal to the ambient pressure and net pressure forces are present on the CV surface. It is only the difference in pressure relative to ambient, $p_i - p_\infty$ and p_o that give pressure forces. These are called gauge pressures. Once again there are only negligible viscous forces on any of the surfaces of the CV.

Now the total force on the CV can be written as:

$$\sum F_x = (p_i - p_\infty)A_i \cos\theta_i - (p_o - p_\infty)A_o \cos\theta_o - R_x = \dot{m}(V_{o_x} - V_{i_x})$$

$$\sum F_y = (p_i - p_\infty)A_i \sin\theta_i - (p_o - p_\infty)A_o \sin\theta_o - R_y = \dot{m}(-V_{o_y} - 0)$$

This same approach can be applied to the analysis of the thrust of a rocket or jet engine or propellor. These will be studied as examples in class.

2.5 Variable Conditions Across Inlet and Outlet

In the analysis so far developed, we have assumed that V and p are constant across the inlets and outlets. If we cannot make that assumption, then we must integrate across these surfaces. If we do this we must change our equations using the following:

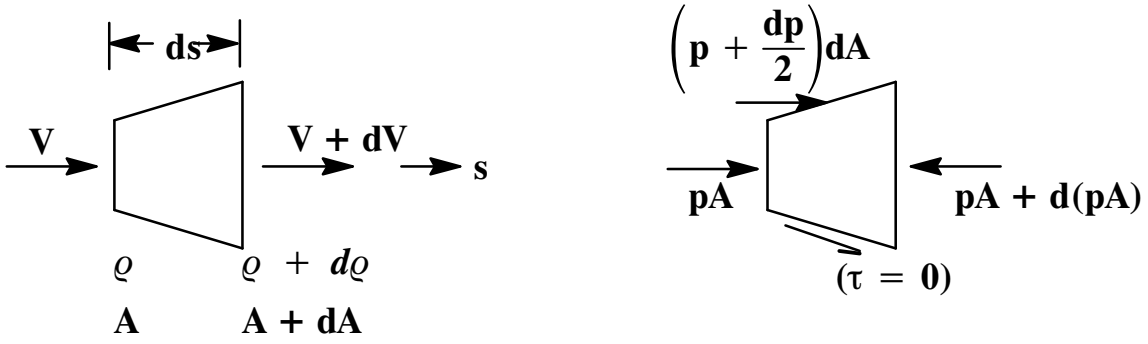
Continuity

$$\dot{m}_o = \int_{A_o} \varrho V_o dA_o \; ; \quad \dot{m}_i = \int_{A_i} \varrho V_i dA_i \qquad 2.4$$

Momentum

$$\text{Pressure forces: } F_{p_x} = \int_{A_o, A_i} p dA_x \; ; \; \text{Momentum fluxes: } \int_{A_o, A_i} \varrho V V_x dA \qquad 2.5$$

3. THE DIFFERENTIAL EQUATIONS OF MOTION − Steady, 1 D



3.1 Continuity

Consider as a control volume, a differential stream tube of length ds, inlet area A, and outlet area A + dA as sketched above. Equation 2.1 now becomes:

$$d(\varrho VA) = 0 \quad \text{or} \quad \frac{d\varrho}{\varrho} + \frac{dV}{V} + \frac{dA}{A} = 0 \qquad \text{3.1a (s,1D)}$$

If we assume constant density, this becomes:

$$d(VA) = 0 \quad \text{or} \quad \frac{dV}{V} + \frac{dA}{A} = 0 \qquad \text{3.1b (s,1D,inc)}$$

3.2 Momentum − Eulers Equation, Bernoulli's Equation

3.2.1 Inviscid Flow Assumption (inv)

If there are no large velocity gradients in the flow, within the CV or on the CS, we can assume there are negligible viscous forces present. This is called the inviscid assumption (inv) and we commonly speak about inviscid flows and treat viscous effects as a separate subject. In these notes we will deal only with inviscid flows and will treat viscous flows separately in class.

3.2.2 Streamline Equations

Having made the assumption of inviscid flow, the only force acting on our differential volume is pressure force. The pressure force on the inlet is pA, and the force on the outlet is pA + d(pA). There is also an average pressure (p+dp/2) acting on the total sloping surface whose area component normal to s is dA. This gives a further force in the s−direction of (p+dp/2)dA. Equation 2.3a now becomes:

$$pA + \left(p + \frac{dp}{2}\right)dA - pA - d(pA) = \varrho VA(V + dV - V)$$

$$-Adp = \varrho VAdV \quad \text{neglecting dpdA}$$

$$dp + \varrho VdV = 0 \qquad \text{3.2a (s,1D,inv)}$$

Equation 3.2a is known as the Euler Equation, and applies along a streamtube or streamline in steady, inviscid flows. If we now also assume constant density, equation 3.2a can be written:

$$dp + d\left(\frac{\varrho V^2}{2}\right) = 0$$

$$p + \frac{\varrho}{2}V^2 = \text{constant} = p_o \qquad \text{3.2b (s,1D,inc,inv)}$$

Equation 3.2b is called the Bernoulli Equation, and the constant, known as Bernoulli's constant, can be determined from the stagnation pressure p_o, the value of the pressure where V = 0. The Bernoulli equation applies along a streamline in steady, incompressible, inviscid flows. The various terms in the equation are identified as follows:

$$p \equiv \text{static pressure}$$

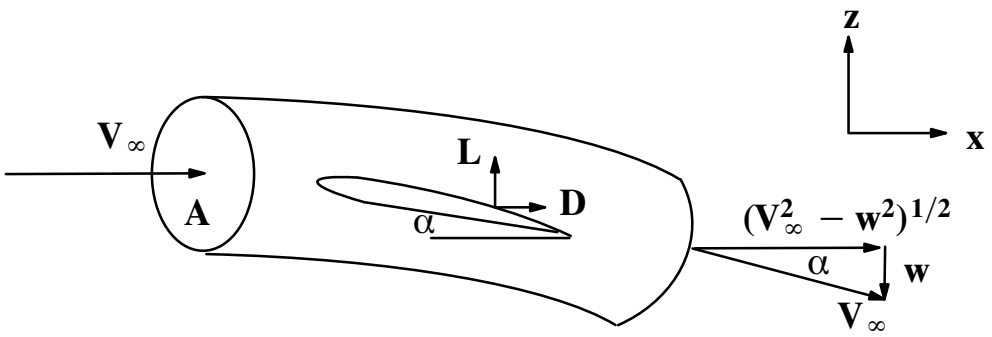
$$p_o \equiv \text{stagnation pressure}$$

$$q = \frac{\varrho}{2}V^2 \equiv \text{dynamic pressure}$$

Stagnation pressure at a point in a flow is defined as the pressure that would occur *if* the flow at that point were brought to rest without any viscous losses. A stagnation streamline is any streamline which originates or terminates at a stagnation point.

4. THE GENERATION OF LIFT (Incompressible Fluids)

4.1 The Origin of Lift Forces



If we use our first inclination to apply the momentum equation to a Control Volume (CV, above) which includes the body, we would assume that the approaching flow is deflected by the body through an angle (approximately equal to the angle of attack α) and thereby give a downward velocity w at the exit plane of the CV. With the usual assumptions that the pressure equals p_∞ everywhere on the edges of the CV, and that no viscous forces are present there, we would get:

$$F_x = \varrho V_\infty A\left[\left(V_\infty^2 - w^2\right)^{1/2} - V_\infty\right] \qquad 4.1$$

$$\equiv \text{D, the force parallel to } V_\infty \text{ called Drag} \qquad 4.2$$

$$F_z = \varrho V_\infty A(-w - 0) = -\varrho V_\infty A w$$

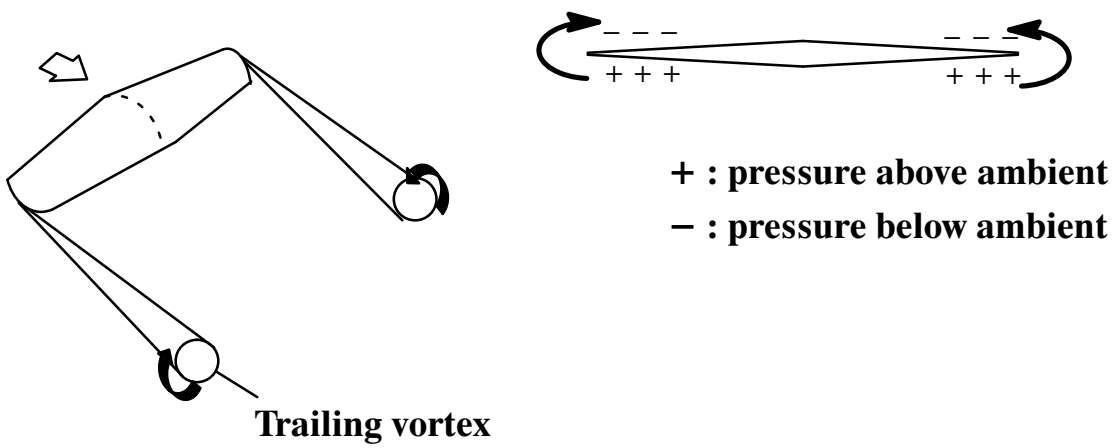
$$\equiv -\text{L, the force normal to } V_\infty \text{ called Lift}$$

F_x and F_z are the forces acting *on* the fluid within the CV (i.e., the pressure forces exerted *on* the fluid *by* the body), and L and D are the fluid forces acting *on* the body.

We have found that in order to obtain a force in the z−direction, we must develop a downward flow velocity, w, behind the body. Where does this flow go? It cannot keep flowing downward indefinitely (to $z = -\infty$) since that would require moving an infinite volume of fluid, an obvious impossibility.

The fluid has been assumed to be incompressible so that it is not just a matter of compressing a finite volume of fluid below the body. (The behavior for compressible flows is markedly different, and we will discuss them later in the course when considering flows with M > 1) This leaves us with only one possibility for an incompressible flow, and that is the one postulated by Lanchester, namely, the flow cannot continue to go downward, but must come back up again somewhere so that the net displacement of fluid is zero. The way this occurs is different in the three−dimensional and two−dimensional cases, i.e. the finite wing and the airfoil section.

4.1.1 Finite Wing (3−D)



In the case of a finite wing, the flow deflected downward just behind the wing comes up again behind the wing tips, and circulates back down again forming a spiraling motion (vortex) trailing behind each wing tip as illustrated above. The flow has no net angular momentum before passage of the wing, and it therefore cannot have any after passage since no external torque is applied and angular momentum must be conserved. Thus the two trailing vortices must spin in opposite directions and be of equal strength. We can see physically why this happens by reference to the sketch above, which shows a wing viewed from the downstream direction. If there is lift generated, the pressure on the upper surface (−) is less than the pressure on the lower surface (+), so that the fluid tends to flow around the tip from high to low pressure. Equations 4.1 and 4.2 can then be used to determine the forces

exerted by a finite wing if we limit our CV to the region enclosing the wing and the two trailing vortices. We will do this in a later section.

4.1.2 Airfoil (2−D)

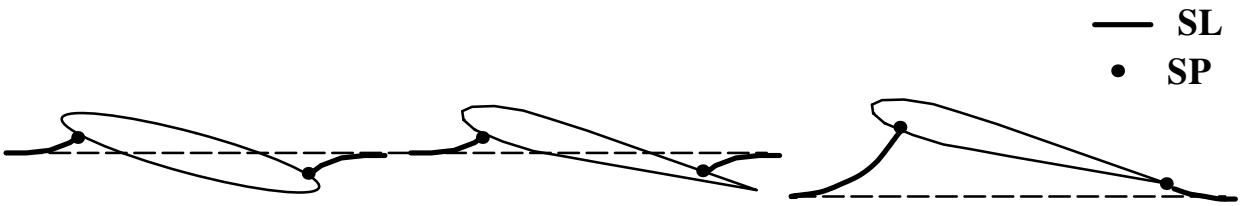
Formation of Starting Vortex Formation of Stopping Vortex
(photographs from Prandtl)

Now consider what happens if the flow is 2−dimensional or has an infinite span. In this case the downward flow cannot come back up at wing tips since they no longer exist, but can only do so upstream or downstream. In fact it must do both since angular momentum must be conserved in this case too. The flow is deflected downwards behind the airfoil and swirls back up again forming a vortex that is swept downstream. The equal and opposite vortex formed upstream as required by conservation of angular momentum, remains bound to the body. Photographs of the starting motion of an airfoil (see the figure above), show us that this actually happens. When an airfoil is started impulsively, a vortex is formed at the trailing edge and is swept downstream. That there must be an oppositely−spinning vortex generated on the body is evidenced by the stopping vortex (not shown here) that is shed when the airfoil stops. The upstream vortex is presumably bound to the body after the flow is started, giving a continual circulating flow, called the "circulation", around the airfoil. The vortex attached to the body is called a "bound vortex", and the one swept downstream is called a "shed" vortex. If the velocity of the airfoil changes, or if its angle of attack changes, a further vortex is shed, and the strength of the bound vortex changes accordingly. You have probably seen this vortex behavior when a paddle is pulled smoothly through the water.

4.2 Circulation

4.2.1 The Kutta Condition

What gives rise to the circulation around an airfoil? Some force is necessary to start the flow swirling. The physical explanation is not so straightforward as it was for the finite wing where the pressure difference between the upper and lower surfaces supplied the necessary force. Now we must look to something more subtle.



a) Bluff Body **b) Airfoil Shape** **c) Airfoil Shape**

INVISCID **VISCOUS**

The other source of aerodynamic force other than pressure, is surface friction due to viscosity, and we have assumed an inviscid flow field. If we must appeal to viscosity for an explanation, we must admit that a truly inviscid, incompressible flow will not generate a lifting force on a 2−dimensional body. Rather, the stagnation streamline (SL) for the body would appear as sketched in a) above, for a blunted body, and as in b) for a body with a sharp trailing edge. The stagnation points (SP) on the upper and lower surfaces are shown, and we see not only no net deflection of SL at $\pm\infty$ but nearer the body, upstream deflection is equal to down stream deflection. If viscosity were allowed, then the flow would still appear much the same for the case in a), where there are no large velocity gradients, but the large changes required to manipulate the sharp trailing edge of b) would be prevented by viscous shearing forces. (Remember that viscous stress is proportional to velocity gradients.) Instead, the stagnation point moves down to the trailing edge under the action of combined viscous and pressure forces, and now leaves the upper and lower surfaces smoothly there, as illustrated in c) thereby making viscous forces there small. This condition at the trailing edge is known as the "Kutta Condition". In c) there is still no net deflection of SL at $\pm\infty$, but now the upstream and downstream deflections near the body are in opposite directions as we would expect for the circulating flow of a bound vor-

tex. Note too, that there is a greater upward deflection of the forward SP than in cases a) and b).

Once steady flow is established and the Kutta condition is satisfied, viscosity is no longer required an we can make the assumption that the flow is inviscid. Note however, that if the angle of attack or the free−stream velocity changes, a further vortex will be shed, and the strength of the bound vortex will change until steady conditions are achieved once again, i.e. until the shed vortex is swept far behind the airfoil. Viscous forces would of course be active during the transition period.

The viscous shear force which acts on the fluid imparts a torque to the flow give it a spinning motion, and thereby moves the rear stagnation point down to the trailing edge.

Recall that $T = \frac{d(\varrho v_\theta r)}{dt}$ **4.3**

for the relation between the torque on unit volume, T, applied by the viscous shear, and the angular momentum $\varrho v_\theta r$ **, which results. A tangential velocity** v_θ**, therefore, will result if the torque is applied for the proper amount of time and** v_θ **will be just sufficient to move the rear stagnation point to the trailing edge. Once that occurs the torque T becomes zero and no further viscous forces are required. The flow field is then left with a constant angular momentum around the airfoil of**

$$\varrho v_\theta r = \text{constant} \qquad \textbf{4.4}$$

For constant density flows, then

$$v_\theta r = \text{constant} \equiv \frac{\Gamma}{2\pi} \qquad \textbf{4.5}$$

where Γ **is called the "circulation"**

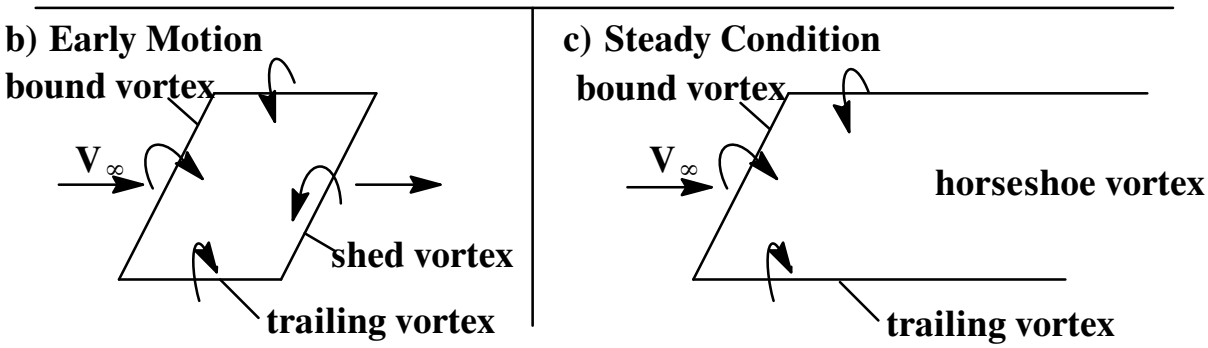
(Actually circulation is defined by $\Gamma = -\oint \vec{V}\cdot d\vec{s}$ **4.6**

where the integration is a line integral around any closed path which surrounds the body. When the path of integration is a circle, then it is easy to show that Equation 4.6 gives $\Gamma = 2\pi r v_\theta$**.)**

Note that we have departed from convention and have taken v_θ positive downwards. This is done to make Γ positive clockwise.

4.2.2 Wing Vortex System

a) Very Near Wing
a sketch by Lanchester



Even for a finite wing, a bound vortex and a shed vortex are formed along with trailing vortices. Actually trailing vortices are formed, generally, all along the span of the wing, but these coalesce (see a) above) to form two trailing tip vortices (see c) above). During the starting motion a vortex is shed from the trailing edge and is swept downstream as in the 2−D case, but since vortices cannot end in space, the trailing vortices are developed at the same time (see b) above). Once the shed vortex is swept far downstream the steady structure appears in a form we call a "horseshoe vortex".

4.3 SUMMARY

Two−Dimensional, Incompressible Flows

1. No lift is generated without viscosity.
2. If the effects of viscosity are included, then the flow leaves the trailing edge smoothly. This condition is known as the Kutta Condition.
3. A body with a rounded trialing edge produces little lift, a sharp trailing edge is required for efficient lift production.
4. Although the effects of viscosity are essential in establishing a flow, or when changes of speed or angle of attack occur, to a good approximation viscous effects may be neglected once steady state is achieved.
5. The starting viscous forces leave the flow field with a steady circulating motion described by $2\pi r v_\theta = \Gamma$, the circulation, constant at large distance from the body. The circulating flow is called a bound vortex of circulation Γ.

Three−Dimensional, Incompressible Flows

6. The bound vortex trails off from the wing and forms two trailing vortices.
7. The trailing vortices and the bound vortex combine to form a horseshoe vortex which extends far behind the wing.
8. The horseshoe vortex generates a downward flow in the region behind the wing (the wake) as well as at the wing itself.
9. The pressure difference between the upper and lower wing surfaces causes flow around the wing tips and generates the trailing vortices.