Formula Sheet EE224 Final

Frequency and Period

$\omega = 2\pi f = 2\pi / T;\ f = 1/T$

Time Delay/Phase Shift

$t = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0}$

$\phi = -\omega t$

Laws of Exponents:

$e^{jx}e^{jy} = e^{j(x+y)}; \left(e^{jx}\right)^y = e^{jxy}; \frac{1}{e^{jx}} = e^{-jx}$

Polar to Rectangular

$x = r\cos\theta; y = r\sin\theta$

Rectangular to Polar

$r = \sqrt{x^2 + y^2}; \theta = \tan^{-1}(y/x)$

Phasor Addition Rule

A series of sinusoids with the same frequency can be added up using complex amplitude and phasors:

$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$

$= A\cos(\omega_0 t + \phi)$

$Ae^{j\phi} = \sum_{k=1}^{N} A_k e^{j\phi_k}$

Values of complex exponentials:

$e^{j0} = e^{j2\pi k} = 1 \quad k$ *is* integer

$e^{j(\theta + 2\pi k)} = e^{j\theta}$

$e^{j\pi/2} = j; e^{-j\pi/2} = -j; e^{j\pi} = -1 = e^{-j\pi}$

$z^N = \left(re^{j\phi}\right)^N = \left(\cos\phi + j\sin\phi\right)^N$

$= \cos N\phi + j\sin N\phi$

Values of Sines and Cosines:

$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta); \sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$

$\cos(-\theta) = \cos(\theta); \sin(-\theta) = -\sin(\theta)$

$\cos(\theta + 2\pi k) = \cos(\theta); k$ *is* integer

$\sin(\pi k) = 0; k$ *is* integer

$\cos(2\pi k) = 1; k$ *is* integer

Popular values

Deg.	Rad.	Cos	sin	Tan=sin/cos
0	0	1	0	0
30	π/6	$\sqrt{3}/2$	$1/2$	$1/\sqrt{3}$
45	π/4	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	π/3	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$
90	π/2	0	1	undefined

Basic Trigonometric Identities

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Euler's Formula

$e^{j(\omega_0 t + \theta)} = \cos(\omega_0 t + \theta) + j\sin(\omega_0 t + \theta)$

$e^{-j(\omega_0 t + \theta)} = \cos(\omega_0 t + \theta) - j\sin(\omega_0 t + \theta)$

Inverse Euler's Formula

$\cos(\omega_0 t + \theta) = \frac{1}{2}\left(e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)}\right)$

$\sin(\omega_0 t + \theta) = \frac{1}{2j}\left(e^{j(\omega_0 t + \theta)} - e^{-j(\omega_0 t + \theta)}\right)$

Complex Numbers

$z_1 = x_1 + jy_1 = r_1 e^{j\phi_1}; z_2 = x_2 + jy_2 = r_2 e^{j\phi_2}$

$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

$z_1 \bullet z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$

$= r_1 r_2 e^{j(\phi_1 + \phi_2)}$

$z_1^* = x_1 - jy_1 = r_1 e^{-j\phi_1}$

$z_1^{-1} = \frac{x_1 - jy_1}{\sqrt{x_1^2 + y_1^2}} = r_1^{-1}{}_1 e^{-j\phi_1}$

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Continuous Fourier Series

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

Discrete-Time Fourier Series

$$a_n = \frac{1}{N_0}\sum_{k=0}^{N_0} x[k]e^{-j\omega_0 kn}$$

Simple Integrals

$$\int e^{at}dt = \frac{1}{a}e^{at}$$

$$\int te^{at}dt = \frac{e^{at}}{a^2}(at-1)$$

Procedure for Finding Multiple Roots of $z^N = c$:

1. Write $z^N = r^N e^{jN\phi}$
2. Write c as $|c|e^{j\theta}e^{j2\pi k}$; k *is* integer
3. Equate and solve for magnitude and angle separately: $r^N e^{jN\phi} = |c|e^{j\theta}e^{j2\pi k}$
4. Magnitude: $r = |c|^{1/N}$
5. Angle: $N\phi = \theta + 2\pi k \Rightarrow \phi = \frac{1}{N}(\theta + 2\pi k)$

Magnitudes are the same, angles are equally spaced around circle, every 2π/N radians

Digital Frequency

$$\hat{\omega} = \frac{2\pi f}{f_s}$$

$\hat{\omega}_0 + 2\pi l; -\hat{\omega}_0 + 2\pi l$; l is an integer

Reconstruction, D to C converter

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

Discrete-time signals

Delta function

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Unit step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Linearity: Scaling and superposition hold

Time-invariance: response doesn't change with time

Discrete-time Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Delta function properties:

$h[n]*\delta[n] = h[n] \qquad h[n]*\delta[n-n_0] = h[n-n_0]$

Frequency Response (DTFT)

$$H\left(e^{j\hat{\omega}}\right) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}$$

Properties of DTFT

1. Digital spectra repeat every 2π

2. Conjugate symmetry $H\left(e^{-j\hat{\omega}}\right) = H^*\left(e^{j\hat{\omega}}\right)$

$$\left|H\left(e^{-j\hat{\omega}}\right)\right| = \left|H\left(e^{j\hat{\omega}}\right)\right| \quad \angle H\left(e^{-j\hat{\omega}}\right) = -\angle H\left(e^{j\hat{\omega}}\right)$$

Cascaded LTI Systems

Time: $h[n] = h_1[n]*h_2[n]$

Frequency: $H\left(e^{j\hat{\omega}k}\right) = H_1\left(e^{j\hat{\omega}k}\right)H_2\left(e^{j\hat{\omega}k}\right)$

LTI Sinusoidal System Response

If $x[n] = Ae^{j\phi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$

Then $y[n] = A\left|H\left(e^{j\hat{\omega}}\right)\right|e^{j\left(\angle H\left(e^{j\hat{\omega}}\right)+\phi\right)}e^{j\hat{\omega}n}$

System Function for Running Average:

$$H\left(e^{j\hat{\omega}}\right) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\omega k} = \frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)}e^{-j\hat{\omega}(L-1)/2}$$

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Fourier Transform Pairs

Time-domain, x(t) to Frequency-Domain: X(jω)		
$e^{-at}u(t)(\mathrm{Re}\,a>0)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{a+j\omega}$
$e^{bt}u(-t)\,(\mathrm{Re}\,b>0)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{b-j\omega}$
$e^{-a\|t\|}\quad(\mathrm{Re}\,a>0)$	$\overset{FT}{\leftrightarrow}$	$\frac{2a}{a^2+\omega^2}$
$te^{-at}u(t)(\mathrm{Re}\,a>0)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{(a+j\omega)^2}$
$\left[u\left(t+\tfrac{1}{2}T\right)-u\left(t-\tfrac{1}{2}T\right)\right]$	$\overset{FT}{\leftrightarrow}$	$\frac{\sin(\omega T/2)}{\omega/2}$
$\frac{\sin(\omega_b t)}{\pi t}$	$\overset{FT}{\leftrightarrow}$	$\left[u(\omega+\omega_b)-u(\omega-\omega_b)\right]$
$\delta(t)$	$\overset{FT}{\leftrightarrow}$	1
$\delta(t-t_d)$	$\overset{FT}{\leftrightarrow}$	$e^{-j\omega t_d}$
$u(t)$	$\overset{FT}{\leftrightarrow}$	$\pi\delta(\omega)+\frac{1}{j\omega}$
1	$\overset{FT}{\leftrightarrow}$	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$\overset{FT}{\leftrightarrow}$	$2\pi\delta(\omega-\omega_0)$
$A\cos(\omega_0 t+\theta)$	$\overset{FT}{\leftrightarrow}$	$\pi A\left[e^{j\theta}\delta(\omega-\omega_0)+e^{-j\theta}\delta(\omega+\omega_0)\right]$
$\cos(\omega_0 t)$	$\overset{FT}{\leftrightarrow}$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$\overset{FT}{\leftrightarrow}$	$j\pi\left[-\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$	$\overset{FT}{\leftrightarrow}$	$\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$
$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\overset{FT}{\leftrightarrow}$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$

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Fourier Transform Properties

Property Name	**Time-domain, x(t) to Frequency-Domain: X(jω)**		
Linearity	$ax(t)+by(t)$	$\overset{FT}{\leftrightarrow}$	$aX(j\omega)+bY(j\omega)$
Conjugation	$x^*(t)$	$\overset{FT}{\leftrightarrow}$	$X^*(-j\omega)$
Time-reversal	$x(-t)$	$\overset{FT}{\leftrightarrow}$	$X(-j\omega)$
Scaling	$x(at)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{\lvert a\rvert}X(j\omega/a)$
Delay	$x(t-t_d)$	$\overset{FT}{\leftrightarrow}$	$e^{-j\omega t_d}X(j\omega)$
Modulation	$x(t)e^{j\omega_0 t}$	$\overset{FT}{\leftrightarrow}$	$X(j(\omega-\omega_0))$
	$x(t)\cos(\omega_0 t)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{2}\left[X(j(\omega-\omega_0))+X(j(\omega+\omega_0))\right]$
Differentiation in time	$\frac{d^k}{dt^k}x(t)$	$\overset{FT}{\leftrightarrow}$	$(j\omega)^k X(j\omega)$
Differentiation in Frequency	$-jtx(t)$	$\overset{FT}{\leftrightarrow}$	$\frac{d}{d\omega}X(j\omega)$
Convolution	$\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$	$\overset{FT}{\leftrightarrow}$	$X(j\omega)H(j\omega)$
Multiplication	$x(t)\,p(t)$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{2\pi}X(j\omega)*P(j\omega)$
Duality	$X(jt)$	$\overset{FT}{\leftrightarrow}$	$2\pi x(-\omega)$
Parseval's Theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt$	$\overset{FT}{\leftrightarrow}$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2\,d\omega$
Symmetry	$x(t)$ *real*	$\overset{FT}{\leftrightarrow}$	$X^*(j\omega)=X(-j\omega)$
	$x(t)$ *imag*	$\overset{FT}{\leftrightarrow}$	$X^*(j\omega)=-X(-j\omega)$