Types of Digital Signals

- Unit step signal

$$u(n) \equiv \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases} .$$

- Unit impulse (unit sample)

$$\delta(n) \equiv \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases} .$$

$$\begin{aligned} u(n) &= \sum_{m=-\infty}^{n} \delta(m) \quad \text{summing}, \\ \delta(n) &= u(n) - u(n-1) \quad \text{differencing}. \end{aligned}$$

- Complex exponentials (cisoids)

$$x(n) = A \exp[j(\omega n + \theta)]$$

obtained by sampling an analog cisoid

$$x_{\mathrm{a}}(t) = A \exp[j(\Omega t + \theta)],$$

i.e. $x(n) = x_{\rm a}(nT)$, where T is the sampling interval. Thus,

$$\omega = \Omega T, \quad \text{or, equivalently,} \quad f = \frac{F}{F_{\rm s}}$$

(using $F_{\rm s} = 1/T$, $\omega = 2\pi f$, $\Omega = 2\pi F$),

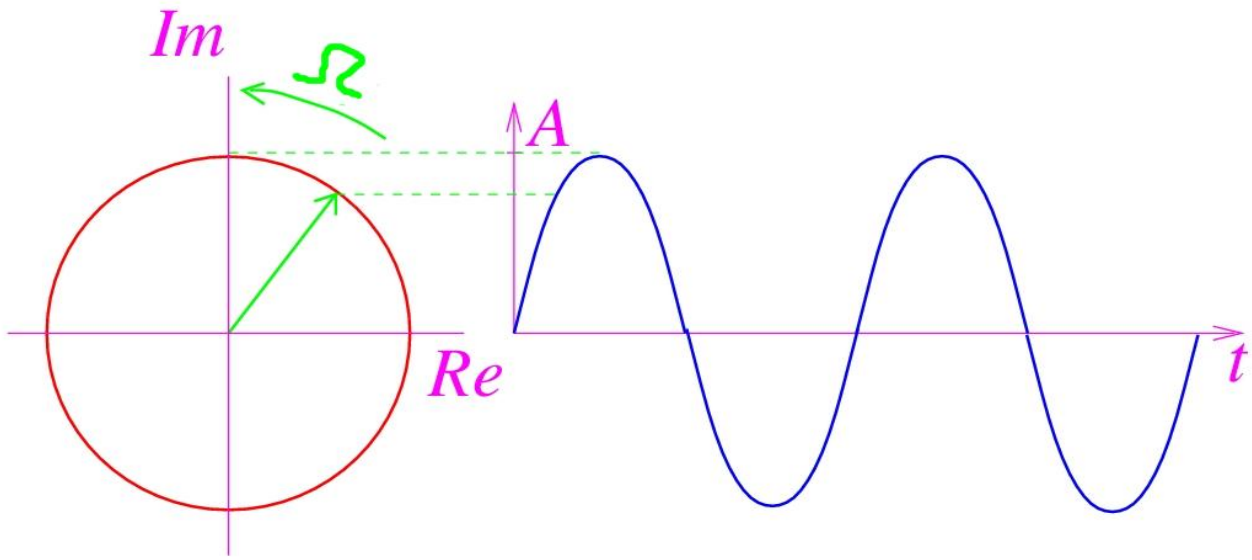
- Sinusoids

$$x(n) = A\sin(\omega n + \theta)$$

Useful properties:

$$\begin{aligned}
\exp[j(\omega n + \theta)] &= \cos(\omega n + \theta) + j\sin(\omega n + \theta), \\
\cos(\omega n + \theta) &= \frac{\exp[j(\omega n + \theta)] + \exp[-j(\omega n + \theta)]}{2}, \\
\sin(\omega n + \theta) &= \frac{\exp[j(\omega n + \theta)] - \exp[-j(\omega n + \theta)]}{2j}.
\end{aligned}$$

A sine wave as the projection of a complex phasor onto the imaginary axis:



Sampled vs. Analog Exponentials

- Analog exponentials and (co)sinusoids are periodic with $T = 2\pi/\Omega$, discrete-time sinusoids are not necessarily periodic (although their values lie on a periodic envelope.)

Periodicity condition: (also for sines and cosines)

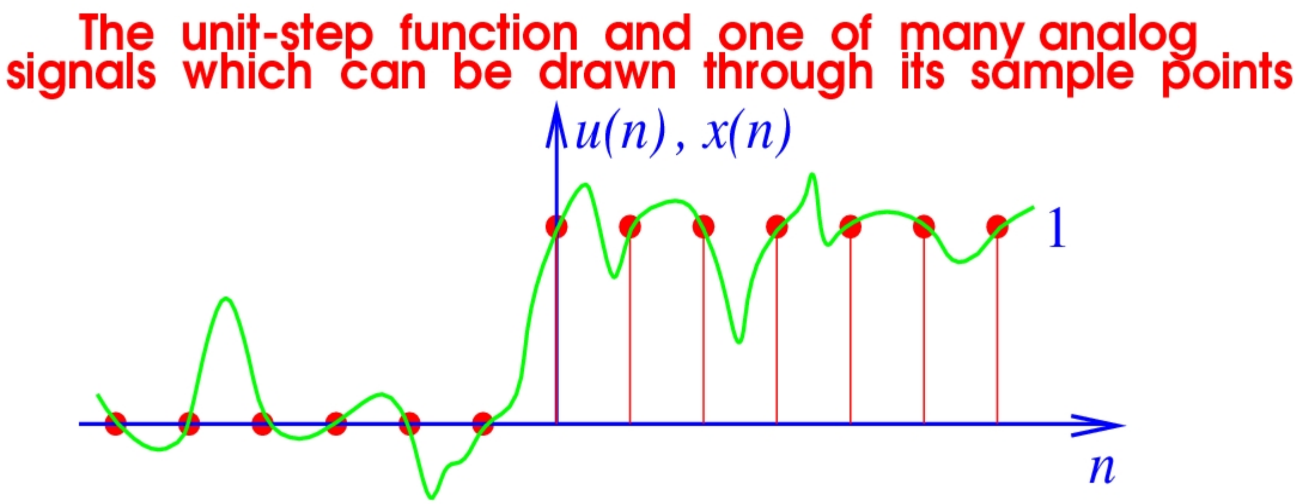
$$x(n) = x(n+N) \Longrightarrow e^{j\omega n} = e^{j\omega(n+N)} \Longrightarrow \exp(j\omega N) = 1$$

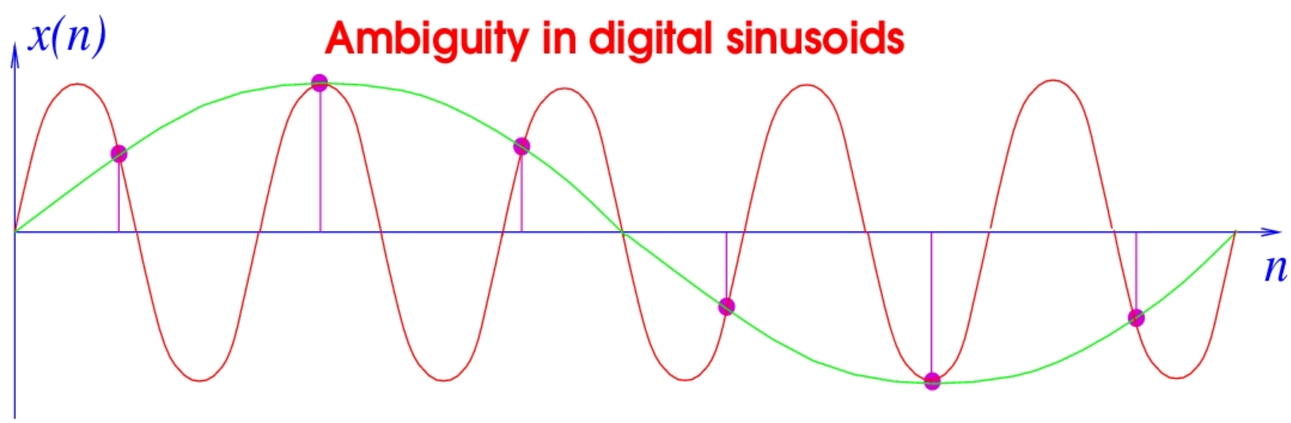
$$\Longrightarrow \omega = \frac{2\pi m}{N} \quad m \text{ integer, or} \quad f = \frac{m}{N} \quad (\omega = 2\pi f).$$

- For sampled exponentials, the frequency ω is expressed in radians, rather than radians/second.

- Digital signals have ambiguity.

Ambiguity in Discrete-time Signals





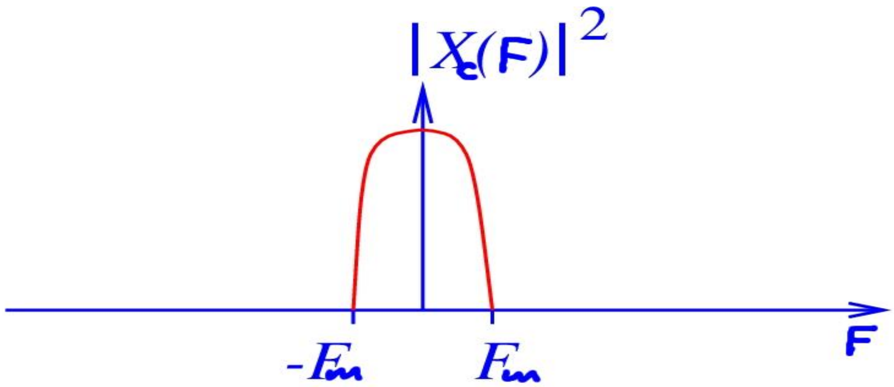
Ambiguity Condition for Discrete-time Sinusoids

$$\sin(\Omega_1 T) = \sin(\Omega_2 T), \quad \Omega_1 \neq \Omega_2 \Rightarrow$$

$$2\pi F_1 T = 2\pi F_2 T + 2\pi m, \quad m = \ldots, -2, -1, 1, 2, \ldots \Rightarrow$$

$$|F_1 - F_2| = \frac{m}{T} = m F_{\mathrm{s}}, \quad m = 1, 2, \ldots$$

Example: lowpass signal (with spectrum $|X(F)|^2$ concentrated in the interval $[-F_{\rm m}, F_{\rm m}]$):



Taking $F_1 = F_{\rm m}$ and $F_2 = -F_{\rm m}$, it follows that there is no ambiguity if the signal is sampled with

$$F_{\rm s} = \frac{1}{T} > 2F_{\rm m}.$$

where $F_{\rm s}$ is the *sampling frequency*. The above equation is a particular form of the *sampling theorem*.

- The frequency $F_{\rm N} = 2F_{\rm m}$ is referred to as the *Nyquist rate*.

- Discrete-time signal ambiguity is often termed as the *aliasing effect*.

Discrete-time Systems

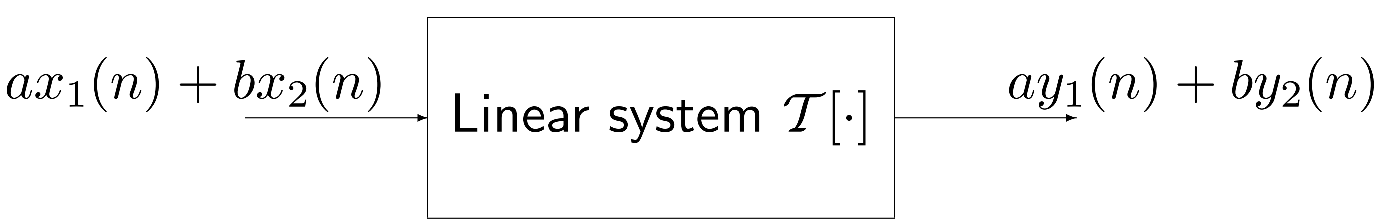
$$y(n) = \mathcal{T}[x(n)]$$

where $\mathcal{T}[\cdot]$ denotes the transformation (operator) that maps an input sequence $x(n)$ into an output sequence $y(n)$.

Linear system: a system is linear if it obeys the *superposition principle*:

The response of the system to the weighted sum of signals $\equiv$ corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Mathematically:

$$\begin{aligned} \mathcal{T}[ax_1(n) + bx_2(n)] &= a\mathcal{T}[x_1(n)] + b\mathcal{T}[x_2(n)] \\ &= ay_1(n) + by_2(n). \end{aligned}$$



Example: (Square-law device) Let $y(n) = x^2(n)$ (i.e. $\mathcal{T}[\cdot] = (\cdot)^2$). Then

$$\begin{aligned} \mathcal{T}[x_1(n) + x_2(n)] &= x_1^2(n) + x_2^2(n) + 2x_1(n)x_2(n) \\ &\neq x_1^2(n) + x_2^2(n). \end{aligned}$$

Hence, the system is nonlinear!

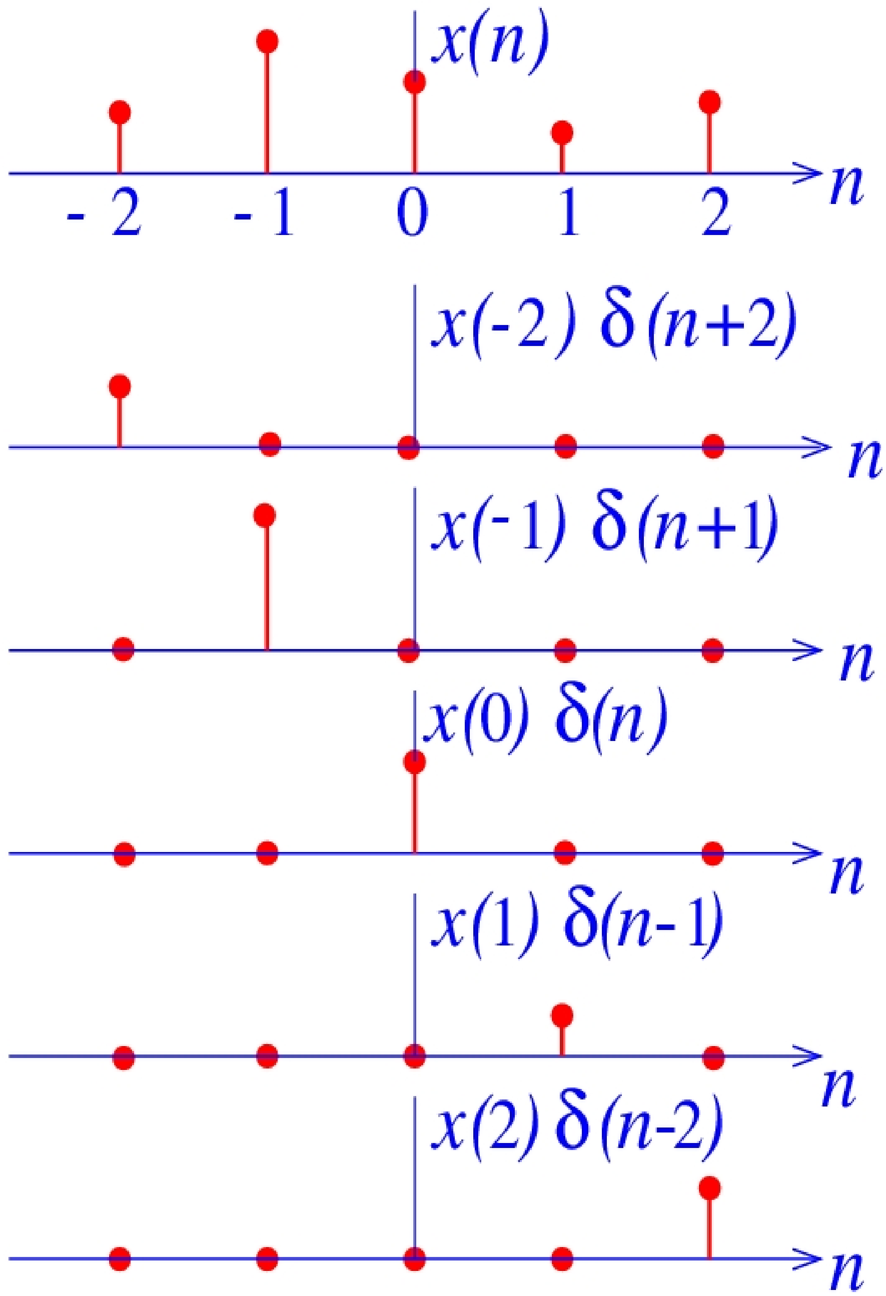
A *time-invariant* (or shift-invariant) system has input-output properties that do not change in time:

$$\text{if} \quad y(n) = \mathcal{T}[x(n)] \Longrightarrow y(n-k) = \mathcal{T}[x(n-k)].$$

Linear time-invariant (LTI) system is a system that is both linear and time-invariant [sometimes referred to as linear shift-invariant (LSI) system].

Discrete-time Signals via Shifted Impulse Functions

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$

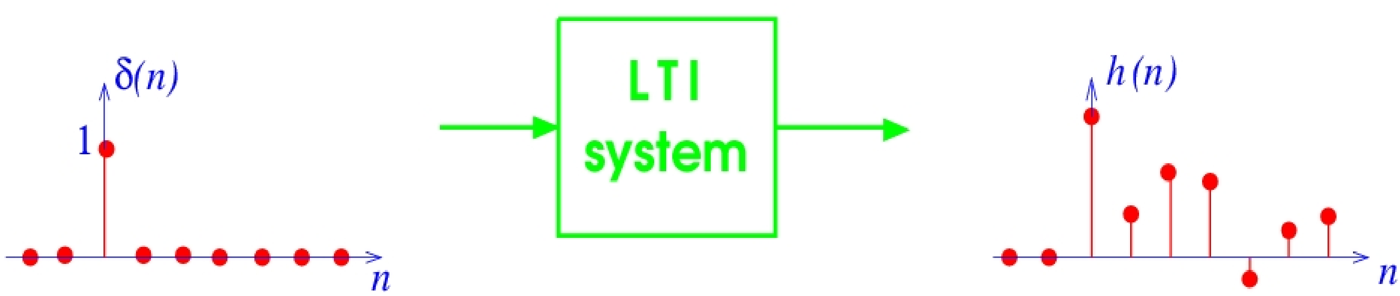


Response of LTI System

Let $h(n)$ be the response of the system to $\delta(n)$. Due to the time-invariance property, the response to $\delta(n-k)$ is simply $h(n-k)$. Thus

$$\begin{aligned} y(n) &= \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \{x(n)\} \star \{h(n)\} \quad \text{convolution sum.} \end{aligned}$$

The sequence $\{h(n)\} \equiv$ *impulse response* of LTI system.



Convolution: Properties

An important property of convolution:

$$\{x(n)\} \star \{h(n)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \{h(n)\} \star \{x(n)\},$$

i.e. the order in which two sequences are convolved is unimportant!

Other properties:

$$\{x(n)\} \star [\{h_1(n)\} \star \{h_2(n)\}] \quad \text{associativity}$$
$$= [\{x(n)\} \star \{h_1(n)\}] \star \{h_2(n)\}.$$

$$\{x(n)\} \star [\{h_1(n)\} + \{h_2(n)\}] \quad \text{distributivity}$$
$$= \{x(n)\} \star \{h_1(n)\} + \{x(n)\} \star \{h_2(n)\}.$$

Stability of LTI Systems

An LTI system is stable if and only if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty,$$

Proof: **(absolute summability $\Rightarrow$ stability)** Let the input $x(n)$ be bounded so that $|x(n)| \leq M_x < \infty, \forall n \in [-\infty, \infty]$. Then

$$\begin{aligned} |y(n)| &= \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \\ &\leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \Rightarrow |y(n)| < \infty \text{ if } \sum_{k=-\infty}^{\infty} |h(k)| < \infty. \end{aligned}$$

$\Longrightarrow$ Now, it remains to prove that, if $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$, then a bounded input can be found for which the output is not bounded. Consider

$$x(n) = \begin{cases} \frac{h^*(-n)}{|h^*(-n)|}, & h(n) \neq 0, \\ 0, & h(n) = 0 \end{cases}.$$

$$y(0) = \sum_{k=-\infty}^{\infty} h(k)x(-k) = \sum_{k=-\infty}^{\infty} |h(k)| \Longrightarrow$$

if $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$, the output sequence is unbounded.

Causality of LTI Systems

Definition. *A system is causal if the output does* **not** *anticipate future values of the input, i.e. if the output at any time depends only on values of the input up to that time. Thus, a causal system is a system whose output $y(n)$ depends only on $\{\ldots, x(n-2), x(n-1), x(n)\}$.*

Consequence: A system $y(n) = \mathcal{T}[x(n)]$ is causal if whenever $x_1(n) = x_2(n)$ for all $n \leq n_0$ then $y_1(n) = y_2(n)$ for all $n \leq n_0$, where $y_1(n) = \mathcal{T}[x_1(n)], \quad y_2(n) = \mathcal{T}[x_2(n)]$.

Comments:

- **All** real-time physical systems are **causal,** because time only moves forward. (Imagine that you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does **not** apply to spatially-varying signals. (We can move both left and right, up and down.)
- Causality does **not** apply to systems processing **recorded** signals (e.g. taped sports games vs. live broadcasts).

Proposition. *An LTI system is causal if and only if its impulse response $h(n) = 0$ for $n < 0$.*

Proof. From the definition of a causal system:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=0}^{\infty} h(k)x(n-k).$$

Obviously, this equation is valid if $\sum_{k=-\infty}^{-1} h(k)x(n-k) = 0$ for all $x(n-k) \Longrightarrow h(n) = 0$ for $n < 0$. The other direction is obvious. $\square$

If $h(n) \neq 0$ for $n < 0$, system is noncausal.

$$\begin{array}{ll} h(n) = 0, & n < -1, \\ h(-1) \neq 0 & \end{array} \Longrightarrow$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) + h(-1)x(n+1) \Longrightarrow$$

$y(n)$ depends on $x(n+1) \Longrightarrow$ noncausal system!

Example:

An LTI system with

$$h(n) = a^n u(n) = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0 \end{cases}.$$

- Since $h(n) = 0$ for $n < 0$, the system is causal.
- To decide on stability, we must compute the sum

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |a|^k = \begin{cases} \frac{1}{1-|a|}, & |a| < 1, \\ \infty, & |a| \geq 1 \end{cases}.$$

Thus, the system is stable only for $|a| < 1$.

Linear Constant-Coefficient Difference (LCCD) Equations

Consider LTI systems satisfying

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad \text{ARMA}$$

Particular cases:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k), \quad \text{MA}$$

$$\sum_{k=0}^{N} a_k y(n-k) = x(n) \quad \text{AR.}$$

Example:

$$y(n) = \sum_{k=-\infty}^{n} x(k) \quad \text{accumulator}$$

$$y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k) = x(n).$$

Property: MA systems are bounded-input bounded-output (BIBO) stable, i.e.

$$|y(n)| = \left| \sum_{k=0}^{M} b_k x(n-k) \right| \leq \sum_{k=0}^{M} |b_k| \cdot |x(n-k)| < \infty$$

for any bounded input $|x(n)| < \infty$ and coefficient sequence $|b_n| < \infty$.

Remark: AR systems may be unstable. For example, the system

$$y(n) = ay(n-1) + x(n)$$

is unstable for $a > 1$, because $y(n)$ is generally unbounded for bounded $x(n)$.

Property: MA systems have finite impulse response (FIR), whereas AR systems have infinite impulse response (IIR):

$$h_{\mathrm{MA}}(n) = \begin{cases} 0, & n < 0, \\ b_n, & 0 \leq n \leq M, \\ 0, & n > M. \end{cases}$$

"Proof" for AR systems: $y(n)$ depends on $y(n-k)$, $k = 1, 2, \ldots \Rightarrow y(n)$ depends on $x(n-k)$, $k = 0, \ldots, \infty \Rightarrow$ the impulse response $h_{\mathrm{AR}}(n)$ is infinite, i.e. is in general nonzero for all $n > 0$.

Suppose that, for a given input $x(n)$, we have found one particular output sequence $y_{\rm p}(n)$ so that a LCCD equation is satisfied. Then, the same equation with the same input is satisfied by any output of the form

$$y(n) = y_{\rm p}(n) + y_{\rm h}(n),$$

where $y_{\rm h}(n)$ is any solution to the LCCD equation with zero input $x(n) = 0$.

Remark: $y_{\rm p}(n)$ and $y_{\rm h}(n)$ are referred to as the particular and homogeneous solutions, respectively.

Proof. From

$$\begin{aligned} \sum_{k=0}^{N} a_k y_{\rm p}(n-k) &= \sum_{k=0}^{N} b_k x(n-k) \\ \sum_{k=0}^{N} a_k y_{\rm h}(n-k) &= 0 \end{aligned}$$

it follows

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

where $y(n) = y_{\rm p}(n) + y_{\rm h}(n)$. $\square$

Property: A LCCD equation does not provide a unique specification of the output for a given input. Auxiliary information or conditions are required to specify uniquely the output for a given input.

Example: Let auxiliary information be in the form of N sequential output values. Then

- later values can be obtained by rearranging LCCD equation as a recursive relation running forward in n,
- prior values can be obtained by rearranging LCCD equation as a recursive relation running backward in n.

LCCD equations as recursive procedures:

$$y(n) = \sum_{k=0}^{M} \frac{b_k}{a_0} x(n-k) - \sum_{k=1}^{N} \frac{a_k}{a_0} y(n-k) \quad \text{forward},$$

$$y(n-N) = \sum_{k=0}^{M} \frac{b_k}{a_N} x(n-k) - \sum_{k=0}^{N-1} \frac{a_k}{a_N} y(n-k) \quad \text{backward}.$$

Example: First-order AR system $y(n) = ay(n-1) + x(n)$ with input $x(n) = b\delta(n-1)$ and the auxiliary condition $y(0) = y_0$.

Forward recursion:

$$\begin{aligned} y(1) &= ay_0 + b, \\ y(2) &= ay(1) + 0 \\ &= a(ay_0 + b) = a^2 y_0 + ab, \\ y(3) &= a(a^2 y_0 + ab) = a^3 y_0 + a^2 b, \\ &\cdots \\ y(n) &= a^n y_0 + a^{n-1} b. \end{aligned}$$

Observe that $y(n-1) = a^{-1}[y(n) - x(n)] \Longrightarrow$

Backward recursion:

$$\begin{aligned} y(-1) &= a^{-1}(y_0 - 0) = a^{-1} y_0, \\ y(-2) &= a^{-2} y_0, \\ y(-3) &= a^{-3} y_0, \\ &\cdots \\ y(-n) &= a^{-n} y_0. \end{aligned}$$

Is this system LTI?

Lemma. *A linear system requires that the output be zero for all time when the input is zero for all time.*

Proof. Represent zero input as $0 \cdot x(n)$, where $x(n)$ is an arbitrary (nonzero) signal. Then

$$y(n) = \mathcal{T}[0 \cdot x(n)] = 0 \cdot \mathcal{T}[x(n)] = 0.$$

□

(Back to Example) Choosing $b = 0$, we have $x(n) = x(-n) = 0$, but $y(n)$ and $y(-n)$ will be nonzero if $a \neq 0$ and $y_0 \neq 0$. Using the above lemma, it follows that the system is not linear!

For an arbitrary n, we can write the system's output as:

$$y(n) = a^n y_0 + a^{n-1} b u(n-1).$$

The shift of the input by n_0 samples, $\widetilde{x}(n) = x(n - n_0) = b\delta(n - n_0 - 1)$, gives

$$\widetilde{y}(n) = a^n y_0 + a^{n-n_0-1} b u(n - n_0 - 1) \neq y(n - n_0).$$

The system is *not* time-invariant!

Example: First-order AR system $y(n) = ay(n-1) + x(n)$ with input $x(n) = b\delta(n-1)$ and the auxiliary condition $y(0) = 0$.

Recursion:

$$\begin{aligned} y(-1) &= 0, \\ y(0) &= 0, \end{aligned}$$

$$\begin{aligned} y(1) &= a \cdot 0 + b = b, \\ y(2) &= ab, \\ &\cdots \\ y(n) &= a^{n-1}b, \end{aligned}$$

which can be rewritten as

$$y(n) = a^{n-1}bu(n-1), \quad \forall n.$$

It is easy to prove that now, the system will be LTI.

Linearity and time-invariance depend on auxiliary conditions!