Decomposition Methods

Preliminary comments

This set of notes is structured as follows.

We will not have time to cover all sections, and so we make the following introductory remarks to help you consider whether you want to review the sections we will not cover.

- You need optimization background to understand decomposition.
- Decomposition is highly applicable in power system planning problems.
- Sections 1-3 gives a good intuitive introduction to the topic that is intended to be fairly easy to follow, independent of background.
- Section 4 illustrates one decomposition method, Benders, via a very simple problem, with intention to show decomposition basics from an analytic perspective.
- Sections 1-4 takes about half of this document. The second half, sections 5-12, addresses various issues, some deeply and others lightly; those of you considering a research topic in this area will do well to carefully read these sections and the references provided in them.

1.0 Introduction

Consider the XYZ corporation that has 3 departments, each of which have a certain function necessary to the overall productivity of the corporation. The corporation has capability to make 50 different products (e.g., power transformers of different voltage ratios and different capacities), but at any particular month, it makes some of them and does not make others. The CEO decides which products to make. The CEO's decision is an integer decision on each of the 50 products. Of course, the CEO's decision depends, in part, on the productivity of the departments and their capability to make a profit given the decision of which products to make.

Each department knows its own particular business very well, and each has developed sophisticated mathematical programs (optimization problems) which provide the best (most profitable) way to use their resources given identification of which products to make.

The organization works like this. The CEO makes a tentative decision on which of the 100 products to make, and when s/he needs them, based on his/her own mathematical program which assumes certain profits from each department based on that decision. S/he then passes that decision to the 3 departments. Each of the departments use that information to determine how it is going to operate in order to maximize profitability. For example, departments learn that the CEO desires:

- Two 100MVA 69/161 kV units,
- Four 50MVA 13.8/69 kV units,
- One 325MVA 230/500 kV unit
- …etc

Then, the departments that make windings, cores, bushings, cooling systems, etc., make their decisions on how to allocate their resources (time and materials) to satisfy at minimum cost what the CEO requires.

Then each department passes their information (cost of satisfying the CEO's request) back to the CEO. Once the CEO gets all the information back from the departments, s/he will observe that some departments have very large costs and some have very small costs, and that a refined selection of products might be wise. So s/he modifies constraints of the CEO-level optimization and re-runs it to select the products, likely resulting in a modified choice of products. This process of CEO-departmental interactions will repeat. At some point, the optimization problem solved by the CEO will not change from one iteration to the next. At this point, the CEO will believe the current selection of products is best.

ASIDE: There are many such problems where the master problem involves choice of integer variables and subproblems involve choice of continuous variables. Such problems conform to the form of a mixed-integer-programming (MIP) problem, which is a kind of problem we often have interest.

This is an example of a *multidivisional problem* [1, pg. 219]. Such problems involve coordinating the decisions of separate divisions, or departments, of a large organization, when the divisions operate autonomously (but all of which make decisions that depend on the CEO's decisions). Solution of such problems often may be facilitated by separating them into a single master problem and subproblems where the master corresponds to the problem addressed by the CEO and the subproblems correspond to the problems addressed by the various departments.

However, the master-subproblem relationship may be otherwise. It may also involve decisions on the part of the CEO to control (by directly modifying) each department's resources. By "resources," we mean amount of time and materials, represented by the right-hand-side of the constraints. Such a scheme is referred to as a *resource-directed* approach.

Alternatively, the master-subproblem relationship may involve decisions on the part of the CEO to indirectly modify resources by charging each department a price for the amount of resources that are used. The CEO then modifies prices, and departments adjust accordingly. Such a scheme is called a *price-directed* approach.

Optimization approaches which reflect these types of structures are referred to as decomposition methods.

2.0 Connection with optimization: problem structure [2]

Linear programming optimization problems are like this:

$$\begin{aligned} &\text{Minimize } f(\underline{x})=c_1x_1+c_2x_2+\ldots+c_nx_n \\ &\text{Subject to } \underline{a}_1\underline{x}\le b_1 \\ &\qquad\qquad\quad \underline{a}_2\underline{x}\le b_2 \qquad\qquad (1) \\ &\qquad\qquad\quad \ldots \\ &\qquad\qquad\quad \underline{a}_m\underline{x}\le b_m \end{aligned}$$

We may place all elements c_i into a column-vector $\underline{c}$, all of the row-vectors $\underline{a}_i$ into a matrix $\underline{A}$, and all elements b_i into a column vector $\underline{b}$, so that our optimization problem is now:

$$\begin{aligned} &\text{Minimize } \underline{c}^T\, \underline{x} \\ &\text{Subject to } \underline{A}\, \underline{x} \le \underline{b} \qquad\qquad (2) \end{aligned}$$

Problems that have special structures in the constraint matrices $\underline{A}$ are typically more amenable to decomposition methods. Almost all of these structures involve the constraint matrix $\underline{A}$ being *block-angular*. A block angular constraint matrix is illustrated in Figure 1. In this matrix, the yellow-cross-hatched regions represent sub-matrices that contain non-zero elements. The remaining sub-matrices, not yellow-cross-hatched, contain all zeros. We may think of each yellow-cross-hatched region as a department. The decision variables $\underline{x}_1$ are important only to department 1; the decision variables $\underline{x}_2$ are important only to department 2; and the decision variables $\underline{x}_3$ are important only to department 3. In this particular structure, we have no need of a CEO at all. All departments are completely independent!

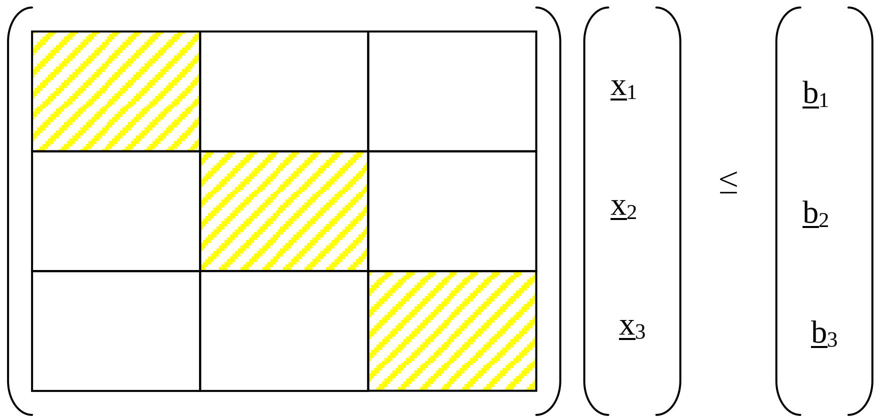


Figure 1: Block-angular structure

In the original description, the CEO choses values for certain variables that affect each department (two 100MVA 69/161 kV units, four 50MVA 13.8/69 kV units, one 325MVA 230/500 kV unit, … etc.). This situation would have different departments linked by these variables. For example, one department would know that the different transformers have different cooling needs that can be satisfied with different numbers and types of fans. Depending on the numbers and types selected, the necessary materials and the necessary labor hours are different. This department has 20 workers whose time can be allocated in various ways among the different tasks necessary to build all of the fans; the department may also hire more workers if necessary.

The structure of the constraint matrix for this situation is shown in Figure 2. The CEO's decisions are integer decisions on the variables contained in $\underline{x}_4$. Variables in $\underline{x}_1$ through $\underline{x}_3$ are decisions that each of departments 1-3, respectively, would need to make.

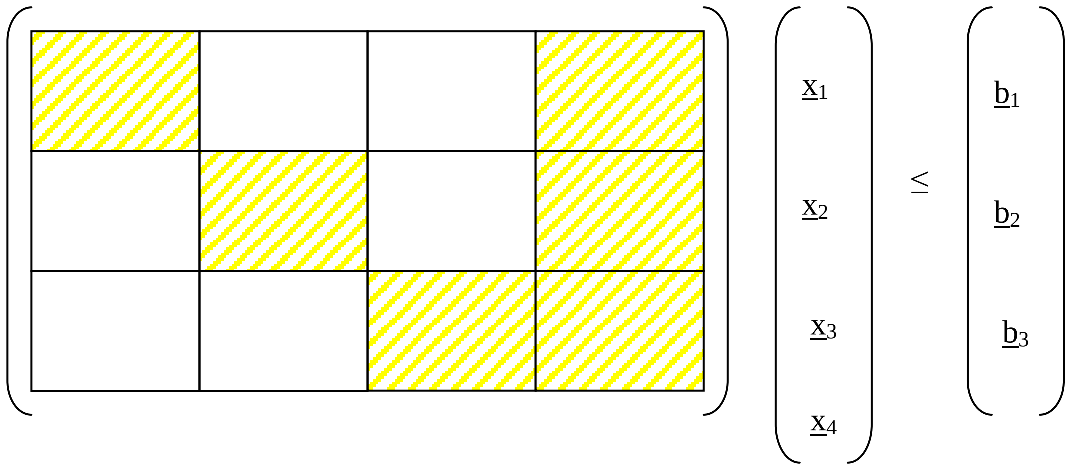


Figure 2: Block-angular structure with linking variables

Alternatively, Figure 3 represents a structure where the decisions linked at the CEO level are such that the CEO must watch out for the entire organization's consumption of resources (e.g., money and labor hours, constrained by $\underline{c}_0$). In this case, the departments are still independent, i.e., they are concerned only with decisions on variables for which no other department is concerned, BUT… the CEO is concerned with constraints that span across the variables for all departments to consume total resources. And so we refer to this structure as block-angular with linking constraints.

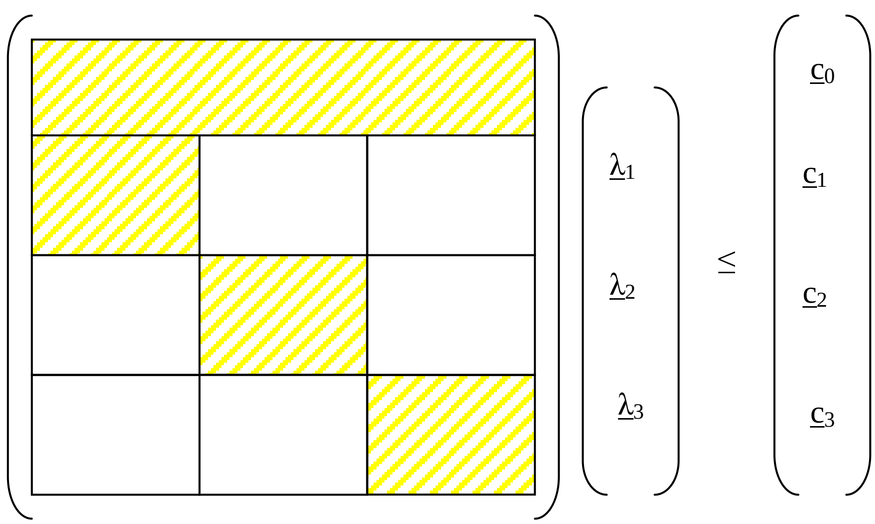


Figure 3: Block-angular structure with linking constraints

3.0 Motivation for decomposition methods: solution speed

To motivate decomposition methods, we consider introducing security constraints to what should be, for power engineers, a familiar problem: the optimal power flow (OPF).

The OPF may be posed as problem P_0.

$$
P_0 \qquad
\begin{array}{lll}
Min & f_0(x_0, u_0) & \\
s.t. & h_k(x_k, u_0) = 0 & k = 0 \\
 & g_k(x_k, u_0) \le g_k^{\max} & k = 0
\end{array}
$$

where $h_k(x_k,u_0)=0$ represents the power flow equations and $g_k(x_k,u_0)\le g_k^{max}$ represents the line-flow constraints. The index k=0 indicates this problem is posed for only the "normal condition," i.e., the condition with no contingencies.

Denote the number of constraints for this problem as N.

Assumption: Let's assume that running time T of the algorithm we use to solve the above problem is proportional to the square of the number of constraints[1], i.e., N^2. For simplicity, we assume the constant of proportionality is 1, so that $T=N^2$.

Now let's consider the security-constrained OPF (SCOPF). Its problem statement is given as problem P_c:

[1] This is a very reasonable assumption for linear programs (LPs) because for LPs, the number of constraints determines the corner points; it is the number of corner points considered within the solution that determines the speed at which the LPs can solve.

$$
P_C \qquad \begin{aligned} & Min && f_0(x_0, u_0) && \\ & s.t. && h_k(x_k, u_0) = 0 && k = 0,1,2,\ldots,c \\ & && g_k(x_k, u_0) \le g_k^{\max} && k = 0,1,2,\ldots,c \end{aligned}
$$

Notice that there are *c* contingencies to be addressed in the SCOPF, and that there are a complete new set of constraints for each of these *c* contingencies. Each set of contingency-related constraints is similar to the original set of constraints (those for problem P_0), except it corresponds to the system with an element removed, and it has a different right-hand-side corresponding to an "emergency" flow limit (k=1,…,c) instead of a "normal" flow limit (k=0).

So the SCOPF must deal with the original N constraints, and also another set of N constraints for every contingency. Therefore, the total number of constraints for Problem P_C is N+cN=(c+1)N.

Under our assumption that running time is proportional to the square of the number of constraints, then the running time will be proportional to $[(c+1)N]^2=(c+1)^2N^2=(c+1)^2T$.

What does this mean?

It means that the running time of the SCOPF is $(c+1)^2$ times the running time of the OPF. So if it takes OPF 1 minute to run, and we want to run SCOPF with 100 contingencies, it will take us 101^2 minutes, or 10,201 minutes to run the SCOPF. This is 170 hours, about 1 week!!!!

Many systems need to address 1000 contingencies. This would take about 2 years!

To address this, we will change the computational procedure of the original problem, as indicated in Fig. 4a, to the computational procedure illustrated in Fig. 4b.

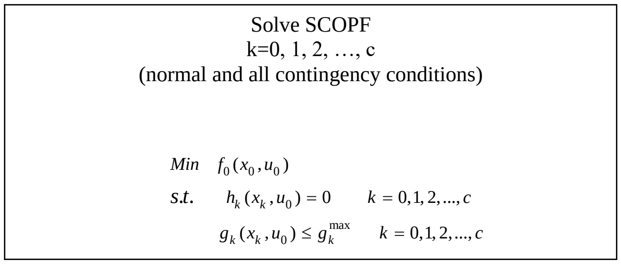


Figure 4: Solution of full SCOPF

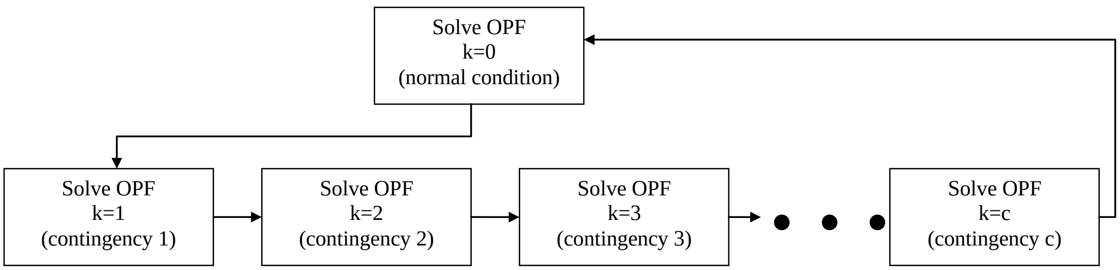


Figure 5: Decomposition solution strategy

The solution strategy first solves the k=0 OPF (master problem) and then takes contingency 1 and re-solves the OPF, then contingency 2 and resolves the OPF, and so on (these are subproblems). For any contingency-OPFs which require a redispatch (relative to the k=0 OPF), an appropriate constraint is generated, and at the end of the cycle these constraints are gathered and applied to the k=0 OPF. Then the k=0 OPF is resolved, and the cycle starts again. Experience has it that such an approach usually requires only 2-3 cycles.

Denote the number of cycles as *m*.

Each of the individual problems has only *N* constraints and therefore requires only T minutes.

There are *(c+1)* individual problems for every cycle.

There are *m* cycles.

So the amount of running time is *m(c+1)T*.

If *c*=100 and *m*=3, *T*=1 minute, this approach requires 303 minutes. That would be about 5 hours (instead of 1 week).

If *c*=1000 and *m*=3, *T*=1 minute, this approach requires about 50 hours (instead of 2 years).

What if it takes 10 cycles instead of 3?
➔If c=1000 and m=10, T=1 minute, this approach requires 167 hours (1 week, instead of 2 years).

What if it takes 100 cycles instead of 3?
➔If c=1000 and m=100, T=1 minute, this approach requires 1668 hours (10 weeks, instead of 2 years).

In addition, this approach is easily parallelizable, i.e., each individual OPF problem can be sent to its own CPU. This will save even more time. Figure 6 compares computing time for a "toy" system. The comparison is between a full SCOPF, a decomposed SCOPF (DSCOPF), and a decomposed SCOPF where the individual OPF problems have been sent to separate CPUs.

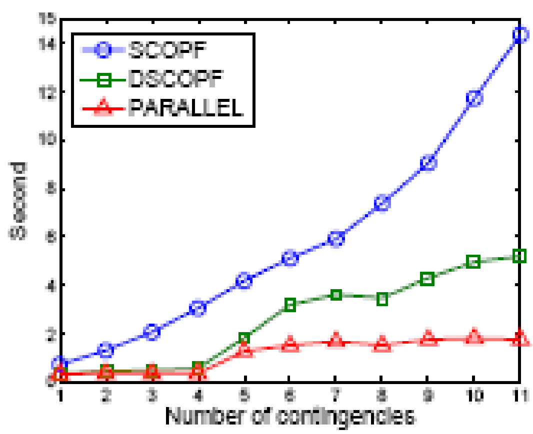
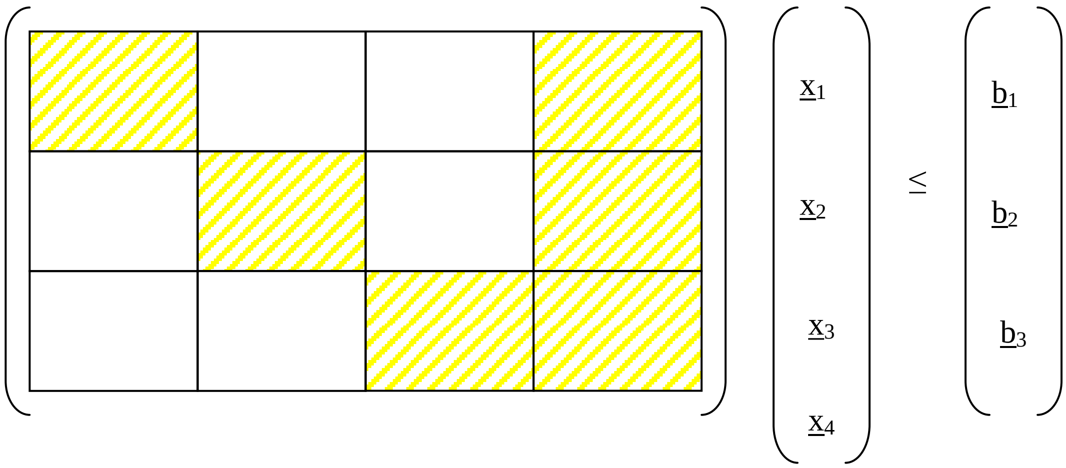


Figure 6

4.0 Benders decomposition

J. F. Benders [3] proposed solving a mixed-integer programming problem by partitioning the problem into two parts – an integer part and a continuous part. It uses the branch-and-bound method on the integer part and linear programming on the continuous part.

The approach is well-characterized by the linking-variable problem illustrated in Figure 2 where here the linking variables are the integer variables. The figure is repeated here for convenience.



In the words of A. Geoffrion [4], "J.F. Benders devised a clever approach for exploiting the structure of mathematical programming problems with complicating variables (variables which, when temporarily fixed, render the remaining optimization problem considerably more tractable)."
Note in the below problem statements, all variables except z_1 and z_2 can be vectors. The problem can be generally specified as follows:

$$\begin{aligned}
&\max z_1 = c_1^T x + c_2^T w \\
&s.t. \\
&Dw \le e \\
&A_1 x + A_2 w \le b \\
&x, w \ge 0 \\
&w \quad \text{integer}
\end{aligned}$$

An example illustrating the matrices of the second constraint might be as follows:

$$A_1 = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{bmatrix}; \quad \mathrm{A}_2 = \begin{bmatrix} 6 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

Thus, the composite matrix would appear as follows, which appears as block-angular with the last variable (*w*, in this case, a scalar) as the linking variable:

$$A_{12} = \begin{bmatrix} 2 & 4 & 0 & 0 & 6 \\ 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 5 & 3 & 1 \end{bmatrix}$$

Define the master problem and primal subproblem as

$$\begin{aligned} &\max z_1 = c_1^T x + c_2^T w \\ &s.t. \\ &Dw \le e \\ &A_1 x + A_2 w \le b \\ &x, w \ge 0 \\ &w \quad \text{integer} \end{aligned} \quad \Rightarrow$$

Master:

$$\begin{aligned} &\max z_1 = c_2^T w + z_2^* \\ &s.t. \\ &Dw \le e \\ &w \ge 0 \\ &w \quad \text{integer} \end{aligned}$$

Primal subproblem:

$$\begin{aligned} &\max z_2 = c_1^T x \\ &s.t. \\ &A_1 x \le b - A_2 w^* \\ &x \ge 0 \end{aligned}$$

We make use of *duality* in what follows. Duality refers to the fact that every linear program (LP), referred to as the *primal problem*, has associated with it a *dual problem*, an equivalent LP, that is related to the primal in certain distinct ways, as identified below.

<u>Some comments on duality for linear programs[2]</u>:

1. If primal objective is to max (min), then dual objective is to min (max).
2. Number of dual decision variables is number of primal constraints. Number of dual constraints is number of primal decision variables.
3. Coefficients of decision variables in dual objective are right-hand-sides of primal constraints.

[2] If you have not taken IE 534, Linear Programming, I encourage you to do so. It is an excellent course.

$$\underbrace{\begin{array}{rl} \multicolumn{2}{l}{\underline{\text{Problem P}}} \\ \multicolumn{2}{l}{\max F = 3x_1 + 5x_2} \\ \text{s.t.} & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{array}{l} \underline{\text{Problem D}} \\ \min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ \text{subject to} \\ \lambda_1 + 3\lambda_3 \ge 3 \\ 2\lambda_2 + 2\lambda_3 \ge 5 \\ \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{array}}_{\text{Dual Problem}}$$

4. Coefficients of decision variables in primal objective are right-hand-sides of dual constraints.

$$\underbrace{\begin{array}{rl} \multicolumn{2}{l}{\underline{\text{Problem P}}} \\ \multicolumn{2}{l}{\max F = 3x_1 + 5x_2} \\ \text{s.t.} & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{array}{l} \underline{\text{Problem D}} \\ \min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ \text{subject to} \\ \lambda_1 + 3\lambda_3 \ge 3 \\ 2\lambda_2 + 2\lambda_3 \ge 5 \\ \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{array}}_{\text{Dual Problem}}$$

5. Coefficients of one variable across multiple primal constraints are coefficients of multiple variables in one dual constraint.

$$\underbrace{\begin{array}{rl} \multicolumn{2}{l}{\underline{\text{Problem P}}} \\ \multicolumn{2}{l}{\max F = 3x_1 + 5x_2} \\ \text{s.t.} & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{array}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{array}{l} \underline{\text{Problem D}} \\ \min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ \text{subject to} \\ \lambda_1 + 0\lambda_2 + 3\lambda_3 \ge 3 \\ 0\lambda_1 + 2\lambda_2 + 2\lambda_3 \ge 5 \\ \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{array}}_{\text{Dual Problem}}$$

Likewise, coefficients of one variable across multiple dual constraints are coefficients of multiple variables in one primal constraint.

All of this means that if the primal constraint matrix is A, the dual constraint matrix is A^T.

6. If primal constraints are ≤ (≥), dual constraints are ≥ (≤).

Let's think about what the above comments 1-5 mean for our LP "general form" problem statements (1) and (2) on pg. 4, repeated here for convenience:

Minimize $f(\underline{x})=c_1x_1+c_2x_2+\ldots+c_nx_n$

Subject to $\underline{a}_1\underline{x}\leq b_1$

$\underline{a}_2\underline{x}\leq b_2$

…

$\underline{a}_m\underline{x}\leq b_m$

which is equivalent to:

Minimize $\underline{c}^T\,\underline{x}$

Subject to $\underline{A}\;\underline{x}\leq\underline{b}$

- Comment 2 means that if the primal has n decision variables and m constraints, then the dual will have m decision variables and n constraints.
- Comment 3 means that the dual objective will be $\underline{b}^T\underline{\lambda}$.
- Comment 4 means that the dual constraints will have right-hand sides of $\underline{c}$.
- Comment 5 means that the dual constraint matrix will be $\underline{A}^T$.

Therefore, the dual problem will be:

Maximize $g(\underline{\lambda})=b_1\lambda_1+b_2\,\lambda_2+\ldots+b_m\lambda_m$

Subject to $\underline{a}_{c1}^T\,\underline{\lambda}\geq c_1$

$\underline{a}_{c2}^T\,\underline{\lambda}\geq c_2$

…

$\underline{a}_{cn}^T\,\underline{\lambda}\geq c_n$

where the notation $\underline{a}_{ck}^T$ refers to the transpose of the column (the "c"-subscript denotes "column") k in the matrix $\underline{A}$. In compact notation, we have:

Maximize $\underline{b}^T\,\underline{\lambda}$

Subject to $\underline{A}^T\,\underline{\lambda}\geq\underline{c}$

Key to understanding the usefulness of the dual is the *strong duality property*, which says that if $\underline{x}^*$ is the optimal solution to the primal and $\underline{\lambda}^*$ is the optimal solution to the dual, then

$$\underline{c}^T\underline{x}^*=\underline{b}^T\underline{\lambda}^* \qquad (5)$$

From this, we can write the dual of our primal subproblem.

The weak duality property says that if $\underline{x}^*$ is a feasible solution to the primal and $\underline{\lambda}^*$ is a feasible solution to the dual, then $\underline{c}^T\underline{x}^*\leq\underline{b}^T\underline{\lambda}^*$. This assumes the primal is a maximization problem.

➔This says that the objective of the dual LP is an upper bound on the objective of the primal LP.

(The sense of the inequality reverses if the primal is a minimization problem, in which case, the objective of the dual LP is a lower bound on the objective of the primal LP).

Primal subproblem :

$$\max z_2 = c_1^T x$$
$$s.t.$$
$$A_1 x \le b - A_2 w^*$$
$$x \ge 0$$

➔

Dual subproblem :

$$\min z_2 = (b - A_2 w^*)^T \lambda$$
$$s.t.$$
$$A_1^T \lambda \ge c_1$$
$$\lambda \ge 0$$

Now consider the master problem and dual subproblem together:

Master :

$$\max z_1 = c_2^T w + z_2^*$$
$$s.t.$$
$$Dw \le e$$
$$w \ge 0$$
$$w \quad \text{integer}$$

Dual subproblem :

$$\min z_2 = (b - A_2 w^*)^T \lambda$$
$$s.t.$$
$$A_1^T \lambda \ge c_1$$
$$\lambda \ge 0$$

My notation generally uses z_2* in the master problem. This is misleading. z_2* is a decision variable in the master problem. Otherwise, use of the "*" notation indicates the variable is optimal, from either the master or the dual.

We make 7 comments about how to solve the original problem using this master-dual subproblem decomposition.

1. *Interdependence*: The master depends on the outcome to the subproblem via generated constraints; the subproblem depends on the master optimal solution w^*. Therefore, the solution to each problem depends on the solution obtained in the other problem.
2. *Iterative procedure*: We will solve the overall problem by iterating between the master and the subproblem. The master will be used to generate a solution w^*, given a value (or a guess) for z_2^*. Then the subproblem will be used to get a new value of z_2^* and λ^* using the solution w^* obtained in the master. This will tell us one very important thing: if we need to resolve the master, we should constrain z_2^* to be no larger than $(b-A_2w^*)^T\lambda^*$, i.e., $z_2^* \le (b-A_2w^*)^T\lambda^*$ in order to ensure that we satisfy the last solution of the subproblem; this directly reduces the master problem objective function and thus, the added constraint is called an "*optimality constraint*".

3. *Upper bound*:
   a. Initial solution: Start the solution procedure by solving the master problem with a guess for an upper bound on z_2*. Since the dual subproblem is going to minimize (lower) z_2, let's be safe and guess a large value of the upper bound on z_2* for this initial master problem solution. Since this value of z_2* is chosen large, we can be sure that the first solution to the master, z_1*, will be above the actual (overall problem optimal) solution, and so we will consider this solution z_1* to be an upper bound on the actual solution.
   b. Successive solutions: As the iterations proceed, we will add constraints to the master problem (adding constraints never improves, or in this case, increases, the optimum), generated by the subproblem, so that the master problem solution z_1*, will continuously decrease towards the actual (overall problem optimal) solution.

   Thus, the value of z_1*, <u>obtained from the master problem</u>, serves as an upper bound on the actual (overall problem optimal) solution.
4. *Lower bound*: The dual problem results in a new value of z_2*, and it can then be added to $c_2^T w$* (where w* was obtained from the last master problem solution) to provide another estimate of z_1*. Since the dual problem minimizes z_2, without the master problem constraints, the term $c_2^T w$* (from master) +z_2* (from dual) will be a lower bound on z_1*.
5. *Feasibility*: An LP primal may result in its solution being optimal, infeasible, or unbounded. These occurrences have implications on what can happen in the dual. And the converse is true: occurrences in the dual have implications regarding what can happen in the primal. Table 1 below summarizes the relationships.

Table 1: Possible combinations of dual and primal solutions

		DUAL		
		Optimal	Infeasible	Unbounded
PRIMAL	Optimal	Possible	Impossible	Impossible
	Infeasible	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

From Table 1, we observe that if the dual problem results in an unbounded solution, then it means the primal problem must be infeasible (note: an infeasible primal implies an unbounded *or* infeasible dual). In Benders, when we solve the dual and obtain unboundedness, it means the primal (which is contained in the master problem) is infeasible (and so the master is infeasible), and we must resolve the master problem with more restrictive constraints on w to force the primal to be feasible. The associated constraints on w are called *feasibility cuts*.

6. *Algorithm*: In what follows, we specify Q as the set of constraints for the master program. It will change via the addition of feasibility and optimality cuts as the algorithm proceeds. Initially, $Q=\{w_k \leq$ large number, for all k, $z_2^* \leq M$, M large$\}$.

Master problem:

$$\begin{aligned}
&\max z_1 = c_2^T w + z_2^* \\
&s.t. \\
&w_k \quad \text{constrained as in } Q \ \ \forall k \\
&z_2^* \le M, \ M \ \text{ is large} \\
&w_k \qquad \text{integer } \ \forall k
\end{aligned}$$

Sub-problem (dual):

$$\begin{aligned}
&\min z_2 = (b - A_2 w^*)^T \lambda \\
&s.t. \\
&A_1^T \lambda \ge c_1 \\
&\lambda_k \ge 0, \qquad \forall k
\end{aligned}$$

There are 3 steps to Benders decomposition.

1. Solve the master problem using Branch and Bound (or any other integer programming method). Designate the solution as w^*.
2. Using the value of w^* found in step 1, solve the sub-problem (the dual) which gives z_2^* and λ^*. There are two possibilities:
   a. If the solution is unbounded (implying the primal is infeasible), adjoin the most constraining feasibility constraint from $(b-A_2w)^T\lambda\geq 0$ to Q, and go to step 1. The constraint $(b-A_2w)^T\lambda\geq 0$ imposes feasibility on the primal because it prevents unboundedness in the dual by imposing non-negativity on the coefficients of each λ_k. We illustrate this challenging concept in an example below.
   b. Otherwise, designate the solution as λ^* and go to step 3.
3. Compare z_1 found in step 1 to $c_2^T w^* + z_2^*$ where w^* is found in step 1; $z_2^* = (b - A_2 w^*)^T \lambda^*$ is found in step 2.There are two possibilities:
   a. If they are equal (or within ε of each other), then the solution (w^*, λ^*) corresponding to the subproblem dual solution, is optimal and the primal variables x^* are found as the dual variables[3] within the subproblem.

[3] Dual variables are the coefficients of the objective function in the final iteration of the simplex method and are provided with the LP solution by a solver like CPLEX. Here, our use of the word

b. If they are not equal, adjoin an <u>optimality constraint</u> to Q given by $z_2^* \le (b - A_2 w^*)^T \lambda^*$ and go to step 1.

Step 3 is a check on Benders optimal rule, stated below.

Figure 7 illustrates the algorithm in block diagram form.

It is useful to study Figure 7 while referring to the statement of Benders optimal rule just above it.

<u>Benders optimal rule</u>: If (z_1^*, w^*) is the optimal solution to the master problem, and (z_2^*, λ^*) is the optimal solution to dual subproblem, and if

$$\overbrace{\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}}}^{\text{Lower bound}} = \overbrace{\underbrace{z_1^*}_{\text{from master problem}}}^{\text{Upper bound}},$$

then (z_1^*, w^*, λ^*) is the optimal solution for the complete problem.

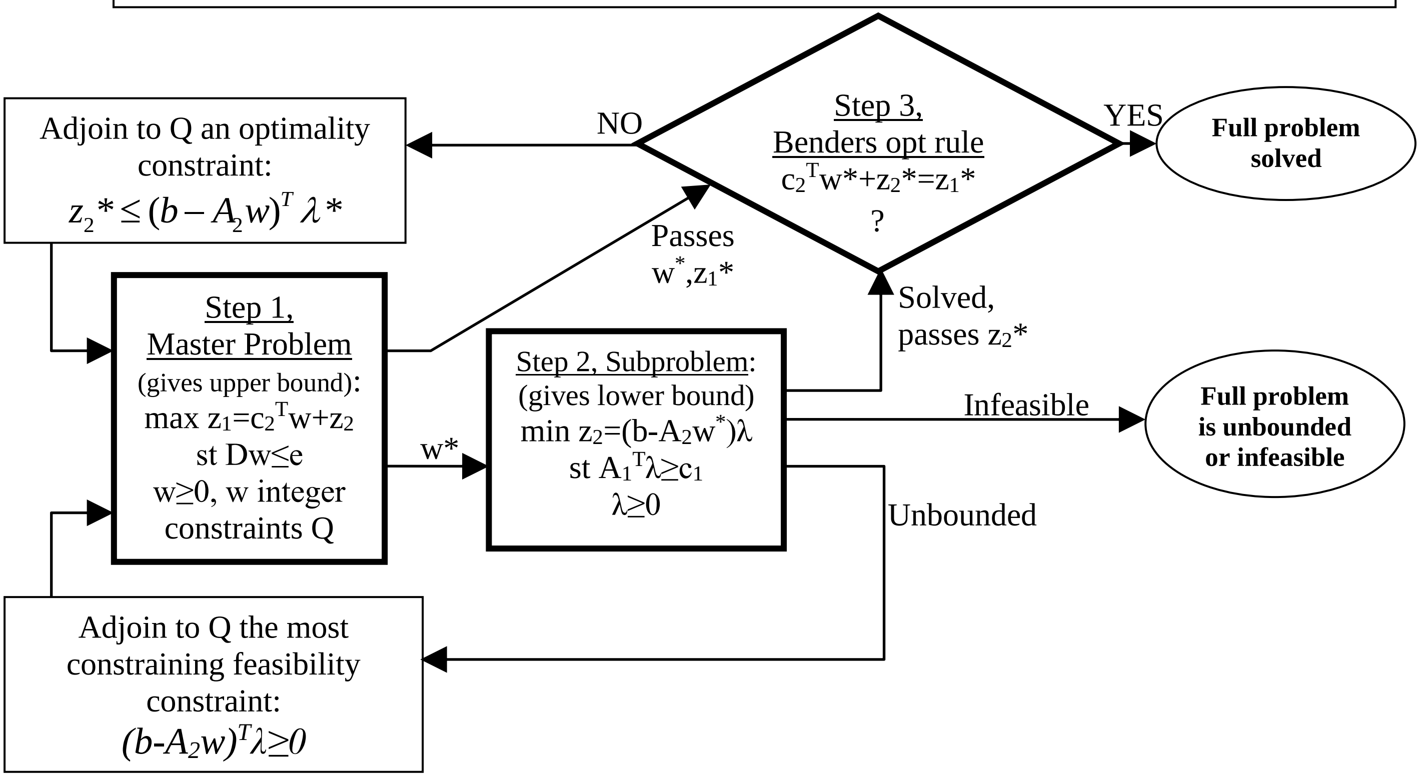


Figure 7: Illustration of Benders Decomposition

"dual" refers to what we previously referred to as the primal. The dual variables x must have values that result in the two objective functions being equal at the optimum: $c^T x = (b - A_2 w^*)^T \lambda^*$.

We will work an example using the formalized nomenclature of the previous summarized steps. But before we do, we introduce the optimization solver CPLEX.

Brief tutorial for using CPLEX.
CPLEX version 12.10.0.0 resides on ISU servers. To access it, you need to logon to an appropriate server (see http://it.engineering.iastate.edu/remote/ for a list of servers with CPLEX). To do that, you need a telnet and ftp facility. You can find instructions on our course website for getting/using the appropriate telnet & ftp facilities (see sec 2 of http://home.eng.iastate.edu/~jdm/ee552/Intro_CPLEX.pdf).

After getting the telnet and ftp facilities set up on your machine, the next thing to do is to construct a file containing the problem. To construct this file, you can use the program called "notepad" under the "accessories" selection of the start button in Windows. Once you open notepad, you can immediately save to your local directory under the filename "filename.lp." You can choose "filename" to be whatever you want, but you will need the extension "lp." To obtain the extension "lp" when you save, do "save as" and then choose "all files." Otherwise, it will assign the suffix ".txt" to your file. Here is what I typed into the file I called "example.lp"…

```
maximize
 12 x1 + 12 x2
subject to
 2 x1 + 2 x2 >= 4
 3 x1 +   x2 >= 3
x1 >= 0
x2 >= 0
end
```

Once you get the above file onto a server having access to CPLEX, you may simply type cplex125

CPLEX> read example.lp

CPLEX> primopt

"Primopt" solves the problem using the primal simplex optimizer. "mipopt" solves mixed integer programs using branch & cut.
More CPLEX info may be found in the tutorial, see above URL.

Consider the following problem P_0 [5]:

$$
P_0 \qquad
\begin{aligned}
\max \quad & z_1 = 4x_1 + 3x_2 + 5w \\
\text{subject to:} \quad & 2x_1 + 3x_2 + w \le 12 \\
& 2x_1 + x_2 + 3w \le 12 \\
& w \le 20 \\
& x_1, x_2, w \ge 0, \\
& w \text{ integer}
\end{aligned}
$$

Clearly P_0 is a mixed integer problem. There are two ways to think about this problem.

First way: Let's redefine the objective function as

$$z_1 = 5w + z_2$$

where

$$z_2 = 4x_1 + 3x_2$$

so that problem P_{1P} below is equivalent to problem P_0 above:

$$
P_{1P} \qquad
\begin{aligned}
\max \quad & z_1 = 5w + \left(\begin{aligned} \max \quad & z_2 = 4x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \le 12 - w \\ & 2x_1 + x_2 \le 12 - 3w \\ & x_1, x_2 \ge 0 \end{aligned} \right) \\
\text{subject to:} \quad & w \le 20 \\
& w \ge 0, \\
& w \text{ integer}
\end{aligned}
$$

This way is similar to the way J. Bloom described his two-stage generation planning problem in [6], which we summarize at the end of these notes.

Second way: Here, we will simply cast the problem into the general form outlined in our three-step procedure.

Comparing this problem (right) to our general formulation (left),

General Formulation		This problem
$\max z_1 = c_1^T x + c_2^T w$		max $\quad z_1 = 4x_1 + 3x_2 + 5w$
$s.t.$		subject to: $\quad 2x_1 + 3x_2 + w \le 12$
$A_1 x + A_2 w \le b$	P$_0$	$2x_1 + x_2 + 3w \le 12$
$x, w \ge 0$		$w \le 20$
w integer		$x_1, x_2, w \ge 0,$
		w integer

we observe that

$$c_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \; c_2 = 5$$

$$A_1 = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; b = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$$

In the next pages, we will go through the first iteration, which is colored green and results in a feasibility cut, and the second iteration, which is colored grey and results in an optimality cut.

Step 1: The master problem is

$$\begin{aligned}&\max z_1 = c_2^T w + z_2^*\\&s.t.\\&Q:\ w_k \quad \text{constrained}\ \forall k,\quad z_2^* \le M,\ M \ \text{ is large}\\&w_k \ge 0 \quad \text{and} \quad \text{integer} \quad \forall k\end{aligned}$$

or

$$\begin{aligned}&\max z_1 = c_2^T w + z_2^*\\&s.t.\\&Q: w \le 20,\ z_2^* \le M\\&w \ge 0 \ \text{ and } \ \text{integer}\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\max z_1 = 5w + z_2^*\\&s.t.\\&Q: w \le 20,\ z_2^* \le M\\&w \ge 0 \ \text{ and } \ \text{integer}\end{aligned}$$

The solution to this problem is trivial: since the objective function is being maximized, we make w and z_2* as large as possible, resulting in w*=20, z_2*=M, and z_1=5*20+M=100+M.

Step 2: Using the value of w found in the master, get the dual:

$$\begin{aligned}&\min z_2 = (b - A_2 w^*)^T \lambda\\&s.t.\\&A_1^T \lambda \ge c_1\\&\lambda \ge 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}1\\3\end{bmatrix} w^*\right)^T \begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\\&s.t.\\&\begin{bmatrix}2 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} \ge \begin{bmatrix}4\\3\end{bmatrix}\\&\lambda_1, \lambda_2 \ge 0\end{aligned}$$

Substituting, from step 1, w*=*20*, the subproblem becomes:

$$\begin{aligned}&\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}1\\3\end{bmatrix} 20\right)^T \begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = -8\lambda_1 - 48\lambda_2\\&s.t.\\&\begin{bmatrix}2 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} \ge \begin{bmatrix}4\\3\end{bmatrix}\\&\lambda_1, \lambda_2 \ge 0\end{aligned}$$

Because the λ_k's are required to be non-negative, all terms in the objective function are negative. Noting the λ_k's are constrained from below by the inequalities, we may make them as large as we like, making the objective function infinitely negative, implying the objective function is unbounded since we are minimizing.

This occurs because the coefficients in the objective function are negative.

➔The coefficients in the objective function are negative because the master problem yielded a poor choice of *w* (in our case, a value of *w* that is too large).

➔The master problem yielded a poor choice of *w* because it was not sufficiently constrained,

We can think of this another way which conforms to Comment #5 made on feasibility (see pg. 16). We know that unboundedness in a dual necessarily implies infeasibility in the primal. In this case, the primal is the problem inside the brackets of Problem P_{1P}. To make this point clear, substitute w=20 into the primal problem resulting in

$$\begin{array}{lll} \max & z_2 = 4x_1 + 3x_2 & \\ \text{subject to} & 2x_1 + 3x_2 \le 12 - w & \\ & 2x_1 + x_2 \le 12 - 3w & \Rightarrow \\ & x_1, x_2 \ge 0 & \end{array} \qquad \begin{array}{ll} \max & z_2 = 4x_1 + 3x_2 \\ \text{subject to} & 2x_1 + 3x_2 \le -8 \\ & 2x_1 + x_2 \le -48 \\ & x_1, x_2 \ge 0 \end{array}$$

Since the right-hand-sides of the inequality constraints are negative, and since the decision variables x_1 and x_2 require non-negativity, then we observe that there is no choice of x_1 and x_2 that will satisfy the inequality constraints. The primal is very definitely infeasible.

If the primal problem, i.e., the problem inside the brackets of P_{1P}, is infeasible, then the whole problem P_{1P} is infeasible.

We need to correct this situation, by taking step 2b, which means we will add a "feasibility constraint" to the master problem. This feasibility constraint is contained in $(b-A_2w)^T\lambda\ge 0$, or

$$\left(\begin{bmatrix}12\\12\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}w\right)^T\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\geq 0$$

or

$$(12-w)\lambda_1+(12-3w)\lambda_2\geq 0$$

We now can see clearly regarding why $(b-A_2w)^T\lambda\geq 0$ is the constraint necessary to ensure feasibility in the primal, and that is because it will avoid unboundedness in the dual. To guarantee that

$$(12-w)\lambda_1+(12-3w)\lambda_2\geq 0$$

without concern for what values of λ_k are chosen, we must make

$$(12-w)\geq 0, \qquad (12-3w)\geq 0$$

resulting in

$$12\geq w, \qquad 4\geq w$$

Alternatively, from a primal point of view, the terms (12-w) and (12-3w) appear on the right-hand-side of the inequalities. Ensuring their non-negativity provides that the primal may be feasible.

Clearly, w must be chosen to satisfy $w\leq 4$. This constraint is added to Q, and we repeat step 1.

<u>Step 1</u>:

$$\max z_1 = 5w + z_2^*$$
$$s.t.$$
$$Q: w\leq 20, \quad w\leq 4, \;\; z_2^*\leq M$$
$$w\geq 0 \quad \text{and integer}$$

The solution is clearly $w=4$, $z_2^*=M$, with $z_1^*=5(4)+M=20+M$.

Step 2: Using the value of *w*=4 found in the master, get the dual.

$$\begin{aligned}&\min z_2 = (b - A_2 w^*)^T \lambda\\ &s.t.\\ &A_1^T \lambda \ge c_1\\ &\lambda \ge 0\end{aligned} \quad \Rightarrow \quad \begin{aligned}&\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}1\\3\end{bmatrix} w^*\right)^T \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}\\ &s.t.\\ &\begin{bmatrix}2 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \ge \begin{bmatrix}4\\3\end{bmatrix}\\ &\lambda_1, \lambda_2 \ge 0\end{aligned}$$

Substituting, from step 1, *w*=4*, the subproblem becomes:

$$\begin{aligned}&\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix} - \begin{bmatrix}1\\3\end{bmatrix} 4\right)^T \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = 8\lambda_1\\ &s.t.\\ &\begin{bmatrix}2 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \ge \begin{bmatrix}4\\3\end{bmatrix}\\ &\lambda_1, \lambda_2 \ge 0\end{aligned}$$

We can use CPLEX LP solver (or any other LP solver) to solve the above, obtaining the solution λ_1*=0, λ_2*=3, with objective function value z_2*=0. Intuitively, one observes that minimization of the objective subject to nonnegativity constraint on λ_1 requires λ_1=0; then λ_2 can be anything as long as it satisfies

$$\begin{aligned}&2\lambda_2 \ge 4 \Rightarrow \lambda_2 \ge 2\\ &\lambda_2 \ge 3\end{aligned}$$

Therefore an optimal solution is λ_1*=0, λ_2*=3. (Although this is a solution, it is a special kind of solution referred to as degenerate because there are many values of λ_2 that are equally good solutions.) Since we have a bounded dual solution (and therefore optimal), our primal is feasible, and we may proceed to step 3 to test for optimality using Benders optimal rule.

<u>Step 3</u>: Compare z_1* found in step 1 to $c_2^T w^* + z_2*$ where $z_2* = (b - A_2 w^*)^T \lambda *$ is found in step 2.

In step 1, solution of the master problem resulted in $z_1*=20+M$.

In step 2, solution of the subproblem resulted in $z_2*=0$.
In both problems, $c_2=5$, and we found (master) or used (sub) $w*=4$.

Benders optimal rule is
$$\underbrace{c_2^T w^*}_{\substack{\text{from master}\\ \text{problem}}} + \underbrace{(b - A_2 w^*)^T \lambda *}_{z_2* \text{ from subproblem}} \overset{?}{=} \underbrace{z_1^*}_{\substack{\text{from master}\\ \text{problem}}}$$

Substitution yields:
$$\underbrace{5 \bullet 4}_{\substack{\text{from master}\\ \text{problem}}} + \underbrace{0}_{z_2* \text{ from subproblem}} \overset{?}{=} \underbrace{20 + M}_{\substack{\text{from master}\\ \text{problem}}}$$

We may think of the left-hand-side as the augmented subproblem objective, and the right-hand-side as the master prob objective. We are asking whether these two are *consistent*. Alternatively, we are asking if z_2* found in the master problem is the same as the objective subproblem objective.

The fact that they are not equal indicates that our solution is not optimal, since it does not satisfy Benders optimal rule. These two problems, the master and the subproblem, are really part of a single problem, and therefore for the single problem to be solved, the solutions to the master and subproblems must be *consistent*. That is, when we maximize $z_1 = c_2^T w^* + z_2*$ in the master, resulting in a value of z_2*, we need to find this value of z_2* to be the same as the solution that the subproblem gives for z_2*. If we do that (since c_2w* is the same for both), the objective function from the master problem, z_1*, will be the same as the sum of $\{c_2^Tw*+z_2*\}$ where z_2* is the objective function from the subproblem.

If we find that z_2* differs in the master and subproblem, as we have found here, then we impose a constraint in the master based on the answer obtained in the subproblem. The fact that this constraint is imposed to satisfy Benders optimal rule means it is imposed to obtain optimality; this makes it an optimality constraint, or in the language of Benders, an *optimality cut*.

We obtain the optimality cut from $z_2^* \le (b - A_2 w)^T \lambda^*$. With

$$b = \begin{bmatrix} 12 \\ 12 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},\ \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$z_2^* \le \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w \right)^T \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 - w & 12 - 3w \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 36 - 9w$$

Now we return to step 1, but before we do, we distinguish between a feasibility cut and an optimality cut:

- *Feasibility cut*: Takes place as a result of finding an unbounded dual subproblem, which, by Table 1, implies an infeasible primal subproblem. **It means that for the value of *w* found in the master problem, there is no possible solution in the primal subproblem**. We address this by adding a feasibility cut (a constraint on w) to the master problem, where that cut is obtained from dual subproblem to avoid its unboundedness, or, alternatively, to avoid the primal subproblem infeasibility.

$$(b - A_2 w)^T \lambda \ge 0.$$

- *Optimality cut*: Takes place as a result of finding that Benders optimal rule is not satisfied, i.e., that

$$\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} < \underbrace{z_1^*}_{\text{from master problem}} \quad \text{instead of} \quad \underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} = \underbrace{z_1^*}_{\text{from master problem}}$$

  It means that the value of z_2^* computed in the master problem (and contained in z_1^*) is larger than the value of z_2^* computed in the subproblem. This must be the case (when Benders optimal rule is not satisfied) since z_1^* is always an upper bound for the solution (see comment 3 regarding "Upper bounds" on pg. 16). We address this by adding an optimality cut (a constraint on z_2^* in terms of w) to the master problem, to force z_2^* in the master problem to be smaller, where that cut is obtained from Benders optimal rule reflecting the maximum which the subproblem allows for z_2^*. The optimality cut is:

$$z_2^* \le (b - A_2 w)^T \lambda^*$$

Observe that $(b - A_2 w)^T$ plays a crucial role in generating both feasibility and optimality cuts. Through this term,

- the feasibility cut, obtained when the dual subproblem is unbounded, imposes additional constraints to limit *w*;
- the optimality cut, obtained when Benders optimal rule is not satisfied, imposes additional constraints to limit z_2^* in terms of *w*.

One last comment is necessary here before we proceed with solving the example. It is also possible that we may find an infeasible subproblem, as shown in Figure 7. Reference to Table 1 indicates that an infeasible dual implies the primal may be either unbounded or infeasible.

If the primal is infeasible, then the situation may at first appear similar to our step 2 situation generated by an unbounded dual, where we were able to avoid the dual unboundedness (and thus the primal infeasibility) by restricting *w* to impose non-negativity on the dual objective coefficients. In other words (focusing on the dual):

- In the case of an unbounded dual, a dual feasible space exists and modifying the dual objective function (by restricting *w*) can avoid the unboundedness.
- But in the case of an infeasible dual, however, there **is no** feasible space, and modifying *w* only affects the dual objective function coefficients (the dual constraints are independent of *w*). The dual space remains infeasible no matter what we do to *w*, and thus the primal remains either infeasible or unbounded, and so the original problem P_{1P} also remains either infeasible or unbounded.

Step 1: Adjoin the optimality cut to *Q*, resulting in the following master problem:

$$\begin{aligned} &\max && z_1 = 5w + \left(z_2^* \quad\right) \\ &\text{subject to :} && Q: \ w \le 20,\ w \le 4,\ z_2^* \le \mathrm{M},\ z_2^* \le 36 - 9w \\ & && w \ge 0,\ w \text{ integer} \end{aligned}$$

This all-integer program can be solved using a branch and bound algorithm (both CPLEX and Matlab have one), but the solution can be identified using enumeration, since *w* can only be 0, 1, 2, 3, or 4. For example, letting *w=0*, we have

$$\max \quad z_1 = \left(z_2^* \quad\right)$$
$$\text{subject to:} \quad Q: \quad z_2^* \le M, z_2^* \le 36$$

The solution is recognized immediately, as z_2*=36, z_1*=36.

Likewise, letting *w=1*, we have

$$\max \quad z_1 = 5 + \left(z_2^* \quad\right)$$
$$\text{subject to:} \quad Q: \quad z_2^* \le M, z_2^* \le 27$$

The solution is recognized immediately, as z_2*=27, z_1*=32.

Continuing on, we find the complete set of solutions are

w=0, z_2*=36, z_1=36
w=2, z_2*=27, z_1=32
w=2, z_2*=18, z_1=28
w=3, z_2*=9, z_1=24
w=4, z_2*=0, z_1=20

Since the first one results in maximizing z_1, our solution is *w**=0, z_2*=36, z_1*=36.

<u>Step 2</u>: Using the value of *w* found in the master, get the dual:

$$\begin{aligned} &\min z_2 = (b - A_2 w^*)^T \lambda \\ &s.t. \\ &A_1^T \lambda \ge c_1 \\ &\lambda \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w^* \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\ &s.t. \\ &\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \ge \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &\lambda_1, \lambda_2 \ge 0 \end{aligned}$$

Substituting, from step 1, *w*=0*, the subproblem becomes:

$$\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} 0 \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 12\lambda_1 + 12\lambda_2$$

$s.t.$

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\lambda_1, \lambda_2 \geq 0$$

We can use CPLEX LP solver (or any other LP solver) to solve the above, obtaining the solution λ_1*=2, λ_2*=0, with objective function value z_2*=24. Since we have a bounded dual solution, our primal is feasible, and we may proceed to step 3.

Step 3: Compare z_1 found in step 1 to $c_2^T w^* + z_2^*$ where $z_2^* = (b - A_2 w^*)^T \lambda^*$ is found in step 2.

In step 1, solution of the master problem resulted in z_1*=*36*

In step 2, solution of the subproblem resulted in z_2*=24.
In both problems, c_2=5, and we found (master) or used (sub) w*=0.

Benders optimal rule is $$\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} \overset{?}{=} \underbrace{z_1^*}_{\text{from master problem}}$$

Substitution yields: $$\underbrace{5 \bullet 0}_{\text{from master problem}} + \underbrace{24}_{z_2^* \text{ from subproblem}} \overset{?}{=} \underbrace{36}_{\text{from master problem}}$$

Benders optimal rule is not satisfied, we need to obtain the optimality cut from $z_2^* \leq (b - A_2 w)^T \lambda^*$. With

$$b = \begin{bmatrix} 12 \\ 12 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$z_2^* \le \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w \right)^T \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 - w & 12 - 3w \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 24 - 2w$$

Now we return to step 1.

<u>Step 1</u>: Adjoin the optimality cut to *Q*, resulting in the following master problem:

$$\begin{aligned} &\max \quad z_1 = 5w + \left(z_2^*\right) \\ &\text{subject to :} \quad Q: \; w \le 20,\; w \le 4,\; z_2^* \le \mathrm{M},\; z_2^* \le 36 - 9w,\; z_2^* \le 24 - 2w \\ &\qquad\qquad w \ge 0,\, w \text{ integer} \end{aligned}$$

This all-integer program can be solved using a branch and bound algorithm (both CPLEX and Matlab have one), but the solution can be identified using enumeration, since *w* can only be 0, 1, 2, 3, or 4. For example, letting $w=0$, we have

$$\begin{aligned} &\max \quad z_1 = \left(z_2^*\right) \\ &\text{subject to:} \quad Q: \quad z_2^* \le \mathrm{M},\; z_2^* \le 36,\; z_2^* \le 24 \end{aligned}$$

The solution is recognized immediately, as z_2*=24, z_1*=24.

Likewise, letting $w=1$, we have

$$\begin{aligned} &\max \quad z_1 = 5 + \left(z_2^*\right) \\ &\text{subject to:} \quad Q: \quad z_2^* \le \mathrm{M},\; z_2^* \le 27,\; z_2^* \le 22 \end{aligned}$$

The solution is recognized immediately, as z_2*=22, z_1*=27.

Continuing on, we find the complete set of solutions is:

w=0, z_2*=24, z_1=24
w=1, z_2*=22, z_1=27
w=2, z_2*=18, z_1=28
w=3, z_2*=9, z_1=24
w=4, z_2*=0, z_1=20

And so the third one results in maximizing z_1, so our solution is w*=2, z_2*=18, z_1*=28.

<u>Step 2</u>: Using the value of *w* found in the master, get the dual:

$$\begin{aligned} &\min z_2 = (b - A_2 w^*)^T \lambda \\ &s.t. \\ &A_1^T \lambda \geq c_1 \\ &\lambda \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w^* \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\ &s.t. \\ &\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &\lambda_1, \lambda_2 \geq 0 \end{aligned}$$

Substituting, from step 1, $w^*=2$, the subproblem becomes:

$$\begin{aligned} &\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} 2 \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 10\lambda_1 + 6\lambda_2 \\ &s.t. \\ &\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &\lambda_1, \lambda_2 \geq 0 \end{aligned}$$

We can use CPLEX LP solver (or any other LP solver) to solve the above, obtaining the solution $\lambda_1^*=0.5$, $\lambda_2^*=1.5$, with objective function value $z_2^*=14$. Since we have a bounded dual solution, our primal is feasible, and we may proceed to step 3.

<u>Step 3</u>: Compare z_1 found in step 1 to $c_2^T w^* + z_2^*$ where $z_2^* = (b - A_2 w^*)^T \lambda^*$ is found in step 2.

In step 1, solution of the master problem resulted in $z_1^*=28$

In step 2, solution of the subproblem resulted in $z_2^*=14$.
In both problems, $c_2=5$, and we found (master) or used (sub) $w^*=2$.

Benders optimal rule is $$\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} \stackrel{?}{=} \underbrace{z_1^*}_{\text{from master problem}}$$

Substitution yields: $$\underbrace{5 \bullet 2}_{\text{from master problem}} + \underbrace{14}_{z_2^* \text{ from subproblem}} \overset{?}{=} \underbrace{28}_{\text{from master problem}}$$

Benders optimal rule is not satisfied, we need to obtain the optimality cut from $z_2^* \le (b - A_2 w)^T \lambda^*$. With

$$b = \begin{bmatrix} 12 \\ 12 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$z_2^* \le \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w \right)^T \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 12 - w & 12 - 3w \end{bmatrix} \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} = 24 - 5w$$

Now we return to step 1.

Step 1: Adjoin the optimality cut to Q, resulting in the following master problem:

$$\begin{array}{ll} \max & z_1 = 5w + \left(z_2^* \right) \\ \text{subject to :} & Q: \; w \le 20, \; w \le 4, \; z_2^* \le \text{M}, \; z_2^* \le 36 - 9w, \; z_2^* \le 24 - 2w, \; z_2^* \le 24 - 5w \\ & w \ge 0, w \text{ integer} \end{array}$$

This all-integer program can be solved using a branch and bound algorithm (both CPLEX and Matlab have one), but the solution can be identified using enumeration, since *w* can only be 0, 1, 2, 3, or 4.

Enumerating the solutions to this problem results in

$w=0$: $z_2^*=24$, $z_1^*=24$
$w=1$: $z_2^*=19$, $z_1^*=24$
$w=2$: $z_2^*=14$, $z_1^*=24$
$w=3$: $z_2^*=9$, $z_1^*=24$
$w=4$: $z_2^*=0$, $z_1^*=20$

We see that w=0, 1, 2, and 3 are equally good solutions

Steps 2 and 3: for each of these solutions, using the value of *w* found in the master, get the dual. Then check Benders rule. The general form of the dual is below.

$$\min z_2 = (b - A_2 w^*)^T \lambda \quad \text{s.t.} \quad A_1^T \lambda \ge c_1, \quad \lambda \ge 0$$

➔

$$\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} w^* \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$s.t.$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \ge \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\lambda_1, \lambda_2 \ge 0$$

Benders optimal rule is $\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} \overset{?}{=} \underbrace{z_1^*}_{\text{from master problem}}$

<u>$w^*=0$, the subproblem becomes</u>:

$$\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} 0 \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 12\lambda_1 + 2\lambda_2$$

$$s.t.$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \ge \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\lambda_1, \lambda_2 \ge 0$$

Solution from CPLEX is $\lambda_1=2$, $\lambda_2=0$, with objective function value $z_2^*=24$.

Benders rule: $\underbrace{5 \bullet 0}_{\text{from master problem}} + \underbrace{24}_{z_2^* \text{ from subproblem}} \overset{?}{=} \underbrace{24}_{\text{from master problem}}$

This solution is optimal. Dual variables obtained from CPLEX are $x_1=6$, $x_2=0$. (These variables are dual variables in the dual problem, therefore they are the variables in our original primal problem).

<u>$w^*=1$, the subproblem becomes</u>:

$$\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}1\right)^T\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}=11\lambda_1+9\lambda_2$$

$$s.t.$$

$$\begin{bmatrix}2 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\geq\begin{bmatrix}4\\3\end{bmatrix}$$

$$\lambda_1,\lambda_2\geq 0$$

Solution from CPLEX is $\lambda_1=0.5$, $\lambda_2=1.5$, with objective function value $z_2^*=19$.

Benders rule: $$\underbrace{5\bullet 1}_{\text{from master problem}} + \underbrace{19}_{z_2^*\text{ from subproblem}} \overset{?}{=} \underbrace{24}_{\text{from master problem}}$$

This solution is optimal. Dual variables obtained from CPLEX are $x_1=4$, $x_2=1$.

<u>$w^*=2$, the subproblem becomes</u>:

$$\min z_2 = \left(\begin{bmatrix}12\\12\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}2\right)^T\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}=10\lambda_1+6\lambda_2$$

$$s.t.$$

$$\begin{bmatrix}2 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\geq\begin{bmatrix}4\\3\end{bmatrix}$$

$$\lambda_1,\lambda_2\geq 0$$

Solution from CPLEX is $\lambda_1=0.5$, $\lambda_2=1.5$, with objective function value $z_2^*=14$.

Benders rule: $$\underbrace{5\bullet 2}_{\text{from master problem}} + \underbrace{14}_{z_2^*\text{ from subproblem}} \overset{?}{=} \underbrace{24}_{\text{from master problem}}$$

This solution is optimal. Dual variables obtained from CPLEX are $x_1=2$, $x_2=2$.

<u>$w^*=3$, the subproblem becomes</u>:

$$\min z_2 = \left(\begin{bmatrix} 12 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} 3 \right)^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 9\lambda_1 + 3\lambda_2$$

$$s.t.$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\lambda_1, \lambda_2 \geq 0$$

Solution from CPLEX is $\lambda_1=0$, $\lambda_2=3$, with objective function value $z_2^*=9$.

Benders rule: $$\underbrace{5 \bullet 3}_{\text{from master problem}} + \underbrace{9}_{z_2^* \text{ from subproblem}} \stackrel{?}{=} \underbrace{24}_{\text{from master problem}}$$

This solution is optimal. Dual variables obtained from CPLEX are $x_1=0$, $x_2=3$.

<u>Problem summary</u>:

Recall our original problem:

$$P_0 \quad \begin{aligned} \max \quad & z_1 = 4x_1 + 3x_2 + 5w \\ \text{subject to}: \quad & 2x_1 + 3x_2 + w \leq 12 \\ & 2x_1 + x_2 + 3w \leq 12 \\ & w \leq 20 \\ & x_1, x_2, w \geq 0, \\ & w \text{ integer} \end{aligned}$$

Optimal solutions to this problem result in an objective function value of $z_1=24$ and are:

- $w=0$, $x_1=6$, $x_2=0$
- $w=1$, $x_1=4$, $x_2=1$
- $w=2$, $x_1=2$, $x_2=2$
- $w=3$, $x_1=0$, $x_2=3$

Some comments about this problem:

1. It is coincidence that the values of x_1 and x_2 for the optimal solution also turn out to be integers.
2. The fact that there are multiple solutions is typical of MIP problems. MIP problems are non-convex.

5.0 Benders simplifications

In the previous section, we studied problems having the following structure:

$$\begin{aligned}
&\max z_1 = c_1^T x + c_2^T w \\
&s.t. \\
&Dw \le e \\
&A_1 x + A_2 w \le b \\
&x, w \ge 0 \\
&w \quad \text{integer}
\end{aligned}$$

and we defined the master problem and primal subproblem as

Master:

$$\begin{aligned}
&\max z_1 = c_2^T w + z_2^* \\
&s.t. \\
&Dw \le e \\
&w \ge 0 \\
&w \quad \text{integer}
\end{aligned}$$

Primal subproblem:

$$\begin{aligned}
&\max z_2 = c_1^T x \\
&s.t. \\
&A_1 x \le b - A_2 w^* \\
&x \ge 0
\end{aligned}$$

However, what if our original problem appears as below, which is the same as the original problem except that it does not contain an "x" in the objective function, although the "x" still remains in one of the constraints.

$$
\begin{aligned}
&\max z_1 = c_2^T w \\
&s.t. \\
&Dw \le e \\
&A_1 x + A_2 w \le b \\
&x, w \ge 0 \\
&w \quad \text{integer}
\end{aligned}
$$

In this case, the master problem and the primal subproblem become:

Master :

$$
\begin{aligned}
&\max z_1 = c_2^T w \\
&s.t. \\
&Dw \le e \\
&w \ge 0 \\
&w \quad \text{integer}
\end{aligned}
$$

Primal subproblem :

$$
\begin{aligned}
&\max z_2 = ??? \\
&s.t. \\
&A_1 x \le b - A_2 w^* \\
&x \ge 0
\end{aligned}
$$

One sees clearly here that the primal subproblem has no z_2 to maximize! One way to address this issue is to introduce a vector of non-negative slack variables *s* having one element for each constraint. We will minimize the sum of these slack variables, so that a non-zero value of this sum indicates the subproblem is infeasible. That is, we replace our primal subproblem with a feasibility check subproblem, as follows:

Master :

$$
\begin{aligned}
&\max z_1 = c_2^T w \\
&s.t. \\
&Dw \le e \\
&w \ge 0 \\
&w \quad \text{integer}
\end{aligned}
$$

Feasibility check subproblem :

$$
\begin{aligned}
&\min \quad v = Ones^T s \\
&s.t. \\
&A_1 x - s \le b - A_2 w^* \\
&x \ge 0, \quad s \ge 0
\end{aligned}
$$

Here, *Ones* is a column vector of 1's, so that $v=Ones^T s$ is the summation of all elements in the column vector *s*. When $v=0$, each constraint in A_1x-$s \le b$-A_2w^* is satisfied so that $A_1x \le b$-A_2w^*, which means the constraints to the original problem are in fact satisfied.

In this case, one observes that if $v=0$, then the problem is solved since Benders optimality rule will always be satisfied.

$$\underbrace{c_2^T w^*}_{\text{from master problem}} + \underbrace{(b - A_2 w^*)^T \lambda^*}_{z_2^* \text{ from subproblem}} = \underbrace{z_1^*}_{\text{from master problem}}$$

Here, z_2 is always zero, and the other two terms come from the master problem, therefore if the problem is feasible, it is optimal, and no step 3 is necessary.

One question does arise, however, and that is what should be the feasibility cuts returned to the master problem if the feasibility check subproblem results in $v>0$? The answer to this is stated in [7] and shown in [8] to be

$$v + \lambda A_2(w^* - w) < 0$$

This kind of problem is actually very common. Figure 5, using a SCOPF to motivate decomposition methods for enhancing computational efficiency, is of this type. This is very similar to the so-called simultaneous feasibility test (SFT) of industry.

The SFT (Simultaneous Feasibility Test) is widely used in SCED and SCUC [9, 10, 11]. SFT is a contingency analysis process. The objective of SFT is to determine violations in all post-contingency states and to produce generic constraints to feed into economic dispatch or unit commitment, where a generic constraint is a transmission constraint formulated using linear sensitivity coefficients/factors.

The ED or UC is first solved without considering network constraints and security constraints. The results are sent to perform the security assessment in a typical power flow. If there is an existing violation, the new constraints are generated using the sensitivity coefficients/ factors and are added to the original problem to solve repetitively until no violation exists. The common flowchart is shown in Figure 8.

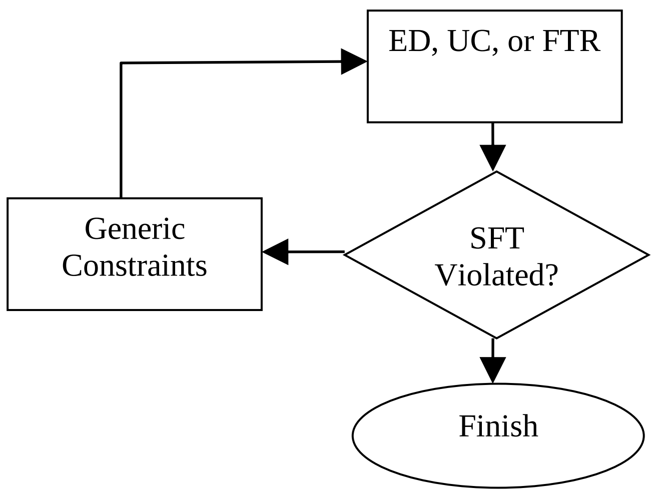


Figure 8

This section has focused on the very common case where the general Benders approach degenerates to a feasibility test problem only, i.e., the optimality test does not need to be done. There are at least two other “degenerate” forms of Benders:

- **<u>No feasibility problem</u>**: In some situations, the optimality problem will be always feasible, and so the feasibility problem is unnecessary.
- **<u>Dual-role feasibility and optimality problem</u>**: In some applications, the feasibility and optimality problem can be the same problem.

Reference [7] provides examples of these degenerate forms of Benders decomposition.

6.0 Application of Benders to other Problem Types

This section is best communicated by quoting from Geoffrion [4] (highlight added), considered the originator of Generalized Benders.

“J.F. Benders devised a clever approach for exploiting the structure of mathematical programming problems with complicating variables (variables which, when temporarily fixed, render the remaining optimization problem considerably more tractable). For the class of problems specifically considered by Benders, fixing the values of the complicating variables reduces the given problem to an ordinary linear program, parameterized, of course, by the value of the complicating variables vector. The algorithm he proposed for finding the optimal value of this vector employs a cutting-plane approach for building up adequate representations of (i) the extremal value of the linear program as a function of the parameterizing vector and (ii) the set of values of the parameterizing vector for which the linear program is feasible. Linear programming duality theory was employed to derive the natural families of cuts characterizing these representations, and the parameterized linear program itself is used to generate what are usually deepest cuts for building up the representations.
In this paper, Benders' approach is generalized to a broader class of programs in which the parametrized subproblem need no longer be a linear program. Nonlinear convex duality theory is employed to derive the natural families of cuts corresponding to those in Benders' case. The conditions under which such a generalization is possible and appropriate are examined in detail.”

The spirit of the above quotations is captured by the below modified formulation of our problem.
The problem can be generally specified as follows:

$$
\begin{aligned}
&\max z_1 = f(x) + c_2^T w \\
&s.t. \\
&Dw \leq e \\
&F(x) + A_2 w \leq b \\
&x, w \geq 0 \\
&w \quad \text{integer}
\end{aligned}
$$

Define the master problem and primal subproblem as

Master:

$$\max z_1 = c_2^T w + z_2^*$$
$$s.t.$$
$$Dw \le e$$
$$w \ge 0$$
$$w \quad \text{integer}$$

Primal subproblem:

$$\max z_2 = f(x)$$
$$s.t.$$
$$F(x) \le b - A_2 w^*$$
$$x \ge 0$$

The Benders process must be generalized to solve the above problem since the subproblem is a nonlinear program (NLP) rather than a linear program (LP). Geoffrion shows how to do this [4].

In the above problem, *w* is integer, the master is therefore a linear integer program (LIP); the complete problem is therefore an integer NLP. If Benders can solve this problem, then it will also solve the problem when *w* is continuous, so that the master is LP and subproblem is NLP. If this is the case, then Benders will also solve the problem where both master and subproblem are LP, which is a very common approach to solving very-large-scale linear programs. Table 2 summarizes the various problems Benders is known to be able to solve.

Table 2

		Master	
		ILP	LP
Subproblem	LP	√	√
	NLP	√	√

One might ask whether Benders can handle a nonlinear integer program in the master, but it is generally unnecessary to do so since such problems can usually be decomposed to an ILP master with a NLP subproblem.

7.0 Generalized Benders for EGEAS

The description of EGEAS provided here is adapted from [12].

The EGEAS computer model was developed by researchers at MIT under funding from the Electric Power Research Institute (EPRI). EGEAS can be run in both the expansion optimization and the production simulation modes. Uncertainty analysis, based on automatic sensitivity analysis and data collapsing via description of function estimation, is also available. A complete description of the model can be found in [13].

The production simulation option consists of production cost/reliability evaluation for a specified generating system configuration during one or more years. Probabilistic production cost/reliability simulation is performed using a load duration curve based model. Customer load and generating unit availability are modeled as random variables to reflect demand fluctuations and generation forced outages. Two algorithmic implementations are available: an analytic representation of the load duration curve (cumulants) and a piecewise linear numerical representation.

EGEAS has three main solution options: Screening curves, dynamic programming, and generalized Benders (GB) decomposition. We discuss here the latter.

GB is a non-linear optimization technique incorporating detailed probabilistic production costing.

- It is based on an iterative interaction of a simplex algorithm master problem with a probabilistic production costing simulation subproblem.
- After a sufficient number of iterations, non-linear production costs and reliability relationship are approximated with as small an error bound as desired by the user.

- It is computationally more efficient than the dynamic programming EGEAS option but produces optimal expansion plans consisting of fractional unit capacity additions.
- It resolves correctly among planning alternative unit sizes, and it models multiple units correctly in terms of expected energy generated and reliability impacts.
- System reliability constraints are modeled according to the probabilistic criterion of expected unserved energy.
- It is suitable for analyses involving thermal, limited energy and storage units, non-dispatchable technology generation, and certain load management activities.
- A unique capability of the GB option is the estimation of incremental costs to the utility associated with meeting allowed unserved energy reliability targets. This capability replaces reliability constraints by an incremental cost of unserved energy to consumers.
- Finally, the GB option has not been developed in its present form to model interconnections or subyearly period production costing/reliability considerations.
- End effects are handled by an extension period model.

The formulation of the GB generating capacity expansion planning problem in EGEAS follows, adapted from [14].

$$\text{minimize}_{\underline{X},\underline{Y}_1,\ldots,\underline{Y}_T} Z = \underline{C}'\underline{X} + \sum_{t=1}^{T} EF_t(\underline{Y}_t) \qquad (1)$$

$$\text{subject to } EG_t(\underline{Y}_t) \leq \epsilon_t \qquad t = 1, \cdots, T \qquad (2)$$

$$0 \leq \underline{Y}_t \leq \delta_t \underline{X} \qquad t = 1, \cdots, T \qquad (3)$$

- $\underline{X}$ = vector of plant capacities, Xj Megawatts (MW) (decision variable);
- j = unique index for each plant;
- $\underline{C}$ = vector of plant present-value capacity costs, Cj $/MW;
- $\underline{Y}$t = vector of plant utilization levels in period t, Yit MW (decision variable);
- i= merit order position of plant in period t;
- EFt($\underline{Y}$t) = present-value expected operating cost function in period t;
- EGt($\underline{Y}$t) = expected unserved energy function in period t;
- εt = desired reliability level in period t, measured in expected MWhr of demand not served;
- δt = matrix which selects and sorts plants, indexed by j, into merit order, indexed by i, in period t;
- T = number of periods (years) in planning horizon.

In this formulation it is assumed that the capacities of all plants are decision variables in order to simplify the notation; however, existing plants of given capacity can be incorporated.

The objective function (1) consists of two components, the capacity costs of the plants and the expected operating costs of the system over the planning horizon.

The constraint (2) represents the reliability standard of the system.

The constraint (3) requires that no plant be operated over its capacity.

Associated with this capacity planning problem there is, for each period t in the planning horizon, an operating subproblem which results from fixing the plant capacities X at trial values.

There is one such subproblem for each period t in the planning horizon; the load duration functions, the operating cost coefficients, and the merit order all depend on the period. The index t has been suppressed below for clarity of notation.

The general subproblem has the following form:

$$\text{minimize}_{Y^1,\dots Y^I}\ EF(\underline{Y}) = \sum_{i=1}^{I} F^i p_i \int_{U^{i-1}}^{U^i} G_i(Q)\,dQ \qquad (4)$$

$$\text{subject to } EG(\underline{Y}) = \int_{U^I}^{\infty} G_{I+1}(Q)\,dQ \le \epsilon \qquad (5)$$

$$0 \le Y^i \le X^i, \qquad i = 1, \cdots, I \qquad (6)$$

- i = index of plant in merit order;
- I= number of plants;
- Yi = utilization level of ith plant, MW (decision variable) (component of vector Y);
- Xi capacity of ith plant, MW (regarded as fixed in the operating problem) (component of the vector δt$\underline{X}$);
- Fi = operating cost of ith plant, $/MWhr;
- pi = 1 - qi= availability of ith plant; qi = FOR of ith plant;
- Gi = equivalent load duration function faced by ith plant;
- Ui = cumulative utilization of first i plants in merit order (Ui-1 is the loading point of the ith plant).

The plant loading points are defined by

$$U^i - U^{i-1} = Y^i, \qquad i = 1, \cdots, I \quad \text{and} \quad U^0 = 0. \qquad (7)$$

Equation (4) is a sum over each units production cost, where individual unit production cost was designated in the course notes on production costing as $C_j(E_j)=b_jE_j$, where E_j is

$$E_j = TA_j \int_{x_{j-1}}^{x_j} F_{D_e}^{(j-1)}(\lambda)d\lambda$$

and $F_{D_e}^{(j-1)}$ is the equivalent load duration curve seen by the j[th] unit, and

$$x_j = \sum_{i=1}^{j} C_i, \qquad x_{j-1} = \sum_{i=1}^{j-1} C_i$$

state the same thing as eq. (7).

This model assumes linearity of the capacity costs with size and of the operating costs with output. In reality, capacity costs for constructing power plants generally exhibit economies of scale and plant operations have decreasing marginal costs at low output levels and increasing marginal costs as output approaches capacity.

The capacity expansion planning problem (1)-(3) can be written in equivalent form as a two-stage optimization

$$\text{minimize}_{\underline{X}\geq 0;\underline{X}\in\Omega}\left\{\underline{C}'\underline{X} + \sum_{t=1}^{T}\left[\begin{array}{l}\text{minimum}_{\underline{Y}_t\geq 0} EF_t(\underline{Y}_t)\\ \text{subject to } EG_t(\underline{Y}_t)\leq \epsilon_t,\ \underline{Y}_t \leq \delta_t\underline{X}\end{array}\right]\right\}$$

where the optimization within the inner brackets is just the operating subproblem (4)-(6). The set Ω consists of all capacity vectors $\underline{X}$ which allow a feasible solution in each of the subproblems.

In addition to [14], Bloom published a number of other papers addressing his work on applying Benders decomposition to the expansion planning problem. These include

- Reference [15]: This paper describes methods of including power plants with limited energy (e.g., hydro) and storage plants in the production costing convolution algorithm we have studied, for use in a Benders decomposition formulation of the expansion planning problem where the production costing problem is the subproblem, and there is one for each period of the planning horizon.
- Reference [16]:
- Reference [17]:

In addition to the work reported on using Benders in EGEAS, there are many other works related to application of Benders decomposition to electric power planning problems. A representative sample of them include [18, 19, 20, 21, 22, 23,…].

8.0 Application of Benders to Stochastic Programming

For good, but brief overviews of Stochastic Programming, see [24] and [25].

In our example problem, we considered only a single subproblem, as shown below.

$$\begin{aligned}
&\max z_1 = c_1^T x + c_2^T w \\
&s.t. \\
&Dw \le e \\
&A_1 x + A_2 w \le b \\
&x, w \ge 0 \\
&w \quad \text{integer}
\end{aligned}$$

To prepare for our generalization, we rewrite the above in a slightly different form, using slightly different notation:

$$
\begin{aligned}
&\max Z_1 = c^T w + d_1^T x_1 \\
&s.t. \\
&Dw \le e \\
&B_1 w + A_1 x_1 \le b_1 \\
&x_1, w \ge 0 \\
&w \quad \text{integer}
\end{aligned}
$$

Now we are in position to extend our problem statement so that it includes more than a single subproblem, as indicated in the structure provided below.

$$
\begin{aligned}
&\max Z_1 = c^T w + d_1^T x_1 + d_2^T x_2 + \dots + d_n^T x_n \\
&s.t. \\
&Dw \le e
\end{aligned}
$$

$$
\begin{array}{llllll}
B_1 w & + A_1 x_1 & & & \le & b_1 \\
B_2 w & & + A_2 x_2 & & \le & b_2 \\
 & & \ddots & & \le & \vdots \\
B_m w & & & + A_n x_n & \le & b_n
\end{array}
$$

$$
x_k, w \ge 0
$$

In this case, the master problem is

$$
\begin{aligned}
&\max Z_1 = c^T w + \sum_{i=1}^{n} z_i(x_i) \\
&s.t. \\
&Dw \le e \\
&w \ge 0
\end{aligned}
$$

where z_i provide values of the maximization subproblems given by:

$$
\begin{aligned}
&\max z_i = d_i^T x_i \\
&s.t. \\
&A_i x_i \le b_i - B_i w \\
&x_i \ge 0
\end{aligned}
$$

Note that the constraint matrix for the complete problem appears as:

$$
\begin{bmatrix} D & & & & \\ B_1 & A_1 & & & \\ B_2 & & A_2 & & \\ \vdots & & & \ddots & \\ B_n & & & & A_n \end{bmatrix}
\begin{bmatrix} w \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\le
\begin{bmatrix} e \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

The constraint matrix shown above, if one only considers D, B_1, and A_1, has an L-shape, as indicated below.

$$
\begin{bmatrix} D & & & & \\ B_1 & A_1 & & & \\ B_2 & & A_2 & & \\ \vdots & & & \ddots & \\ B_n & & & & A_n \end{bmatrix}
\begin{bmatrix} w \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\le
\begin{bmatrix} e \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Consequently, methods to solve these kinds of problems, when they are formulated as stochastic programs, are called L-shaped methods.

But what is, exactly, a stochastic program [24]?

- A stochastic program is an optimization approach to solving decision problems under uncertainty where we make some choices for "now" (the current period) represented by w, in order to minimize our present costs.
- After making these choices, event i happens, so that we take *recourse*[4], represented by x_i, in order to minimize our costs under each event i that could occur in the next period.

[4] Recourse is the act of turning or applying to a person or thing for aid.

- Our decision must be made *a-priori*, however, and so we do not know which event will take place, but we do know that each event i will have probability p_i.
- Our goal, then, is to minimize the cost of the decision for "now" (the current period) plus the expected cost of the later recourse decisions (made in the next period).

An application of this problem for power systems is the security-constrained optimal power flow (SCOPF) with corrective action.

- In this problem, we dispatch generation to minimize costs for the network topology that exists in this 15 minute period. Each unit generation level is a choice, and the complete decision is captured by the vector w. The dispatch costs are represented by $c^T w$.
- These "normal" conditions are constrained by the power flow equations and by the branch flow and unit constraints, all of which are captured by $Dw \leq e$.
- Any one of i=1,…,n contingencies may occur in the next 15 minute period. Given that we are operating at w during this period, each contingency i requires that we take corrective action (modify the dispatch, drop load, or reconfigure the network) specified by x_i.
- The cost of the corrective action for contingency i is $d_i^T x_i$, so that the expected costs over all possible contingencies is $\Sigma p_i d_i^T x_i$.
- Each contingency scenario is constrained by the post-contingency power flow equations, and by the branch flow and unit constraints, represented by $B_i w + A_i x_i \leq b_i$. The dependency on w (the pre-contingency dispatch) occurs as a result of unit ramp rate limitations, to constrain each unit's redispatch to an amount that can be achieved within a given time frame, i.e., for each unit, the vector w would contain P_0 (pre-contingency dispatch), the vector x would contain ΔP^+ (increases) and ΔP^- (decreases), so that the equations

  $$P_0 + \Delta P^+ \leq b^+, \quad -P_0 + \Delta P^- \leq b^-$$

  would be represented in $B_i w + A_i x_i \leq b_i$.

A 2-stage recourse problem is formulated below:

$$\begin{aligned}
&\min z_1 = c^T w + \sum_{i=1}^{n} p_i d_i^T x_i \\
&s.t. \\
&Dw \le e \\
&\begin{array}{lllll}
B_1 w & + A_1 x_1 & & & \le \quad b_1 \\
B_2 w & & + A_2 x_2 & & \le \quad b_2 \\
 & & \ddots & & \le \quad \vdots \\
B_m w & & & + A_n x_n & \le \quad b_n
\end{array} \\
&x, w \ge 0
\end{aligned}$$

where p_i is the (scalar) probability of event i, and d_i is the vector of costs associated with taking recourse action x_i. Each constraint equation $B_i w + A_i x_i \le b_i$ limits the recourse actions that can be taken in response to event i, and depends on the decisions w made for the current period.

Formulation of this problem for solution by Benders (the L-shaped method) results in the master problem as

$$\begin{aligned}
&\min z_1 = c^T w + \sum_{i=1}^{n} z_i \\
&s.t. \\
&Dw \le e \\
&w \ge 0
\end{aligned}$$

where z_i is minimized in the subproblem given by:

$$\begin{aligned}
&\min z_i = p_i d_i^T x_i \\
&s.t. \\
&A_i x_i \le b_i - B_i w \\
&x_i \ge 0
\end{aligned}$$

Note that the first-period decision, w, does not depend on which second-period scenario actually occurs (but does depend on a probabilistic weighting of the various possible futures). This is called the *non-anticipativity property*. The future is uncertain and so today's decision cannot take advantage of knowledge of the future.

Recourse models can be extended to handle multistage problems,

- where a decision is made "now" (in the current period),
- we wait for some uncertainty to be resolved,
- and then we make another decision based on what happened.

The objective is to minimize the expected costs of all decisions taken. This problem can be appropriately thought of as the coverage of a decision tree, as shown in Fig. 7, where each "level" of the tree corresponds to another stochastic program.

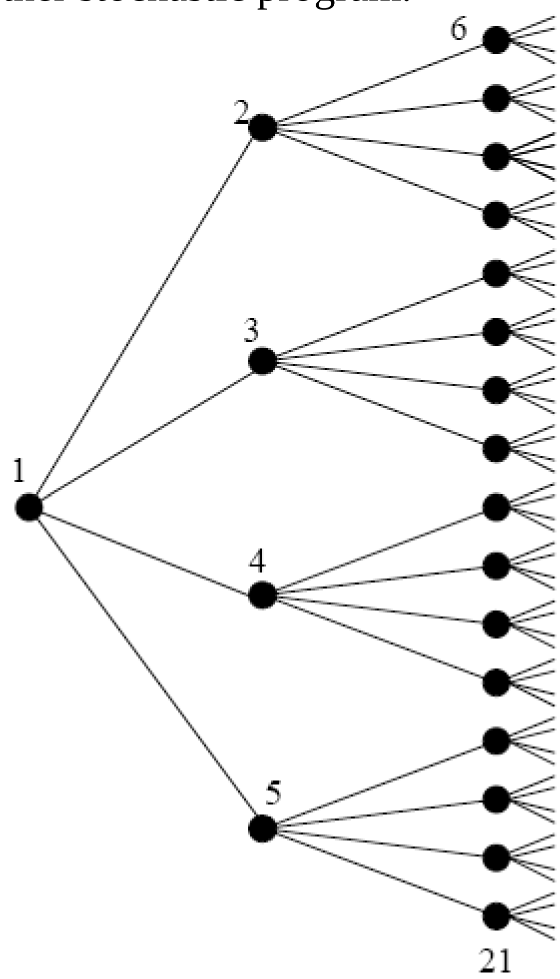


Fig. 7

Multistage stochastic programs have been applied to handle uncertainty in planning problems before. This is a reasonable approach; however, one should be aware that computational requirements increase with number of time periods and number of scenarios (contingencies in our example) per time period. Reference [26] by J. Beasley provides a good but brief overview of multistage stochastic programming. Reference [27], notes for an entire course, provides a comprehensive treatment of stochastic programming including material on multistage stochastic programming. Dr. Sarah Ryan of ISU's IMSE department teaches a course on this topic, described below.

> I E 633X. Stochastic Programming. **(3-0) Cr. 3. S.** ***Prereq: I E 513 or STAT 447, I E 534 or equivalent*. Mathematical programming with uncertain parameters; modeling risk within optimization; multi-stage recourse and probabilistically constrained modes; solution and approximation algorithms including dual decomposition and progressive hedging; and applications to planning, allocation and design problems.**

9.0 Two related problem structures

We found the general form of the stochastic program to be

$$
\begin{aligned}
&\max z_1 = c^T w + d_1^T x_1 + d_2^T x_2 + \ldots + d_n^T x_n \\
&s.t. \\
&Dw \le e \\
&\begin{array}{lllll}
B_1 w & + A_1 x_1 & & & \le & b_1 \\
B_2 w & & + A_2 x_2 & & \le & b_2 \\
 & & \ddots & & \le & \vdots \\
B_m w & & & + A_n x_n & \le & b_n
\end{array} \\
&x_k, w \ge 0
\end{aligned}
$$

so that the constraint matrix appears as

$$
\begin{bmatrix} D & & & & \\ B_1 & A_1 & & & \\ B_2 & & A_2 & & \\ \vdots & & & \ddots & \\ B_n & & & & A_n \end{bmatrix}
\begin{bmatrix} w \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\le
\begin{bmatrix} e \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

If we move the w vector to the bottom of the decision vector, the constraint matrix appears as

$$
\begin{bmatrix} A_1 & & & & B_1 \\ & A_2 & & & B_2 \\ & & \ddots & & \vdots \\ & & & A_n & B_n \\ & & & & D \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ w \end{bmatrix}
\le
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ e \end{bmatrix}
$$

Notice that this is the structure that we introduced in Fig. 3 at the beginning of these notes, repeated here for convenience. We referred to this structure as "*block angular with linking variables*." Now we will call it the *Benders structure*.

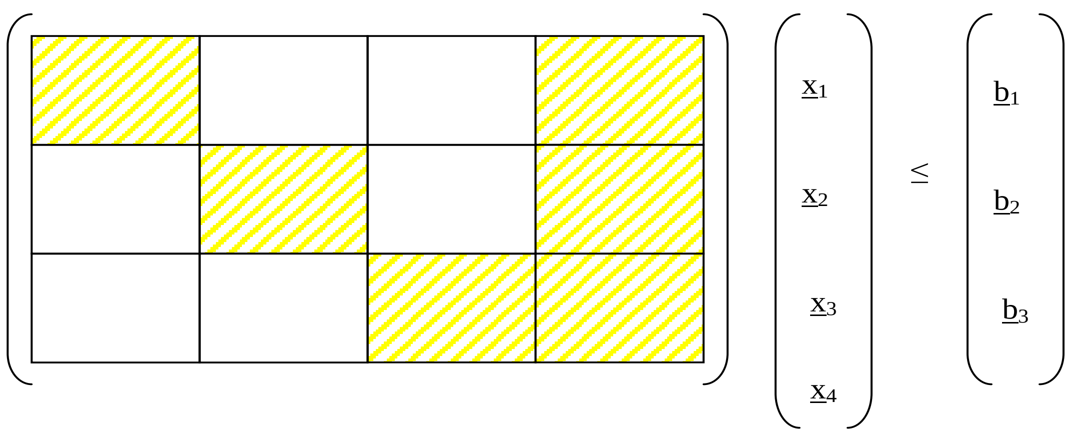


Fig. 3

This means that coupling exists between what would otherwise be independent optimization problems, and that coupling occurs via the variables, in this case w.

For example, in the SCOPF with corrective action, the almost-independent optimization problems are the *n* optimization problems related to the *n* contingencies. The dependency occurs via the dispatch determined for the existing (no-contingency) condition.

The first example at the beginning of these notes shows this structure. In this example, the CEO decided which of 100 products would be made by the company, and then each department has to optimize its resources accordingly. The CEO's decisions are 1-0 decisions on the 100 variables contained in x_4. The variables in x_1 through x_3 would be the decisions that each of departments 1-3 would need to make.

Recall from our discussion of duality that one way that primal and dual problems are related is that

- coefficients of one variable across multiple primal constraints
- are coefficients of multiple variables in one dual constraint,

as illustrated below.

$$\underbrace{\begin{array}{l} \underline{\text{Problem P}} \\ \max F = 3x_1 + 5x_2 \\ \text{s.t.}\ x_1 \qquad\quad \le 4 \\ \qquad\quad 2x_2 \le 12 \\ \quad 3x_1 + 2x_2 \le 18 \\ \quad x_1 \ge 0, x_2 \ge 0 \end{array}}_{\text{Primal Problem}} \Leftrightarrow \underbrace{\begin{array}{l} \underline{\text{Problem D}} \\ \min G = 4\lambda_1 + 12\lambda_2 + 18\lambda_3 \\ \text{subject to} \\ \lambda_1 + 3\lambda_3 \ge 3 \\ 2\lambda_2 + 2\lambda_3 \ge 5 \\ \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{array}}_{\text{Dual Problem}}$$

This can be more succinctly stated by saying that the dual constraint matrix is the transpose of the primal constraint matrix. You should be able to see, then, that the structure of the dual problem to a problem with the Benders structure looks like Fig. 2, repeated here for convenience.

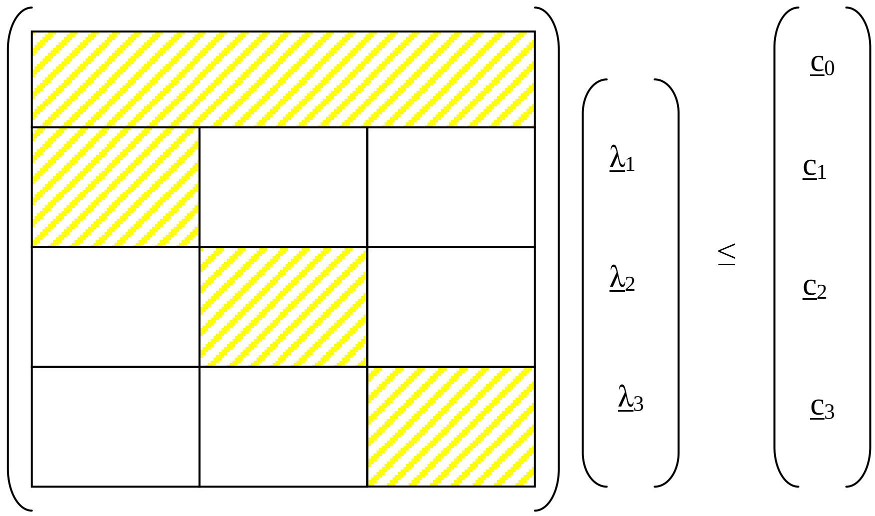


Fig. 2

This structure differs from that of the Benders structure (where the subproblems were linked by variables) in that now, the subproblems are linked by constraints. This problem is actually solved most effectively by another algorithm called Dantzig-Wolfe (DW) decomposition. We refer to the above structure as the DW structure. It is illustrated in the next section using a GEP problem.

10.0 A GEP formulation resulting in a DW structure

Consider the following network for which a generation expansion planning (GEP) problem will be solved.

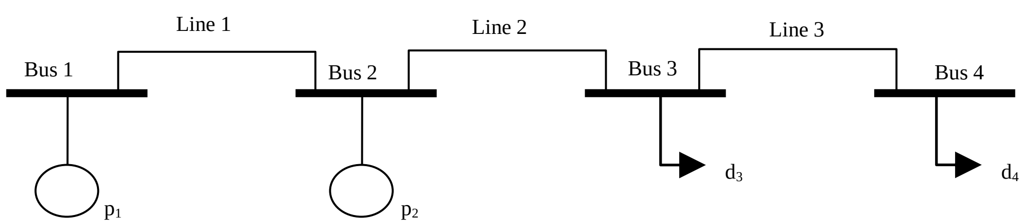


Fig. 8

This GEP problem has the following features:

1. There are four buses, buses 1 and 2 have only generation, and buses 3 and 4 have only load.
2. There are two periods, t=1, 2.

3. There are two existing generating units, k=1, 2, having capacities at time t=0 of C_{10} and C_{20}, respectively.
4. Expansion can only occur at the two generation units, in either periods 1 or 2. Thus, the decision variables for investment are x_{kt} and represent the additional capacity added to unit k at time t. The specific investment-related decision variables are then x_{11}, x_{21}, x_{12}, and x_{22}.
5. The operation-related decision variables are the generation levels at each unit k at time t, p_{kt}. The specific generation level variables are then p_{11}, p_{21}, p_{12}, and p_{22}.
6. The operation-related load parameters are d_{31}, d_{41}, d_{32}, and d_{42}.
7. The operation-related bus angles are θ_{11}, θ_{21}, θ_{31}, θ_{41}, θ_{12}, θ_{22}, θ_{32}, and θ_{42}.
8. The DC power flow matrix relating bus angles to bus injections for period 1 is $\underline{B}_1$ and for period 2 is $\underline{B}_2$. (In this problem formulation, there is no transmission investment, and we will not consider outages, therefore these matrices will be the same). For our 4-bus system, these matrices are dimension 4x4 with elements b_{ijt}.
9. There are three lines; they are flow-constrained to flows $\underline{P}_{L1}$ for period 1 and to flows $\underline{P}_{L2}$ for period 2. Flows are computed as a function of angles using $\underline{DA\theta}$, where $\underline{D}$ is the square diagonal matrix of susceptances and $\underline{A}$ is the node-arc incidence matrix. Again, because there is no transmission investment, $\underline{D}_1$ and $\underline{D}_2$ are identical, and $\underline{A}_1$ and $\underline{A}_2$ are identical. With 3 lines and 4 buses, the matrix $\underline{DA}$ is 3x4, with elements denoted by s_{jkt} (line j, bus k, period t).
10. We have the following equations for our system:

Period 1 generation constraints:	$\underline{p}_1-\underline{x}_1 \leq \underline{C}_0$
Period 2 generation constraints:	$\underline{p}_2-\underline{x}_1-\underline{x}_2 \leq \underline{C}_0$
Period 1 DC power flow eqts:	$\underline{p}_1-\underline{B}_1\underline{\theta}_1=\underline{0}$
Period 1 Line flow constraints:	$\underline{D}_1\underline{A}_1\underline{\theta}_1 \leq \underline{P}_{B1}$
Period 2 DC power flow eqts:	$\underline{p}_2-\underline{B}_2\underline{\theta}_2=\underline{0}$
Period 2 Line flow constraints:	$\underline{D}_2\underline{A}_2\underline{\theta}_2 \leq \underline{P}_{B2}$

We have written the above equations to conform to how we want to order them in our constraint matrix.

11. Define α_1 and α_2 as the discount factors for periods 1 and 2, respectively, and c_1 and c_2 as the cost coefficients associated with units 1 and 2 generation, respectively.

Given the above points 1-11, we write our optimization problem as

$$\min\left\{\alpha_1\left(\underbrace{\overbrace{x_{11}+x_{21}}^{\text{investment costs}}+\overbrace{c_1p_{11}+c_1p_{21}}^{\text{operational costs}}}_{\text{period 1 costs}}\right)+\alpha_2\left(\underbrace{\overbrace{x_{12}+x_{22}}^{\text{investment costs}}+\overbrace{c_1p_{12}+c_1p_{22}}^{\text{operational costs}}}_{\text{period 2 costs}}\right)\right\}$$

Subject to[5]

$$\left[\begin{array}{cc:cccccc:cc:cccccc}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & -1 & 0 & b_{111} & b_{121} & b_{131} & b_{141} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & b_{211} & b_{221} & b_{231} & b_{241} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_{311} & b_{321} & b_{331} & b_{341} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_{411} & b_{421} & b_{431} & b_{441} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & s_{111} & s_{121} & s_{131} & s_{141} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & s_{211} & s_{221} & s_{231} & s_{241} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & s_{311} & s_{321} & s_{331} & s_{341} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & b_{112} & b_{122} & b_{132} & b_{142}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & b_{212} & b_{222} & b_{232} & b_{242}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{312} & b_{322} & b_{332} & b_{342}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{412} & b_{422} & b_{432} & b_{442}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_{112} & s_{122} & s_{132} & s_{142}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_{212} & s_{222} & s_{232} & s_{242}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_{312} & s_{322} & s_{332} & s_{342}
\end{array}\right]
\begin{bmatrix} x_{11}\\ x_{21}\\ p_{11}\\ p_{21}\\ \theta_{11}\\ \theta_{21}\\ \theta_{31}\\ \theta_{41}\\ x_{12}\\ x_{22}\\ p_{12}\\ p_{22}\\ \theta_{12}\\ \theta_{22}\\ \theta_{32}\\ \theta_{42} \end{bmatrix}
\begin{matrix}\le\\=\end{matrix}
\begin{bmatrix} C_{10}\\ C_{20}\\ C_{10}\\ C_{20}\\ 0\\ 0\\ -d_{31}\\ -d_{41}\\ P_{L11}\\ P_{L21}\\ P_{L31}\\ 0\\ 0\\ -d_{32}\\ -d_{42}\\ P_{L12}\\ P_{L22}\\ P_{L32} \end{bmatrix}$$

Comparing to Fig. 2, we observe the DW structure in the above constraint matrix. The yellow area at the top identifies the linking constraints, and the yellow submatrices below indicate two different subproblems corresponding to the operating conditions in the two different periods.

There are several observations/questions/thoughts to consider now:

[5] The constraint equation uses both "≤" and "=" to indicate both types of constraints exist. The constraints with "=" (e.g., the DC power flow equations $\underline{p}=\underline{B}\underline{\theta}$) can be expressed as $\underline{p}\le\underline{B}\underline{\theta}$ and $\underline{p}\ge\underline{B}\underline{\theta}$; doing so increases the size of each subproblem block in the constraint matrix, but it does not change the structure of the constraint matrix.

1. Observation: The subproblems in the above DW structure are associated with the network equations in different periods.
2. Observation: The fact that the above problem decomposes by period is encouraging with respect to increasing the fidelity of the production cost representation.
3. Question: Bloom used Benders to solve his problem, and so we assume he obtained a Benders structure. Is it important, relative to obtaining a Benders structure, that Bloom did not account for the network? That is, if we include the network, does that influence our ability to obtain a Benders structure?
4. Question: Is the DW decomposition method efficient? Consider the following comment from [28]:

   > "*Thus, although (DW) has a smaller number of constraints, its number of variables can be huge since the number of extreme points and extreme rays of a polyhedron can be very large. To use this idea to effectively solve large scale LP problems, we need to avoid considering all extreme points and extreme rays of $Cx \geq d$. This is when the idea of column generation comes into play, i.e., we start by including only a few number of extreme points and extreme rays in the problem and we add more on the fly in a as needed basis.*"

   There is another reference, [29], that raises questions about using either DW or Benders for decomposed, parallelized solutions of large linear programs (in contrast to large MIPs). For example, he writes, "*At present Dantzig-Wolfe and Benders methods in the original versions are rarely used for solving large scale continuous linear programming problems. The reason is their too slow convergence*."
5. Observation: The DW decomposition procedure is described in Section 11.0. It is good to review Section 11.0 to begin to get familiar with DW, trying to think of reasons it might not work very well. Following review of Section 11.0, if it still looks promising, we should identify it as a promising method and list it among other possibly promising methods. At this point, it

might be good to start to review individual references about DW, e.g., [2, 5, 46] and others.

6. Thought: Consider the following additional approaches:
    a. Nested decomposition: Use a nested decomposition algorithm [30] and solving it iteratively in a forward and backward manner using forward and backward passes. In essence, the forward pass iteratively solves over each time period to yield an upper bound for each period over the full problem, and the backward pass provides a lower bound by generating cuts from relaxed sub-problems. The convergence criteria is determined by a pre-defined tolerance in terms of the difference between the upper and lower bounds.
    b. CPLEX's barrier algorithm: One statement from [31] is interesting: "*The CPLEX barrier optimizer is appropriate and often advantageous for large problems, for example, those with more than 100,000 rows or columns. It is not always the best choice, though, for sparse models with more than 100,000 rows. It is effective on problems with staircase structures or banded structures in the constraint matrix. It is also effective on problems with a small number of nonzeros per column (perhaps no more than a dozen nonzero values per column). In short, denseness or sparsity are not the deciding issues when you are deciding whether to use the barrier optimizer. In fact, its performance is most dependent on these characteristics:*
        - *the number of floating-point operations required to compute the Cholesky factor;*
        - *the presence of dense columns, that is, columns with a relatively high number of nonzero entries*."

    But what about parallelization? CPLEX offers the parallel barrier optimizer. A good treatment of CPLEX algorithms is in [32]. Some insights on using CPLEX for large LPs can be found in [33]. Some related questions:

- Does use of GAMS offer the full capabilities of the parallel barrier algorithm?
- Is AIMMS better at using the CPLEX parallel barrier algorithm than GAMS?
- Does GORUBI have a better parallel barrier algorithm than CPLEX?
- Does Pyomo have a better parallel barrier algorithm than CPLEX?

c. Problem structure: There may be a natural decomposition based on our particular adaptive expansion planning problem. For example, can we parallelize the operating problems for each future (i.e., scenario)?

d. Lagrangian relaxation (LR): LR is also an effective approach for problems that have the DR structure (i.e., problems that have block structure with complicating constraints). References [34, 35] provide good tutorial treatment of LR. Paraphrasing from [34]:

   ➔*In LR, the complicating constraints are removed from the constraint equations and then dualized, i.e., added to the objective function, with a penalty term (the Lagrange multiplier) proportional to the amount of violation of the dualized constraints. It is a relaxation of the original problem because (a) the removal of some constraints relaxes the original feasible space; (b) the solution to the LR problem will bound the original problem (from above if maximizing and from below if minimizing) because the addition of the complicating constraints will always cause the objective function to increase (if maximizing) or decrease (if minimizing).*

e. Progressive hedging: Reference [36] indicates that "*PH, sometimes referred to as a horizontal decomposition method because it decomposes stochastic programs by scenarios rather than by time stages, possesses theoretical convergence properties when all decision* *variables are continuous*." Reference [37] indicates, "*The progressive hedging algorithm (PHA) has emerged as an effective method for solving multi-stage stochastic programs, particularly those with discrete decision variables in every stage. The PHA mitigates the computational difficulty associated with large problem instances by decomposing the extensive form according to scenario, and iteratively solving penalized versions of the sub-problems to gradually enforce implementability*."

f. Hybrid decomposition methods (HDM): HDM use a combination of Benders, DW, LR, and/or some other decomposition methods. For example, so-called cross-decomposition uses Benders and LR [38, 39, 40].

7. Question: How would our decision change if we implemented the production costing approach with a higher degree of fidelity? Relatedly, what level of fidelity is required in the expansion planning production cost model to adequately represent storage? These are important questions because of the need to include flexibility services[6] within expansion planning. A very good basic review of UC is contained in [41], and a high-fidelity model for storage for production cost is given in [42]. There have been several good papers written recently on this topic, e.g., [43, 44, 45]. In gaining understanding of how to include production cost modeling fidelity within expansion planning, it is useful to review these papers, paying attention to the papers they reference for possible additional resources to study.

[6] We define flexibility services as (1) transient frequency response (0-20 seconds following loss of generation); (2) frequency regulation (continuous steady-state frequency control at ~4 second intervals); (3) contingency reserve provision (capacity reserves having the ability to compensate for loss of generation within 10-30 minutes); (4) load following or ramping reserve provision (capacity reserves having the ability to compensate for 30-min to 4 hour daily changes in new load); and (5) planning reserve provision (capacity reserves to satisfy the annual peak).

The above thinking revolves around a view of optimization design which is captured by Fig. 9 below. The solution speed is not determined by any one design feature but rather by their combination.

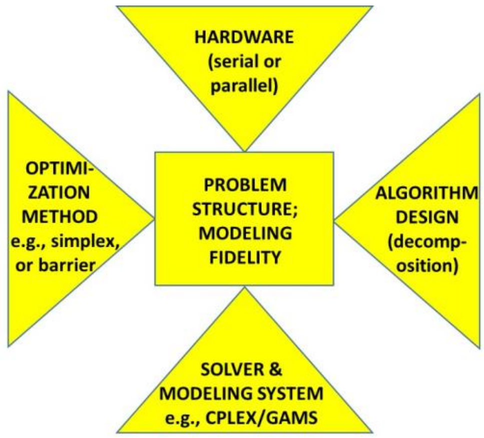


Fig. 9: Influences on compute time

11.0 The Dantzig-Wolfe Decomposition Procedure

Most of the material from this section is adapted from [5].

We attempt to solve the following problem P.

$$
\text{P}\quad
\begin{array}{llllllll}
\max & z = & c_1^T x_1 & + c_2^T x_2 & + \cdots & + c_h^T x_h & & \\
\text{subject to} & A_{00}x_0 & + A_{01}x_1 & + A_{02}x_2 & + \cdots & + A_{0h}x_h & = b_0 & (0) \\
 & & A_{11}x_1 & & & & = b_1 & (1) \\
 & & & A_{22}x_2 & & & = b_2 & (2) \\
 & & & & \ddots & & \vdots & (3) \\
 & & & & & A_{hh}x_h & = b_h & (4)
\end{array}
$$

$$x_1, x_2, \ldots, x_h \geq 0$$

where

- c_j and x_j have dimension $n_j \times 1$
- A_{ij} has dimension $m_i \times n_j$
- b_i has dimension $m_i \times 1$
- This problem has $\sum_{i=0}^{h} m_i$ constraints and $\sum_{i=0}^{h} n_i$ variables.

Note:

- Constraints *1, 2, 3, …,h* can be thought of as constraints on individual departments.
- Constraint 0 can be thought of as a constraint on total corporate (organizational) resources (sum across all departments).

<u>Definition 1</u>: For a linear program (LP), an *extreme point* is a "corner point" and represents a possible solution to the LP. The solution to any feasible linear program is always an extreme point. Figure 10 below illustrates 10 extreme points for some linear program.

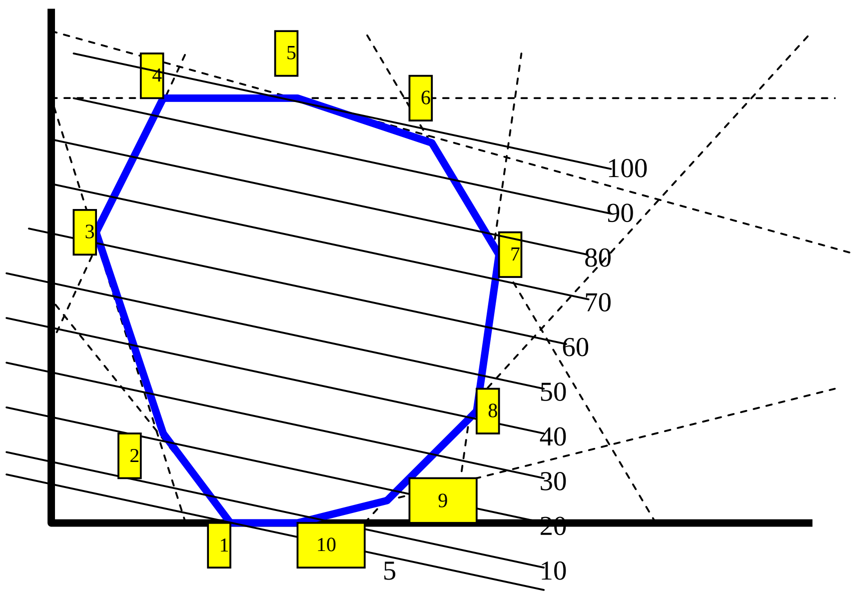


Fig. 10

<u>Observation 1</u>: Each individual subproblem represented by

$$\max z = c_i^T x_i$$
$$\text{subject to } A_{ii} x_i = b_i$$

has its own set of extreme points. We refer to these extreme points as the *subproblem i extreme points,* denoted by x_i^k, $k = 1, \ldots, p_i$ where p_i is the number of extreme points for subproblem *i*.

Observation 2: Corresponding to each extreme point solution x_i^k, the amount of corporate resources used is $A_{0i}x_i^k$, an $m_0 \times 1$ vector.

Observation 3: The contribution to the objective function of the extreme point solution is $c_i^T x_i^k$, a scalar.

Definition 2: A convex combination of extreme points is a point $\sum_{k=1}^{p_i} x_i^k y_i^k$, where $\sum_{k=1}^{p_i} y_i^k = 1$, so that y_i^k is the fraction of extreme point x_i^k in the convex combination. Figure 11 below shows a white dot in the interior of the region illustrating a convex combination of the two extreme points numbered 4 and 9.

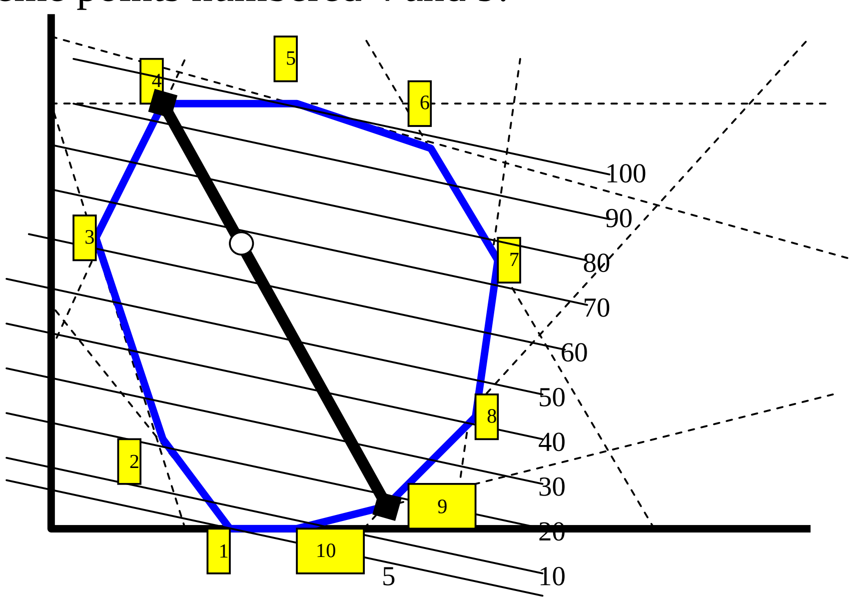


Fig. 11

Fact 1: Any convex combination of extreme points must be feasible to the problem. This should be self-evident from Fig. 11 and can be understood from the fact that the convex combination is a "weighted average" of the extreme points and therefore must lie "between" them (for two extreme points) or "interior to" them (for multiple extreme points).

Fact 2: Any point in the feasible region may be identified by appropriately choosing the y_i^k.

Observation 4a: Since the convex combination of extreme points is a weighted average of those extreme points, then the total resource usage by that convex combination will also be a weighted average of the resource usage of the extreme points, i.e., $\sum_{k=1}^{p_i}\left(A_{0i}x_i^k\right)y_i^k$. The total resources for the complete problem is the summation over all of the subproblems, $\sum_{i=1}^{h}\sum_{k=1}^{p_i}\left(A_{0i}x_i^k\right)y_i^k$.

Observation 4b: Since the convex combination of extreme points is a weighted average of those extreme points, then the contribution to the objective function contribution by that convex combination will also be a weighted average of the objective function contribution of the extreme points, i.e., $\sum_{k=1}^{p_i}\left(c_i^T x_i^k\right)y_i^k$. The total objective function can be expressed as the summation over all subproblems, $\sum_{i=1}^{h}\sum_{k=1}^{p_i}\left(c_i^T x_i^k\right)y_i^k$.

Based on Observations 4a and 4b, we may now transform our optimization problem P as a search over all possible combinations of points within the feasible regions of the subproblems to maximize the total objective function, subject to the constraint that the y_i^k must sum to 1.0 and must be nonnegative, i.e.,

$$
\text{P-T}\qquad
\begin{aligned}
&\max z = \sum_{i=1}^{h}\sum_{k=1}^{p_i}\left(c_i^T x_i^k\right)y_i^k \\
&\text{subject to } \ A_{00}x_0 + \sum_{i=1}^{h}\sum_{k=1}^{p_i}\left(A_{0i}x_i^k\right)y_i^k = b_0 && m_0 \text{ constraints} \\
&\qquad \sum_{k=1}^{p_i} y_i^k = 1, \quad i = 1,\dots,h && h \text{ constraints} \\
&\qquad y_i^k \ge 0, \quad i = 1,\dots,h, \quad k = 1,\dots,p_i
\end{aligned}
$$

Note that this new problem P-T (P-transformed) has m_0+h constraints, in contrast to problem P which has $\sum_{i=0}^{h} m_i$ constraints. Therefore it has far fewer constraints. However, whereas problem P has only $\sum_{i=0}^{h} n_i$ variables, this new problem has as many variables as it has total number of extreme points across the h subproblems, $\sum_{i=0}^{h} p_i$, and so it has a much larger number of variables.

The DW decomposition method solves the new problem without explicitly considering all of the variables.

Understanding the DW method requires having a background in linear programming so that one is familiar with the revised simplex algorithm. We do not have time to cover this algorithm in this class, but it is standard in any linear programming class to do so.

Instead, we provide an economic interpretation to the DW method.

In the first constraint of problem P-T, b_0 can be thought of as representing shared resources among the various subproblems $i=1,\ldots,h$.

Let the first m_0 dual variables of problem P-T be contained in the $m_0\times 1$ vector π. Each of these dual variables provide the change in the objective as the corresponding right-hand-side (a resource) is changed.

➔ That is, if b_{0k} is changed by $b_{0k}+\Delta$, then the optimal value of the objective is modified by adding $\pi_k\Delta$.

➔ Likewise, if the i^{th} subproblem (department) increases its use of resource b_{0k} by Δ (instead of increasing the amount of the resource by Δ), then we can consider that that subproblem (department) has incurred a "charge" of $\pi_k\Delta$. This "charge" worsens its contribution to the complete problem P-T objective, and accounting for all shared resources b_{0k}, $k=1, m_0$, the contribution to the objective is $c_i^T x_i - \pi^T A_{0i} x_i$ where $A_{0i} x_i$ is the amount of shared resources consumed by the i^th subproblem (department).

One may think of these dual variables contained in π as the "prices" that each subproblem (department) must pay for use of the corresponding shared resources.

Assuming each subproblem $i=1,\ldots,h$ represents a different department in the CEO's organization, the DW-method may be interpreted in the following way, paraphrased from [2]:

- If each department i worked independently of the others, then each would simply minimize its part of the objective function, i.e., $c_i^T x_i$.
- However, the departments are not independent but are linked by the constraints of using resources shared on a global level.
- The right-hand sides b_{0k}, $k=1,\ldots,m_0$, are the total amounts of resources to be distributed among the various departments.
- The DW method consists of having the CEO make each department pay a unit price, π_k, for use of each resource k.
- Thus, the departments react by including the prices of the resources in its own objective function. In other words,
    - Each department will look for new activity levels x_i which minimize $c_i^T x_i - \pi^T A_{0i} x_i$.
    - Each department performs this search by solving the following problem:

$$\begin{aligned} \max \quad & c_i^T x_i - \pi^T A_{0i} x_i \\ \text{subject to} \quad & A_{ii} x_{\mathrm{i}} = b_i \\ & x_{ik} \geq 0 \quad \forall k \end{aligned}$$

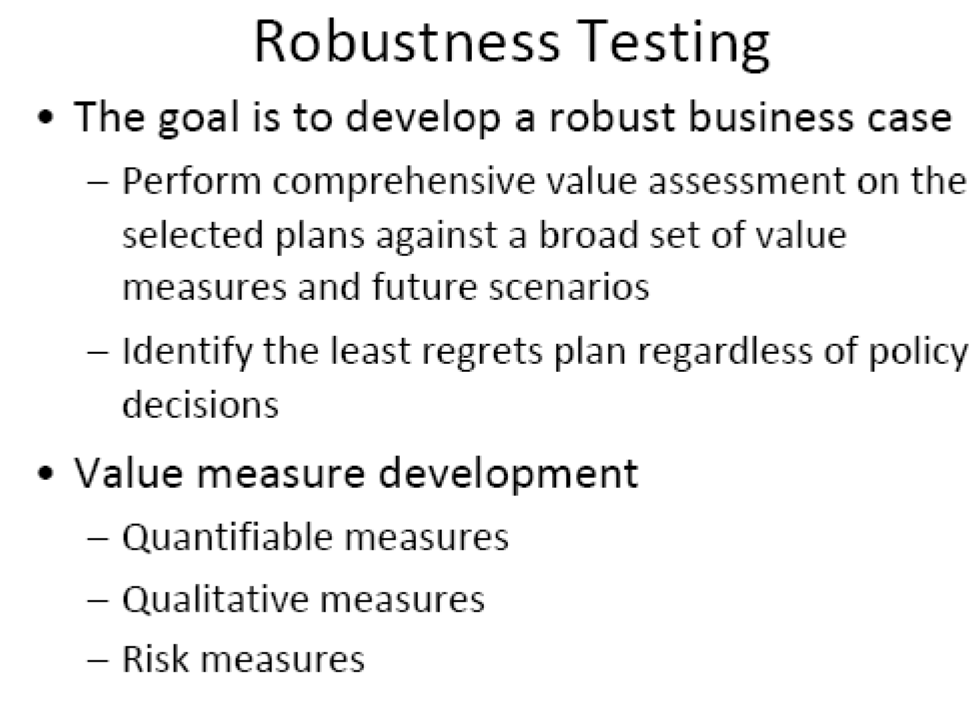
- The departments make proposals of activity levels x_i back to the CEO, and the CEO then determines the optimal weights y_i^k for the proposals by solving problem P-T, getting a new set of prices π, and the process repeats until all proposals remain the same.

Reference [2], p. 346, and [46], pp. 349-350, provide good articulations of the above DW economic interpretation.

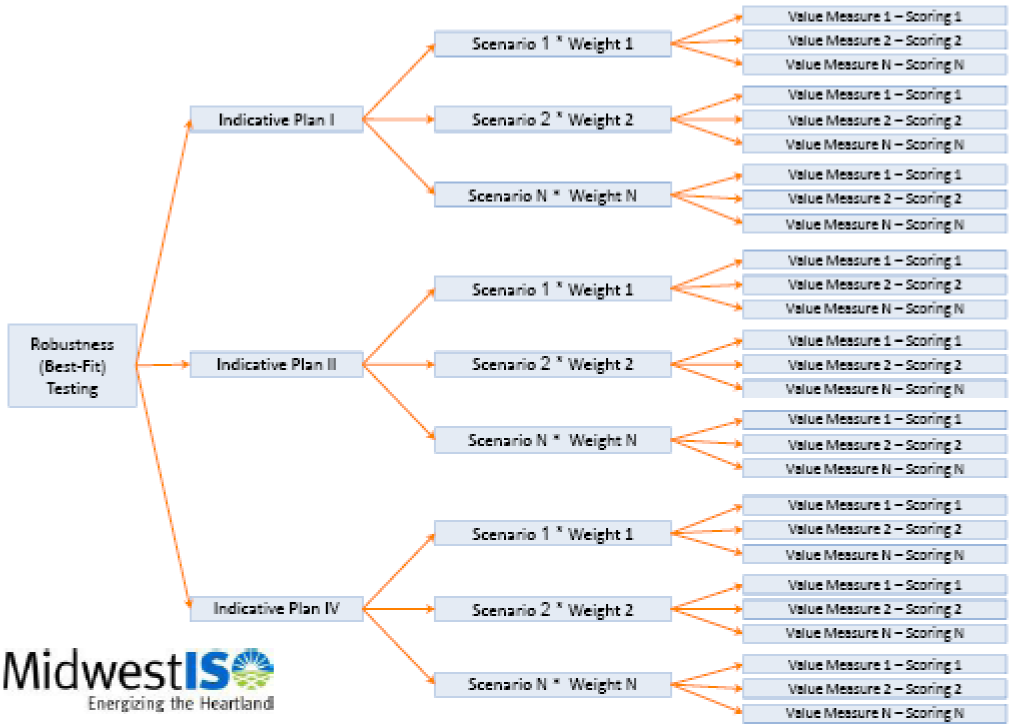
12.0 Other ways of addressing uncertainty in planning

Stochastic programming is an elegant mathematical tool for addressing uncertainty in planning, but it is computationally burdensome. Other ways include Monte Carlo simulation and robust optimization.

A few slides presented by engineers from the Midwest ISO are interesting.



Indicative Robustness (Best-Fit) Testing Application



MidwestISO
Energizing the Heartland

How Do You Decide On a Strategy?

- Robustness testing
  - How does an alternative perform in a variety of future scenarios?
  - Are significantly greater economic benefits projected in one case over the other?
- Faith based scenario evaluation
  - What would you have to believe?
  - Actively test important assumptions
- Delay choosing as long as possible
  - Without jeopardizing legal requirements
  - Without risking wasted investment

What Would You Have To Believe…

- For the **higher mileage lower voltage** strategy to be superior ?
  - Energy policies will not expand beyond the current laws, thus the actual construction will "most likely" stop short of full implementation

- For the **higher voltage lower mileage** strategy to be superior?
  - Increasingly aggressive Energy Policy objectives will significantly expand the objective and thus providing a more robust system up front leads to significant flexibility and saved time

Uncertainty	Unit	Low (L)	Mid (M)	High (H)
Alternative Capital Costs				
Coal	($/KW)	2,113	2,452	2,945
CC	($/KW)	936	1,059	1,210
CT	($/KW)	652	737	826
Nuclear	($/KW)	3,485	3,947	4,974
Wind-Onshore	($/KW)	1,857	2,195	2,408
IGCC	($/KW)	2,356	2,750	2,992
IGCC w/Sequestration	($/KW)	3,312	3,669	4,093
CC w/Sequestration	($/KW)	1,839	1,879	1,989
Pumped Storage	($/KW)	2,400	3,000	3,600
Compressed Air Energy Storage	($/KW)			
Photovoltaic	($/KW)	5,190	6,434	6,979
BioMass	($/KW)	3,237	4,013	4,559
Hydro	($/KW)	1,927	2,389	2,596
Wind-Offshore	($/KW)	3,310	4,104	4,459
Distributive Generation-Peak	($/KW)	1,414	1,753	1,905
Demand Response Level	%		GEP	
Energy Efficiency Level	%		GEP	
FGD			NERC Cost Curves	
SCR			NERC Cost Curves	
Cooling Tower			NERC Cost Curves	
Coal Ash Pond Liner			NERC Cost Curves	
Demand and Energy				
Demand Growth Rate	%	0.30	0.75	1.60
Energy Growth Rate	%	0.30	1.00	2.19
Fuel Prices (Starting Values)				
Gas	($/MBtu)	4.35	6.22	8.71
Oil	($/MBtu)	Powerbase default - 20%	Powerbase default	Powerbase default + 20%
Coal	($/MBtu)	Powerbase default - 20%	Powerbase default	Powerbase default + 20%
Uranium	($/MBtu)	0.90	1.12	1.34
Fuel Prices (Escalation Rates)				
Gas	%	1.74	2.91	4.00
Oil	%	1.74	2.91	4.00
Coal	%	1.74	2.91	
Uranium	%	1.74	2.91	
Emissions				
SO_2	($/ton)	Powerbase Default - 25%	Powerbase Default	Powerbase Default +25%
NOx	($/ton)	Powerbase Default - 25%	Powerbase Default	Powerbase Default +25%
CO_2	($/ton)	0	50	100
CATR			Rule as Proposed	
CWIS 316 (b)			Rule as Proposed	
MACT			Rule as Proposed	
CCR			Rule as Proposed	
Economic Variables				
Inflation Rate	%	1.74	2.91	4.00
Potential Coal Retirement			Known Public	Forced retirements
Renewable Penetration as a percentage of total energy delivered				
State mandates	%	No State Mandates	Use existing state requirements in the MISO footprint	Use exisiting standards or pending proposals / goals
National	%	0	20	30
Carbon Reduction Requirements from Baseline Level				
CO2	%	0	1/2 of Waxman-Markey Bill: 2012 - 1.5%; 2020 - 8.5%; 2030 - 21%	Waxman-Markey Legislation up to 2030; 2012 3%/2020 17%/2030 42%

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