Notes on Mutual Inductance and Transformers
J. McCalley

1.0 Use of Transformers

Transformers are one of the most common electrical devices. Perhaps the most familiar application today is for small electronic devices such as laptop computers. The circuit diagram for such an application is given in Fig. 1.

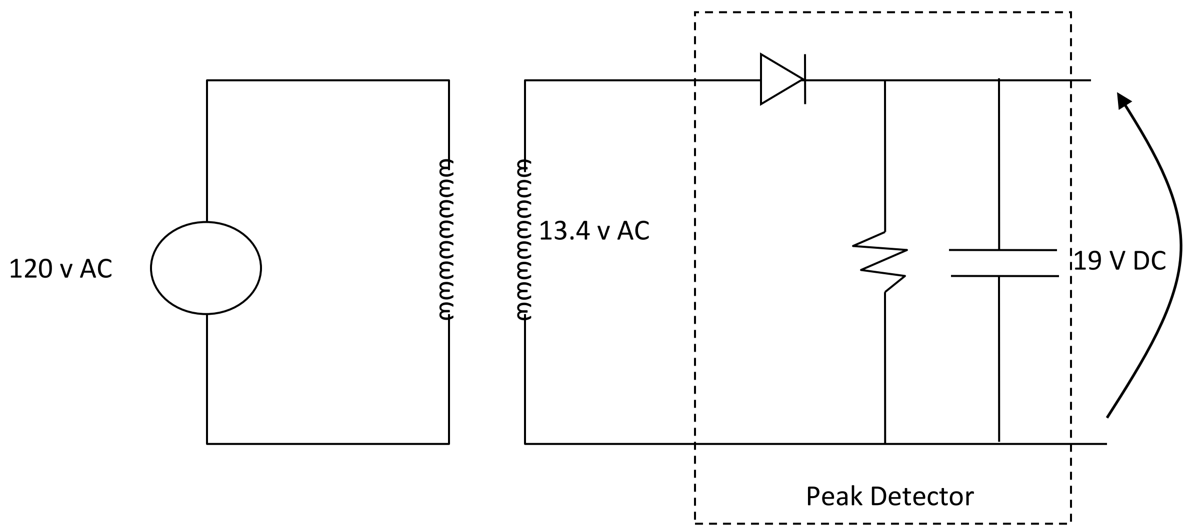


Fig. 1

Transformers are also used in devices with rechargeable batteries, e.g., drills, screwdrivers, cordless phones. Another major transformer application is the ballast used in florescent lighting, as shown in Fig. 2.

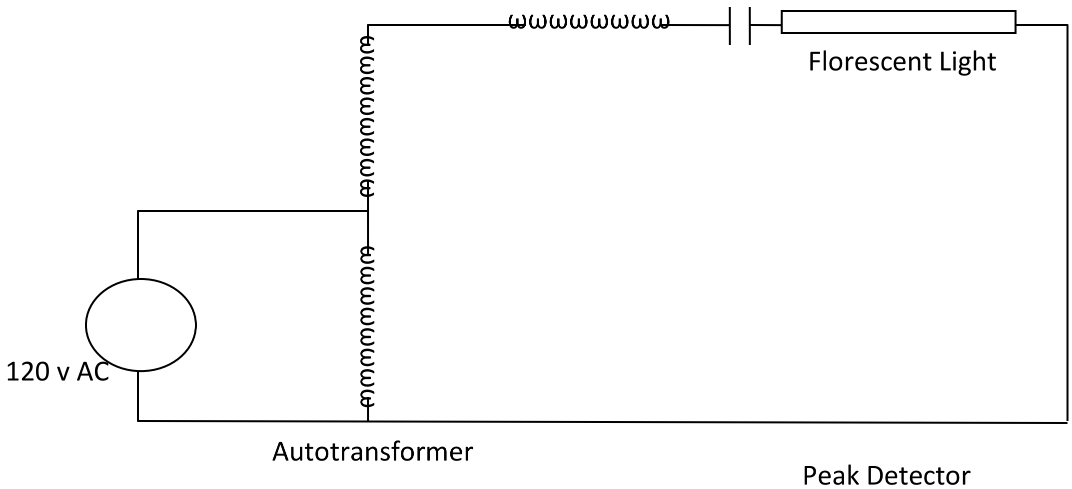


Fig. 2

In the electric power industry, several types of transformers are utilized, including the power transformer and the instrument transformer. Our interest is the power transformer. Power transformers are used in the following ways:

- Stepping up the voltage from a generator to high voltage transmission levels;
- Stepping down the voltage to distribution primary voltage levels;
- Stepping down the voltage to distribution secondary voltage levels;
- Interconnecting different system voltage levels in the HV and EHV systems.

2.0 Self inductance

Consider the arrangement of Fig. 3.

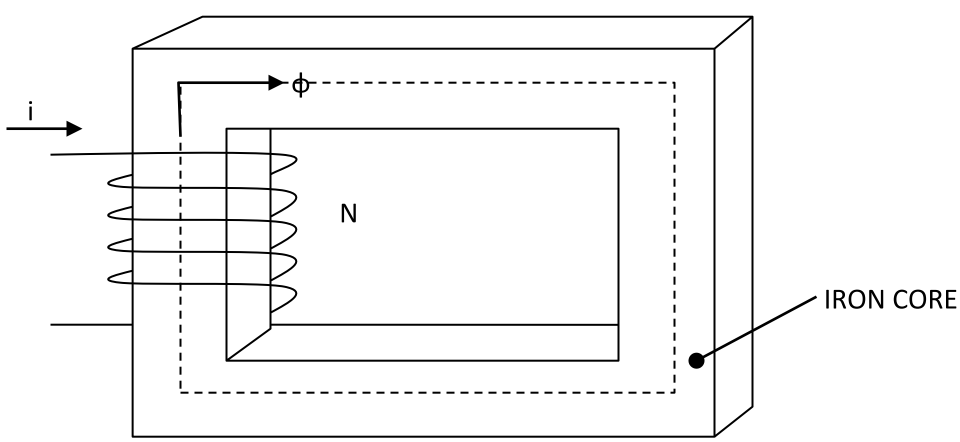


Fig. 3

Ampere's law is

$$\oint \vec{H} \bullet d\vec{L} = I \tag{1}$$

Ampere's law says that the line integral of magnetic field intensity H about any closed path equals the current enclosed by that path. When (1) is applied to the arrangement of Fig. 3,

- The path is the dotted line;
- The magnetic field intensity is along the direction of ϕ, which is in the same direction as dL;
- The left hand side of (1) therefore becomes just Hl, where l is the mean length of the path around the core.
- The right-hand-side of (1) is the number of turns times the current, Ni.

Therefore, we obtain

$$Hl = Ni \tag{2}$$

We recall from basic electromagnetics that

$$B = \mu H \tag{3}$$

where B is the magnetic flux density (webers/m^2), and μ is the permeability of the iron with units of Henry/m or Newtons/$ampere^2$ (μ for a given material is the amount of flux density B that will flow in that material for a unit value of magnetic field strength H. For most types of iron used in transformers, $\mu=5000\mu_0$ N/A^2, where $\mu_0=4\pi\times10^{-7}$N/A^2 is the permeability of free space).

We also know that flux ϕ (webers) is related to flux density by

$$\phi = BA \tag{4}$$

where A is the cross-sectional area of the iron core. Solving for B in (4) and substituting into (3), solving for H, and substituting into (2) yields

$$\frac{\phi}{\mu A} l = Ni \tag{5}$$

Solving for ϕ results in

$$\phi = \frac{\mu A}{l} Ni \tag{6}$$

Now we define:

- Magnetomotive Force, $\mathscr{F} = Ni$
- Reluctance: $\mathscr{R} = \dfrac{l}{\mu A}$

Then (6) becomes

$$\varphi = \frac{\mathscr{F}}{\mathscr{R}} \tag{7}$$

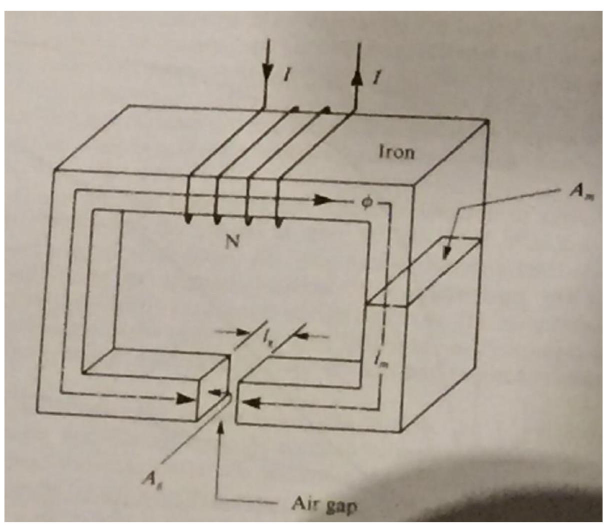
Equation (7) should remind you of a familiar relation… Ohm's Law!

Ohm's Law is I=V/R and so the analogy is

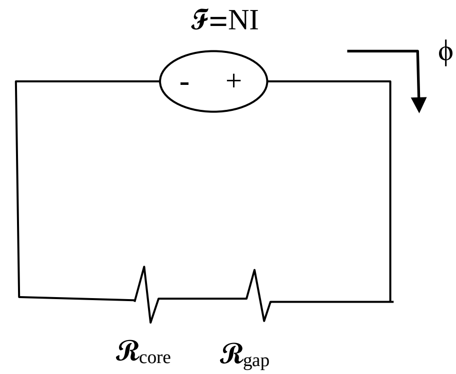
- I → ϕ (flux "flows" like current)
- V → F (MMF provides the "push" like voltage)
- R → R (Reluctance "resists" like resistance)

==

Example 1 [1]: The magnetic circuit shown in the below figure has N=100 turns, a cross-section area of A_m=A_g=40cm^2, an air gap length of l_g=0.5mm, and a mean core length of l_c=1.2m. The relative permeability of the iron is μ_r=2500. The current in the coil is I_{DC}=7.8 amperes. Determine the flux and flux density in the air gap.



Solution: We may think of this magnetic circuit in terms of its electric analogue, as shown below.



The electric analogue makes the solution immediately clear, where

$$\varphi = \frac{\mathcal{F}}{\mathcal{R}}$$

The MMF is computed as

$$\mathcal{F} = NI = 100 * 7.8 = 780 \text{ ampere-turns}$$

The reluctance of the air-gap is computed as

$$\mathcal{R}_{gap} = \frac{l_g}{\mu A_g} = \frac{0.5/1000}{(4\pi \times 10^{-7})(40/100^2)} = 99,472 \text{amperes/Weber}$$

The reluctance of the core is computed as

$$\mathcal{R}_{core} = \frac{l_c}{\mu A_c} = \frac{1.2}{2500(4\pi \times 10^{-7})(40/100^2)} = 95,492 \text{amperes/Weber}$$

The flux is then computed as

$$\varphi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{780}{99,472 + 95,492} = 0.004 \text{Webers}$$

The flux density is given by

$$B = \frac{\varphi}{A} = \frac{0.004}{40/(100)^2} = 1 \text{ Tesla}$$

===

Now let's return to (6) and multiply both sides by N to obtain

$$N\phi = \frac{\mu A}{l} N^2 i \tag{8}$$

Define:

- Flux linkage:

$$\lambda = N\phi \tag{9}$$

- Self inductance:

$$L = \frac{\mu A N^2}{l} = \frac{N^2}{R} \tag{10}$$

Substituting (9) and (10) into (8), we obtain

$$\lambda = Li \tag{11}$$

We will introduce some additional notation that will help us later, as follows:

$$L_{11} = \frac{\lambda_{11}}{i_1} \tag{12}$$

And so we can see that the self-inductance L_{11} is the ratio of

- the flux from coil 1 linking with coil 1, λ_{11}
- to the current in coil 1, i_1.

Observe from (12) that a large L_{11} means that a little current i_1 generates a lot of flux linkages λ_{11}. What makes L_{11} large? Recall:

$$L_{11} = \frac{\mu A N_1^2}{l}$$

And so we see that to make self inductance large, we need to

- make N_1, μ, and A large
- make l small

And so a large L_{11} results from

- many turns (N_1)
- large cross section (A)
- compact construction (small l)
- large μ (e.g., core made of iron)

Recalling that reluctance is given by $\mathscr{R} = \dfrac{l}{\mu A}$, we see that a magnetic circuit characterized by a large self-inductance will have a small magnetic path reluctance.

Example 2 [1]: Compute the self-inductance of the magnetic circuit given in Example 1.

Solution: Here, we need to recognize that the magnetic field intensity, H, will be different in the iron core than in the air gap. We can see that this must be so because the air gap is in series with the core and so the flux ϕ in the air gap must be the same as the flux in the core. Since the cross-sectional area in the air gap and in the core are the same, the flux densities B must also be the same. But because B=μH, and the permeability of the air gap differs from the permeability of the core, the magnetic field intensities must differ as well. Thus, equation (2) will change to

$$H_c l_c + H_g l_g = Ni$$

Using B=μH, we have

$$\frac{B}{\mu_c} l_c + \frac{B}{\mu_g} l_g = Ni \Rightarrow B\left[\frac{l_c}{\mu_c} + \frac{l_g}{\mu_g}\right] = Ni$$

And using B=ϕ/A, we have

$$\frac{\phi}{A}\left[\frac{l_c}{\mu_c} + \frac{l_g}{\mu_g}\right] = Ni$$

Solving for ϕ, we obtain

$$\phi = \frac{Ni}{\dfrac{l_c}{A\mu_c} + \dfrac{l_g}{A\mu_g}} = \frac{Ni}{\mathscr{R}_c + \mathscr{R}_g}$$

Using subscripted notation to identify the flux from coil 1 linking with coil 1 results in

$$\phi_{11} = \frac{N_1 i_1}{\frac{l_c}{A\mu_c} + \frac{l_g}{A\mu_g}} = \frac{N_1 i_1}{\mathcal{R}_c + \mathcal{R}_g}$$

Recalling that self-inductance is given by

$$L_{11} = \frac{\lambda_{11}}{i_1}$$

and that $\lambda_{11}=N_1\phi_{11}$, we can write that

$$L_{11} = \frac{\lambda_{11}}{i_1} = \frac{N_1\phi_{11}}{i_1}$$

Substitution for ϕ_{11} results in

$$L_{11} = \frac{N_1^2}{\mathcal{R}_c + \mathcal{R}_g}$$

Recalling from Example 1 that $N_1=100$ and

$$\mathcal{R}_{gap} = 99,472\text{amperes/Weber}$$

$$\mathcal{R}_{core} = 95,492\text{amperes/Weber}$$

the self-inductance becomes:

$$L_{11} = \frac{100^2}{95492 + 99472} = 0.0513 \text{ henries}$$

3.0 Mutual inductance

Let's consider another arrangement as shown in Fig. 4.

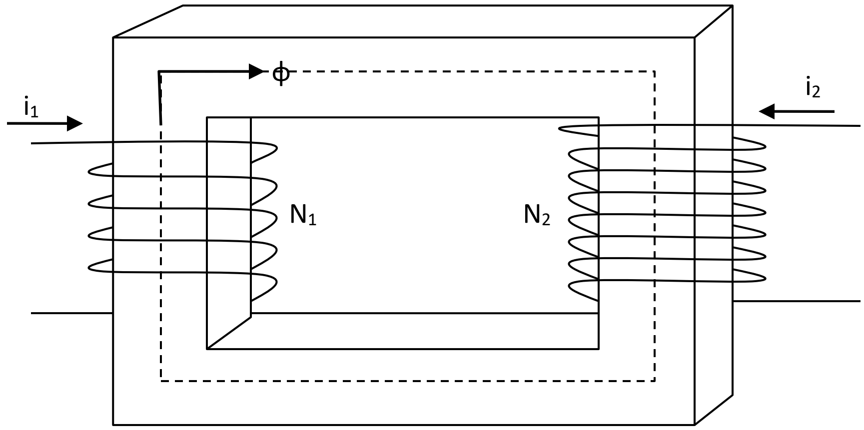


Fig. 4

We have for each coil:

$$L_{11} = \frac{\lambda_{11}}{i_1} \tag{13a}$$

$$L_{22} = \frac{\lambda_{22}}{i_2} \tag{13b}$$

We can also define L_{12} and L_{21}.

L_{12} is the ratio of

- the flux from coil 2 linking with coil 1, λ_{12}
- to the current in coil 2, i_2.

That is,

$$L_{12} = \frac{\lambda_{12}}{i_2} \tag{14a}$$

where the first subscript, 1 in this case, indicates "links with coil 1" and the second subscript, 2 in this case, indicates "flux from coil 2."

Here, we also have that

$$\lambda_{12} = N_1\phi_{12} \Rightarrow L_{12} = \frac{N_1\phi_{12}}{i_2} \tag{14b}$$

Likewise, we have that

$$L_{21} = \frac{\lambda_{21}}{i_1} \tag{15a}$$

$$\lambda_{21} = N_2\phi_{21} \Rightarrow L_{21} = \frac{N_2\phi_{21}}{i_1} \tag{15b}$$

Now let's assume that all flux produced by each coil links with the other coil. The implication of this is that there is no leakage flux, as illustrated in Fig. 5.

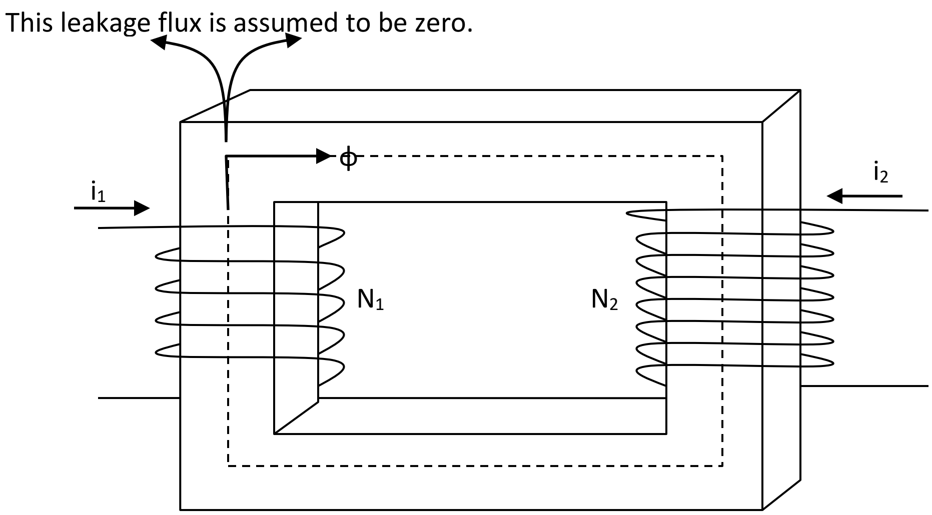


Fig. 5

Although in reality there is some leakage flux, it is quite small because the iron has much less reluctance than the air. With this assumption, then we can write that

- the flux from coil 2 linking with coil 1 is equal to the flux from coil 2 linking with coil 2, i.e.,

$$\phi_{12} = \phi_{22} = \frac{\mu A}{l} N_2 i_2 \tag{16a}$$

- the flux from coil 1 linking with coil 2 is equal to the flux from coil 1 linking with coil 1, i.e.,

$$\phi_{21} = \phi_{11} = \frac{\mu A}{l} N_1 i_1 \tag{16b}$$

Substitution of (16a) and (16b) into (14b) and (15b), respectively, results in:

$$L_{12} = \frac{N_1 \varphi_{12}}{i_2} = \frac{N_1 \frac{\mu A}{l} N_2 i_2}{i_2} = N_1 N_2 \frac{\mu A}{l} = \frac{N_1 N_2}{\mathcal{R}} \tag{17a}$$

$$L_{21} = \frac{N_2 \varphi_{21}}{i_1} = \frac{N_2 \frac{\mu A}{l} N_1 i_1}{i_1} = N_2 N_1 \frac{\mu A}{l} = \frac{N_2 N_1}{\mathcal{R}} \tag{17b}$$

Examination of (17a) and (17b) leads to

$$L_{21} = L_{12} = \frac{N_1 N_2}{\mathcal{R}} \tag{18}$$

Also recall

$$L = \frac{N^2}{\mathcal{R}} \tag{10}$$

or in subscripted notation

$$L_{11} = \frac{N_1{}^2}{\mathcal{R}} \tag{19a}$$

$$L_{22} = \frac{N_2{}^2}{\mathcal{R}} \tag{19b}$$

Solving for N_1 and N_2in (19a) and (19b) results in

$$N_1 = \sqrt{L_{11} \mathcal{R}} \tag{20a}$$

$$N_2 = \sqrt{L_{22} \mathcal{R}} \tag{20b}$$

Now substitute (20a) and (20b) into (18) to obtain

$$L_{21} = L_{12} = \frac{\sqrt{L_{11}\mathscr{R}}\sqrt{L_{22}\mathscr{R}}}{\mathscr{R}} = \sqrt{L_{11}L_{22}} \tag{21}$$

Definition: $L_{12}=L_{21}$ is the mutual inductance and is normally denoted M.

Mutual inductance gives the ratio of

- flux from coil k linking with coil j, λ_{jk}
- to the current in coil k, i_k,

That is,

$$M = \begin{cases} \dfrac{\lambda_{12}}{i_2} \\ \dfrac{\lambda_{21}}{i_1} \end{cases} \tag{22}$$

4.0 Polarity and dot convention for coupled circuits

Consider Fig. 6 illustrating two coupled circuits. Assume the voltage v_1 is DC, but you have a dial you can turn to increase v_1. Also assume that the secondary is open (i.e., the dashed line connecting the secondary terminals to a load is not really there). The coil 1 has very small resistance so that in the steady-state, the current is not infinite.

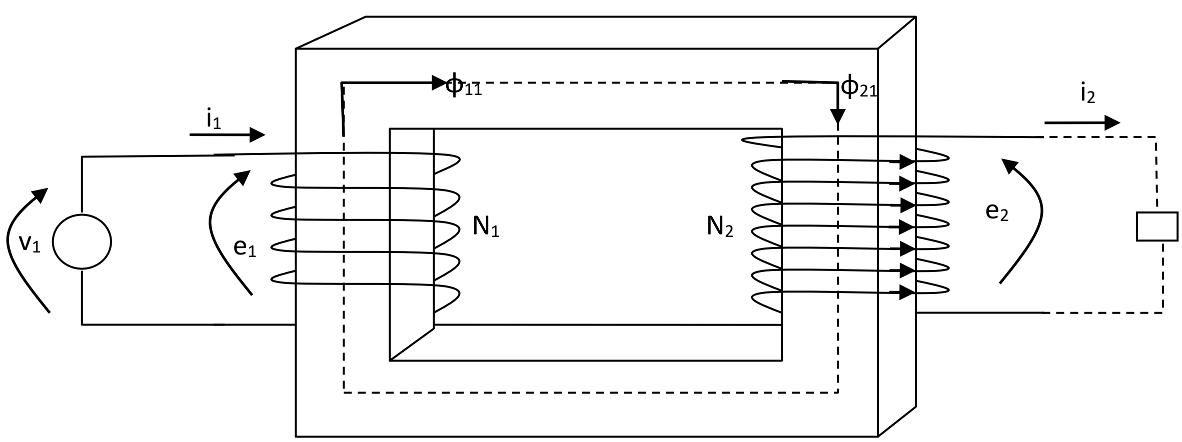


Fig. 6

Now assume that we increase the voltage v_1 to some higher value. This causes the current i_1 to increase with time which causes the flux from coil 1, ϕ_{11}, to also increase with time (which means that the flux linkages λ_{11} also increase with time). By Faraday's Law,

$$e_1 = N_1 \frac{d\phi_{11}}{dt} = \frac{d(N_1\phi_{11})}{dt} = \frac{d\lambda_{11}}{dt} = L_{11}\frac{di_1}{dt} \tag{23}$$

In addition, the coil 2 sees that same flux increase, which we denote by ϕ_{21} (and correspondingly, the flux linkages are denoted as λ_{21}). Again, by Faraday's Law,

$$e_2 = N_2 \frac{d\phi_{21}}{dt} = \frac{d(N_2\phi_{21})}{dt} = \frac{d\lambda_{21}}{dt} = L_{21}\frac{di_1}{dt} = M\frac{di_1}{dt} \tag{24}$$

Question: How do we know the sign of the right-hand-side of (24)? That is, how do we know which of the below are correct?

$$\begin{aligned} e_2 &= +M\frac{di_1}{dt} \\ e_2 &= -M\frac{di_1}{dt} \end{aligned} \tag{25}$$

Here is another way to ask our question:

➔Does the assumed polarity of our e_2 match the actual polarity of the voltage that would be induced by the changing current i_1? If so, we should choose the equation in (25) with the positive sign. If not, we should choose the equation in (25) with the negative sign.

And so what is the answer? To obtain the answer, we need to recall Lenz's Law. This law states that the induced voltage e_2 must be in a direction so as to establish a current in a direction to produce a flux opposing the change in flux that produced e_2. (You can find a good explanation/illustration of Lenz's Law at https://www.khanacademy.org/science/physics/magnetic-forces-and-magnetic-fields/magnetic-flux-faradays-law/v/lenzs-law).

When e_1 increases, i_1 increases, and by the right-hand-rule (RHR), ϕ_{21} increases.

Our assumed polarity of e_2 would cause current to flow into the load in the direction shown. How do we know if this polarity is correct or not? We know it is correct because the RHR says that a current in the direction of i_2 would cause flux in the direction opposite to the direction of the ϕ_{21} increase. Thus, for the given polarity of e_2, the sign of (25) should be positive, i.e.,

$$e_2 = +M\frac{di_1}{dt}$$

How might we obtain a different answer?

There are two ways.

First way: Switch the sign of e_2, as in Fig. 7.

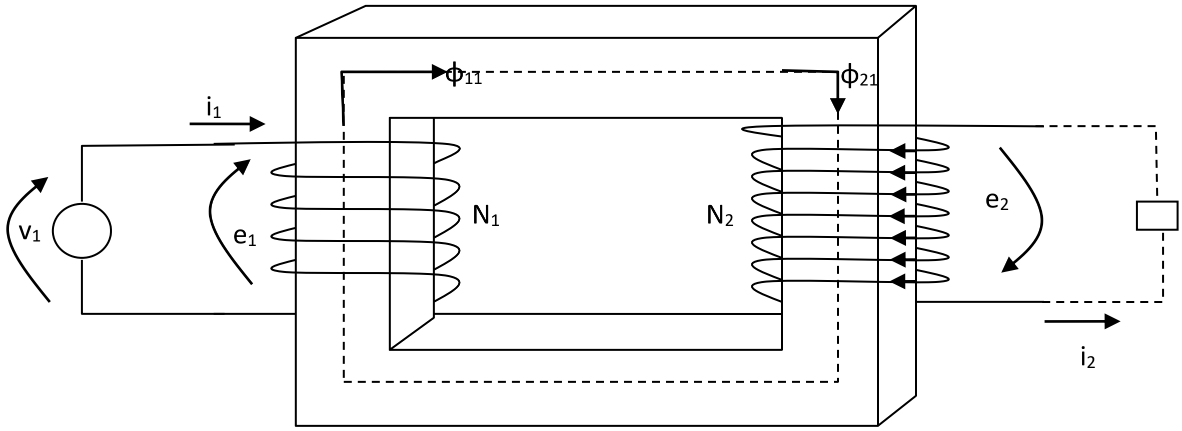


Fig. 7

In Fig. 7, the current i_2, by the RHR, would produce a flux in the same direction as the ϕ_{21} increase. In this case, we should use the negative sign in (25) according to:

$$e_2 = -M \frac{di_1}{dt}$$

Second way: Switch the sense of the coil 2 wrapping, as in Fig. 8.

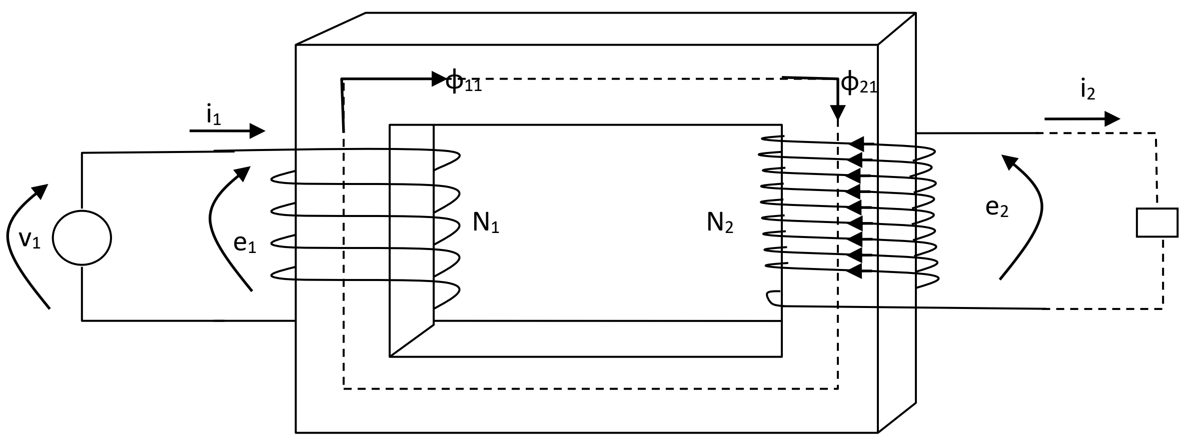


Fig. 8

In Fig. 8, the current i_2, by the RHR, would produce a flux in the same direction as the ϕ_{21} increase. In this case, we should again use the negative sign in (25).

The main point here is that we want to be able to know which secondary terminal, when defined with positive voltage polarity, results in using the form of (25) with a positive sign.

On paper, there are two approaches for doing this. The first is to draw the physical winding and to go through the Lenz's Law analysis as we have been doing above.

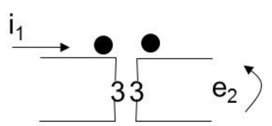
The second approach is easier, and it is to use the so-called "dot convention."

In the dot convention, we will mark one terminal on either side of the transformer so that

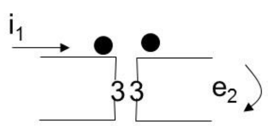
- when e_2 is defined positive at the dotted terminal of coil 2 and
- i_1 is into the dotted terminal of coil 1, then

$$e_2 = +M\frac{di_1}{dt}$$

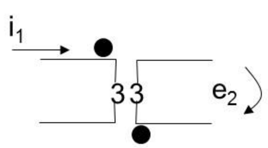
Example 3:



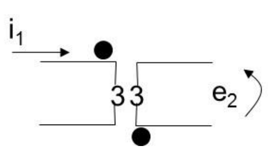
$$e_2 = +M\frac{di_1}{dt}$$



$$e_2 = -M\frac{di_1}{dt}$$



$$e_2 = +M\frac{di_1}{dt}$$



$$e_2 = -M\frac{di_1}{dt}$$

So far, we have focused on answering the following question: given the dotted terminals, how to determine the sign to use in (25)?

Here is another question: If you are given the physical layout, how do you obtain the dot-markings? There are two approaches:

First approach: Use Lenz's Law and the right-hand-rule (RHR) to determine if a defined voltage direction at the secondary produces a current in the secondary that generates flux opposing the flux change that caused that voltage.

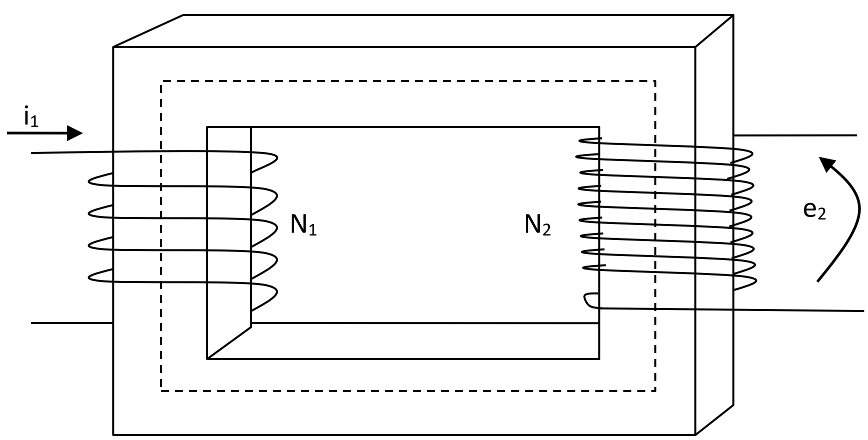
Second approach: Do it by steps:

1. Arbitrarily pick a terminal on one side and dot it.
2. Assign a current into the dotted terminal.
3. Use RHR to determine flux direction for current assigned in step 2.
4. Arbitrarily pick a terminal on the other side and assign a current *out of* **(into)** it.
5. Use RHR to determine flux direction for current assigned in Step 4.
6. Compare the direction of the two fluxes (the one from Step 3 and the one from Step 5). If the two flux directions are *opposite* **(same)**, then the terminal chosen in Step 4 is correct. If the two flux directions are *same* **(opposite)**, then the terminal chosen in Step 4 is incorrect – dot the other terminal.

This approach depends on the following principle (consistent with words in italics in above steps): Current entering one dotted terminal and leaving the other dotted terminal should produce fluxes inside the core that are in opposite directions.

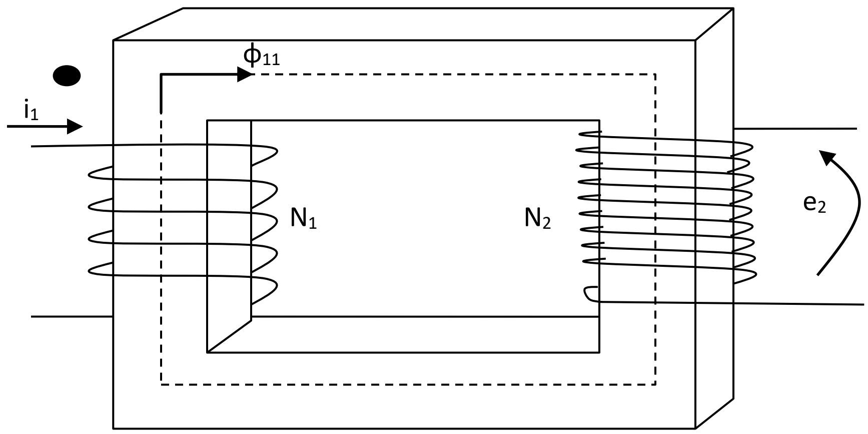
An alternative statement of this principle is as follows (consistent with words in underline bold in above steps): Currents entering the dotted terminals should produce fluxes inside the core that are in the same direction.

Example 4: Determine the dotted terminals for the configuration below, and then write the relation between i_1 and e_2.

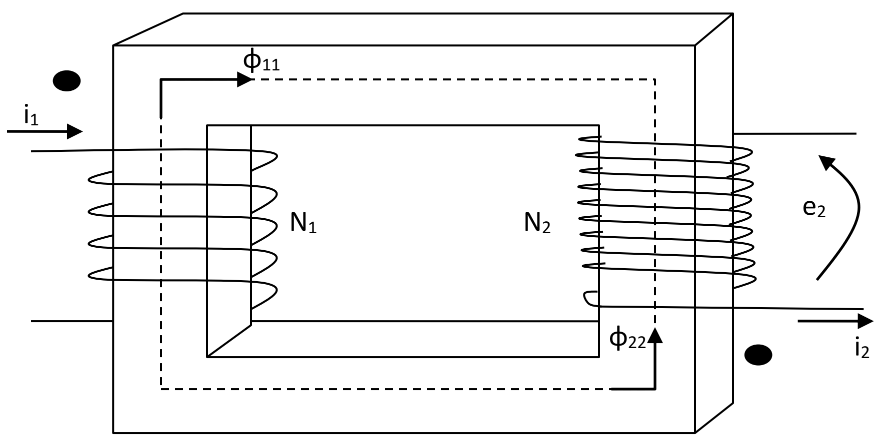


Solution:

Steps 1-3:



Steps 4-6:



Now we can write the equation for the above coupled circuits. Recall that in the dot convention, we will mark one terminal on either side of the transformer so that

- when e_2 is defined positive at the dotted terminal of coil 2 and
- i_1 is into the dotted terminal of coil 1, then

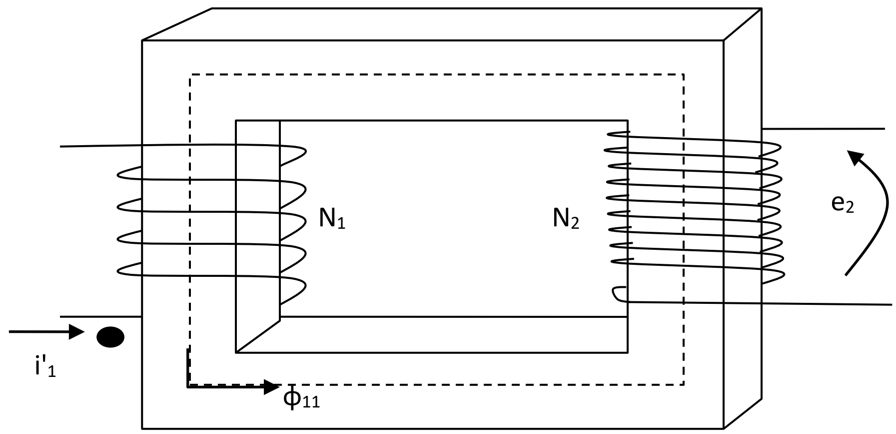
$$e_2 = +M\,\frac{di_1}{dt}$$

In the above, however, although i_2 is into the dotted terminal of coil 1, e_2 is defined negative at the dotted terminal of coil 2. Therefore

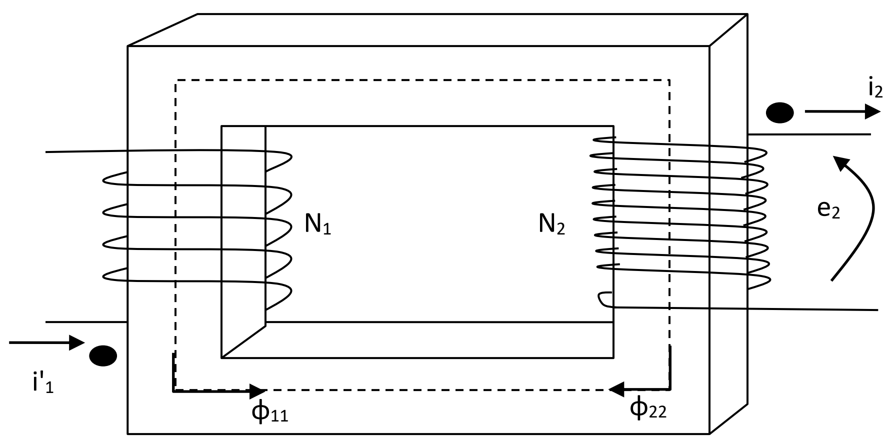
$$e_2 = -M\frac{di_1}{dt}$$

But note, there is another way we could have solved this problem, as follows:

Steps 1-3:



Steps 4-6:



Now we can write the equation for the above coupled circuits. Recall that in the dot convention, we will mark one terminal on either side of the transformer so that

- when e_2 is defined positive at the dotted terminal of coil 2 and
- i_1 is into the dotted terminal of coil 1, then

$$e_2 = +M \frac{di_1}{dt}$$

In the above, we have i'_1 flowing into the dotted terminal of coil 1, and e_2 is defined positive at the dotted terminal of coil 2. Therefore

$$e_2 = +M \frac{di'_1}{dt}$$

If, however, we wanted to express e_2 as a function of i_1 (observing that $i_1=-i'_1$) then we would have

$$e_2 = -M \frac{di_1}{dt}$$

Example:

5.0 Writing circuit equations for coupled coils

…. (see in-class notes)

6.0 Derivation of transformer turns ratio relations

…(we skipped this in class)

7.0 Power for ideal transformer and referring quantities

…(see in-class notes)

8.0 Exact and approximate transformer models

…(see in-class notes)

9.0 Three-phase transformers

A three-phase transformer will have six windings: three for the primary (phases A, B, and C on the primary) and three for the secondary (phases A, B, and C for the secondary).

There are two very different approaches to developing a three-phase transformer. One approach is to use just one “three phase bank,” where here the word “bank” refers to a single core, i.e., a single magnetic circuit. Figure 9 [1] illustrates, where each pair of primary and secondary windings (there are three pairs, one for each phase) are on the same leg.

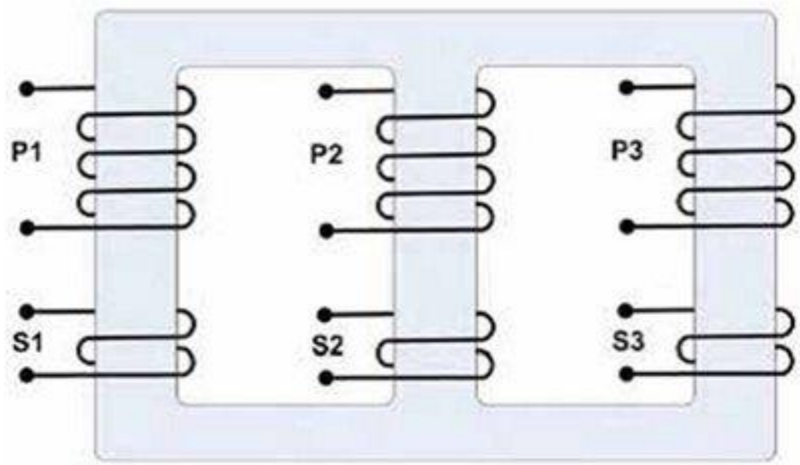


Figure 9: A three-phase transformer bank

The three-phase bank illustrated in Fig. 9 is utilizing a "core-type" of construction. In the core-type, primary and secondary windings are wound outside and surround their leg. In another type of construction, the so-called "shell-type," windings pass inside the core, forming a shell around the windings. Figure 10 [2] illustrates the difference.

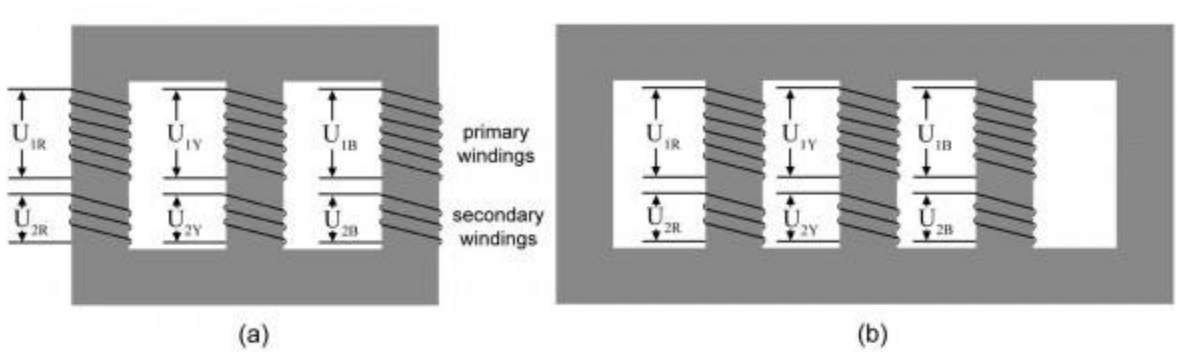


Figure 10: Core type (a) and shell type (b)

Another approach to developing a three-phase transformer is to interconnect three single-phase transformers. This approach is illustrated in Fig. 11 below.

[1] http://www.gamatronic.com/three-phase-transformers/
[2] http://www.itacanet.org/basic-electrical-engineering/part-15-transformers/

Note that wires that cross are not connected unless they have a dot ⇒ —+—

There are actually coils in each phase of these pictures.

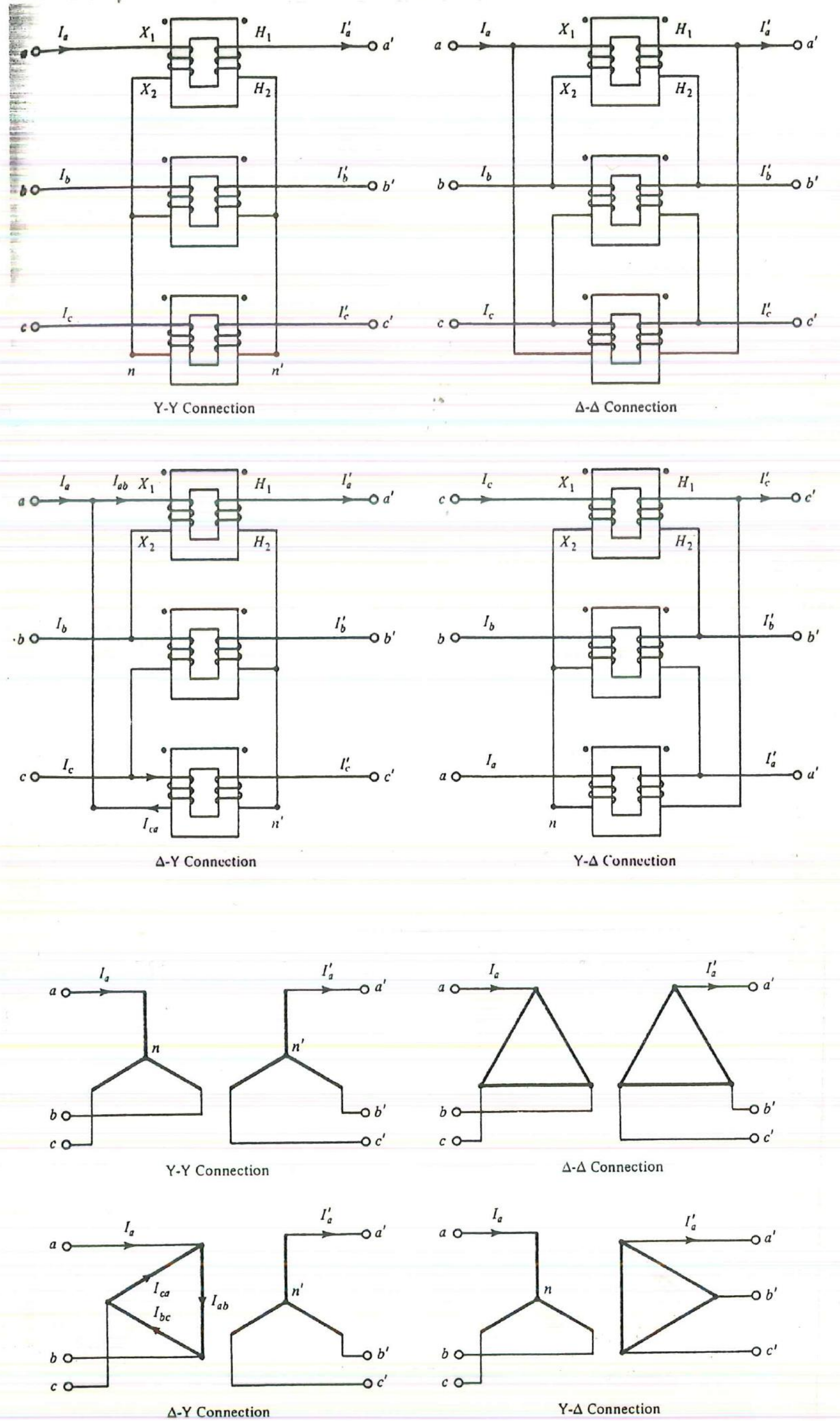


Fig. 11: Single-phase transformer connections to form a three-phase transformer

The voltage transformation ratio for three-phase transformers is always given as the ratio of the line-to-line voltage magnitude on either side. If a transformer is connected Y-Y or Δ-Δ, this ratio will be the same as the ratio of the winding voltages on either side (which is the same as the turns ratio N_1/N_2).

However, for Y-Δ or Δ-Y connected transformers, the ratio of the line-to-line voltages on either side is ***not*** the same as the ratios of the winding voltages. This means that you can take three single-phase transformers, each with the same turns ratio, and connect them for a three-phase configuration such that the three-phase configuration will have a different line-to-line ratio than the phase-to-phase ratio!

…see in-class examples.

It is the line-to-line ratio that you should use in performing per-phase analysis.