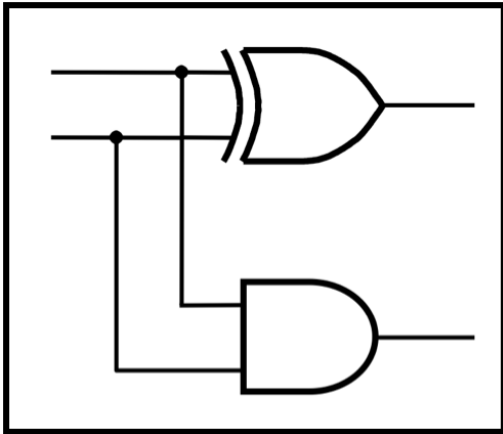




CprE 2810: Digital Logic

Instructor: Alexander Stoytchev

<http://www.ece.iastate.edu/~alexs/classes/>

Minimization with K-Maps

*CprE 2810: Digital Logic
Iowa State University, Ames, IA
Copyright © Alexander Stoytchev*

Administrative Stuff

- **HW4 is out**
- **It is due on Monday Sep 22 @ 10 pm**
- **It is posted on the class web page**
- **I also sent you an e-mail with the link.**

Quick Review

Expressions for the minterms

$$0 \ 0 \ 0 \quad m_0 = \bar{x} \ \bar{y} \ \bar{z}$$

$$0 \ 0 \ 1 \quad m_1 = \bar{x} \ \bar{y} \ z$$

$$0 \ 1 \ 0 \quad m_2 = \bar{x} \ y \ \bar{z}$$

$$0 \ 1 \ 1 \quad m_3 = \bar{x} \ y \ z$$

$$1 \ 0 \ 0 \quad m_4 = x \ \bar{y} \ \bar{z}$$

$$1 \ 0 \ 1 \quad m_5 = x \ \bar{y} \ z$$

$$1 \ 1 \ 0 \quad m_6 = x \ y \ \bar{z}$$

$$1 \ 1 \ 1 \quad m_7 = x \ y \ z$$

The bars coincide
with the 0's
in the binary expansion
of the minterm sub-index

Expressions for the Maxterms

$$0 \ 0 \ 0 \quad M_0 = x + y + z$$

$$0 \ 0 \ 1 \quad M_1 = x + y + \bar{z}$$

$$0 \ 1 \ 0 \quad M_2 = x + \bar{y} + z$$

$$0 \ 1 \ 1 \quad M_3 = x + \bar{y} + \bar{z}$$

$$1 \ 0 \ 0 \quad M_4 = \bar{x} + y + z$$

$$1 \ 0 \ 1 \quad M_5 = \bar{x} + y + \bar{z}$$

$$1 \ 1 \ 0 \quad M_6 = \bar{x} + \bar{y} + z$$

$$1 \ 1 \ 1 \quad M_7 = \bar{x} + \bar{y} + \bar{z}$$

The bars coincide
with the 1's
in the binary expansion
of the maxterm sub-index

Expressions with three variables (for three-variable K-maps)

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	0
	1	0	0	0	0

$\overline{x_1} \overline{x_2} \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	0	0
	1	0	0	0	0

$\overline{x_1} x_2 \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	0
	1	0	0	0	0

$x_1 x_2 \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	1
	1	0	0	0	0

$x_1 \overline{x_2} \overline{x_3}$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	1	0	0	0

$$\overline{x_1} \overline{x_2} x_3$$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	0	1	0	0

$$\overline{x_1} x_2 \overline{x_3}$$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	0	0	1	0

$$x_1 x_2 x_3$$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	0	0	0	1

$$\overline{x_1} \overline{x_2} x_3$$

**Expressions with two variables
(for three-variable K-maps)**

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	0
	1	1	0	0	0

$\overline{x_1} \overline{x_2}$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	0	0
	1	0	1	0	0

$\overline{x_1} x_2$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	0
	1	0	0	1	0

$x_1 x_2$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	1
	1	0	0	0	1

$x_1 \overline{x_2}$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	0
	1	0	0	0	0

$\overline{x_1} \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	0
	1	0	0	0	0

$x_2 \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	1
	1	0	0	0	0

$x_1 \overline{x_3}$

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	1
	1	0	0	0	0

$\overline{x_2} \overline{x_3}$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	1	1	0	0

$\overline{x_1} x_3$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	0	1	1	0

$x_2 x_3$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	0	0	1	1

$x_1 x_3$

		x_1x_2			
		00	01	11	10
x_3	0	0	0	0	0
	1	1	0	0	1

$\overline{x_2} x_3$

**Expressions with one variable
(for three-variable K-maps)**

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	0
	1	1	1	0	0

$\overline{x_1}$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	0
	1	0	1	1	0

x_2

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	1
	1	0	0	1	1

x_1

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	1
	1	1	0	0	1

$\overline{x_2}$

Groupings and Expressions

x_1x_2		00	01	11	10
x_3	0	1	1	1	1
	1	0	0	0	0

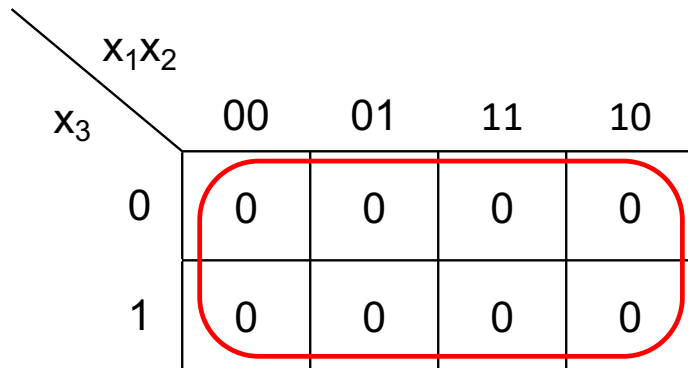
$\overline{x_3}$

x_1x_2		00	01	11	10
x_3	0	0	0	0	0
	1	1	1	1	1

x_3

**Expressions with zero variables
(for three-variable K-maps)**

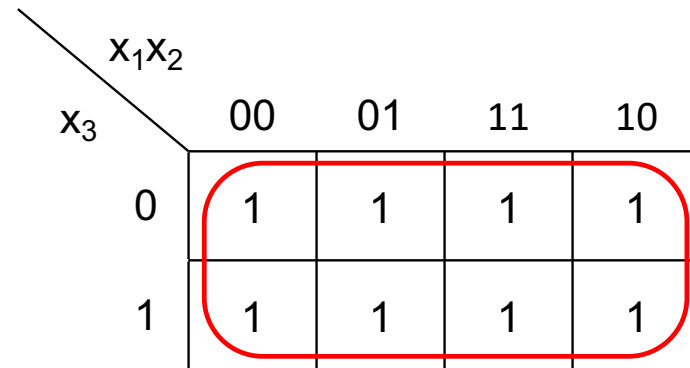
Groupings and Expressions



A Karnaugh map for a function of three variables x_1, x_2, x_3 . The map is a 2x4 grid. The columns are labeled x_1x_2 with values 00, 01, 11, 10. The rows are labeled x_3 with values 0 and 1. All cells in the map contain the value 0. A red rounded rectangle groups all four cells in the top row ($x_3 = 0$).

$x_3 \backslash x_1x_2$	00	01	11	10
0	0	0	0	0
1	0	0	0	0

0



A Karnaugh map for a function of three variables x_1, x_2, x_3 . The map is a 2x4 grid. The columns are labeled x_1x_2 with values 00, 01, 11, 10. The rows are labeled x_3 with values 0 and 1. All cells in the map contain the value 1. A red rounded rectangle groups all four cells in the top row ($x_3 = 0$).

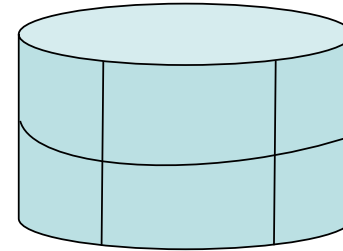
$x_3 \backslash x_1x_2$	00	01	11	10
0	1	1	1	1
1	1	1	1	1

1

Adjacency Rules

x_1x_2					
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

adjacent
columns

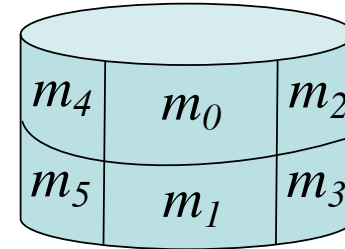


As if the K-map were
drawn on a cylinder

Adjacency Rules

x_1x_2					
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

adjacent
columns

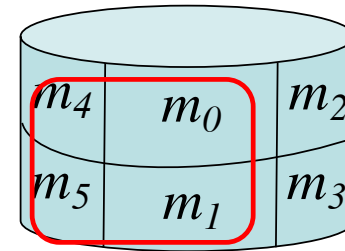


As if the K-map were
drawn on a cylinder

Adjacency Rules

		x_1x_2			
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

adjacent
columns

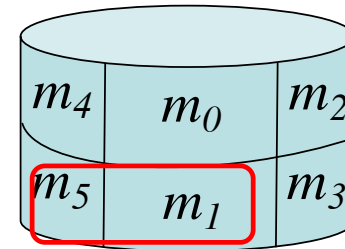


As if the K-map were
drawn on a cylinder

Adjacency Rules

x_1x_2		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

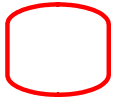
adjacent
columns



As if the K-map were
drawn on a cylinder

Grouping Size v.s. Term Size (for 3-variable K-maps)

Grouping Size v.s. Term Size



3-variable term



2-variable term

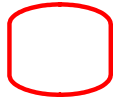


1-variable term

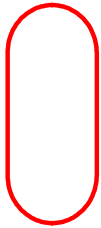


0-variable term

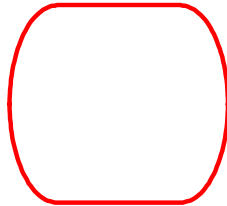
Grouping Size v.s. Term Size



3-variable term



2-variable term



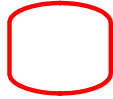
1-variable term



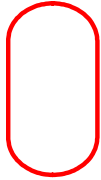
0-variable term

Grouping Size v.s. Term Size (for 4-variable K-maps)

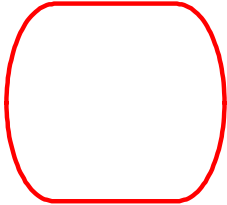
Grouping Size v.s. Term Size



4-variable term



3-variable term



2-variable term

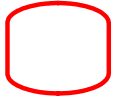


1-variable term



0-variable term

Grouping Size v.s. Term Size



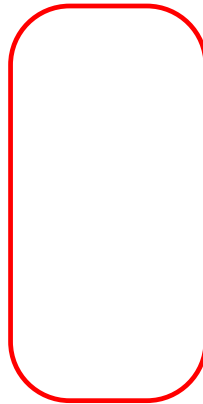
4-variable term



3-variable term



2-variable term



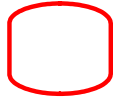
1-variable term



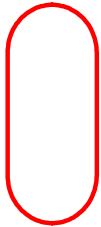
0-variable term

Grouping Size v.s. Term Size (for 2-variable K-maps)

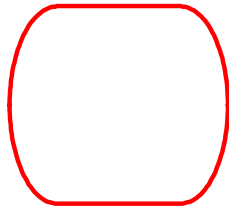
Grouping Size v.s. Term Size



2-variable term

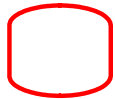


1-variable term



0-variable term

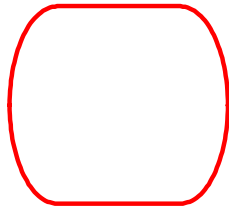
Grouping Size v.s. Term Size



2-variable term

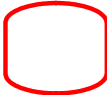
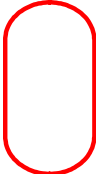
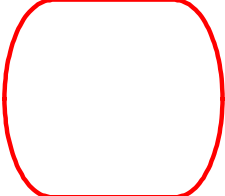


1-variable term



0-variable term

Grouping Size v.s. Term Size

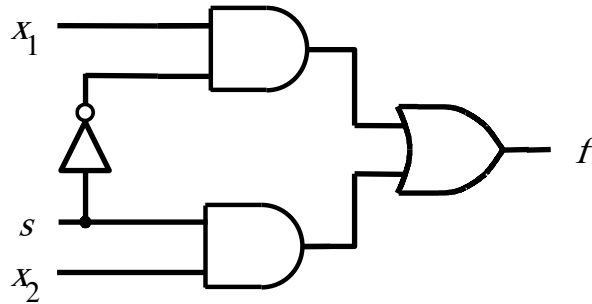
	2-variable K-map	3-variable K-map	4-variable K-map
	2	3	4
	1	2	3
	0	1	2
	N/A	0	1
	N/A	N/A	0

Example:
K-Map for the 2-to-1 Multiplexer

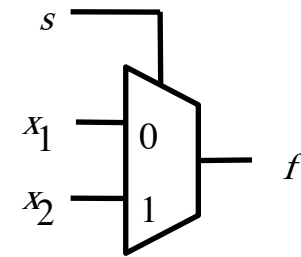
2-to-1 Multiplexer (Definition)

- **Has two inputs: x_1 and x_2**
- **Also has another input line s**
- **If $s=0$, then the output is equal to x_1**
- **If $s=1$, then the output is equal to x_2**

Circuit for 2-to-1 Multiplexer



(b) Circuit



(c) Graphical symbol

Truth Table for a 2-to-1 Multiplexer

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let's Draw the K-map

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let's Draw the K-map

	$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

Let's Draw the K-map

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$x_1 x_2$			
		00	01	11	10
s	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

Let's Draw the K-map

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$x_1 x_2$			
		00	01	11	10
s	0	0	1	0	0
	1	0	1	1	1

Let's Draw the K-map

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

$s \backslash x_1 x_2$					
		00	01	11	10
0	0	0	1	0	0
1	0	0	1	1	1

Let's Draw the K-map

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

$s \backslash x_1 x_2$					
		00	01	11	10
0		0	1	0	0
1		0	1	1	1

$$f(s, x_1, x_2) = \bar{x}_1 x_2 + s x_1$$

Let's Draw the K-map

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

$s \backslash x_1 x_2$	00	01	11	10
0	0	1	0	0
1	0	1	1	1

$$f(s, x_1, x_2) = \bar{x}_1 x_2 + s x_1$$

Something is wrong!

Compare this with the SOP derivation

Let's Derive the SOP form

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Let's Derive the SOP form

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Let's Derive the SOP form

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Where should we
put the negation signs?

$s \ x_1 \ x_2$

$s \ x_1 \ x_2$

$s \ x_1 \ x_2$

$s \ x_1 \ x_2$

Let's Derive the SOP form

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

$$\overline{s} \ x_1 \ \overline{x}_2$$

$$\overline{s} \ x_1 \ x_2$$

$$s \ \overline{x}_1 \ x_2$$

$$s \ x_1 \ x_2$$

Let's Derive the SOP form

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$	
0 0 0	0	
0 0 1	0	
0 1 0	1	$\overline{s} \ x_1 \ \overline{x}_2$
0 1 1	1	$\overline{s} \ x_1 \ x_2$
1 0 0	0	
1 0 1	1	$s \ \overline{x}_1 \ x_2$
1 1 0	0	
1 1 1	1	$s \ x_1 \ x_2$

$$f(s, x_1, x_2) = \overline{s} \ x_1 \ \overline{x}_2 + \overline{s} \ x_1 \ x_2 + s \ \overline{x}_1 \ x_2 + s \ x_1 \ x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \overline{s} x_1 \overline{x_2} + \overline{s} x_1 x_2 + s \overline{x_1} x_2 + s x_1 x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \bar{s} x_1 \bar{x}_2 + \bar{s} x_1 x_2 + s \bar{x}_1 x_2 + s x_1 x_2$$

$$f(s, x_1, x_2) = \bar{s} x_1 (\bar{x}_2 + x_2) + s (\bar{x}_1 + x_1) x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \bar{s} x_1 \bar{x}_2 + \bar{s} x_1 x_2 + s \bar{x}_1 x_2 + s x_1 x_2$$

$$f(s, x_1, x_2) = \bar{s} x_1 (\bar{x}_2 + x_2) + s (\bar{x}_1 + x_1) x_2$$

$$f(s, x_1, x_2) = \bar{s} x_1 + s x_2$$

Let's Draw the K-map again

$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let's Draw the K-map again

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

Let's Draw the K-map again

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

$s \backslash x_1 x_2$					
		00	01	11	10
0		m_0	m_2	m_6	m_4
1		m_1	m_3	m_7	m_5

Let's Draw the K-map again

	$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

		x_1x_2				
		s	00	01	11	10
0		m_0	m_2	m_6	m_4	
1		m_1	m_3	m_7	m_5	

Let's Draw the K-map again

	$s \ x_1 \ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

		$s \ x_1$ x_2			
		00	01	11	10
0		m_0	m_2	m_6	m_4
1		m_1	m_3	m_7	m_5

The order of the labeling matters.

Let's Draw the K-map again

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

Let's Draw the K-map again

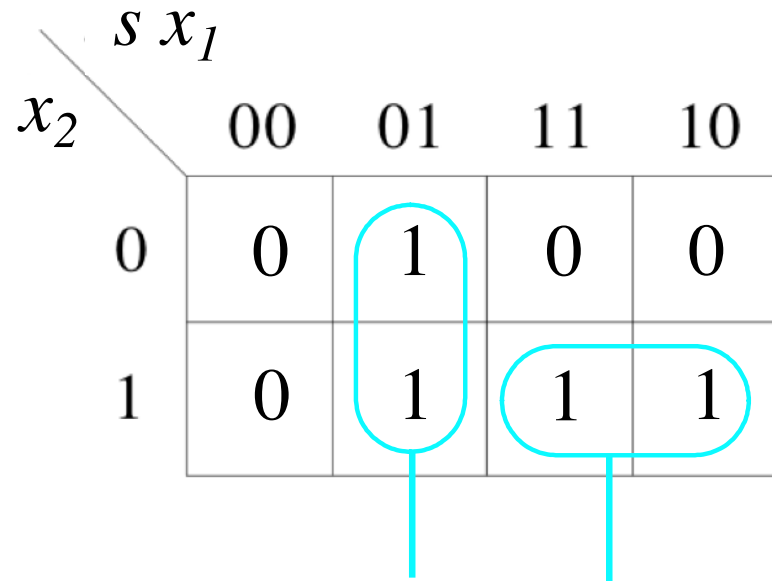
	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	0	1	0	0
	1	0	1	1	1

Let's Draw the K-map again

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	0	1	0	0
	1	0	1	1	1



Let's Draw the K-map again

	s	x_1	x_2	$f(s, x_1, x_2)$
m_0	0	0	0	0
m_1	0	0	1	0
m_2	0	1	0	1
m_3	0	1	1	1
m_4	1	0	0	0
m_5	1	0	1	1
m_6	1	1	0	0
m_7	1	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	0	1	0	0
	1	0	1	1	1

$$f(s, x_1, x_2) = \overline{s} x_1 + s x_2$$

This is correct!

Two Different Ways to Draw the K-map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

		$x_1 x_2$			
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

(b) Karnaugh map

		$x_2 x_3$			
		00	01	11	10
x_1	0	m_0	m_1	m_3	m_2
	1	m_4	m_5	m_7	m_6

Another Way to Draw 3-variable K-map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

		$x_1 \ x_2$			
		x_3			
		00	01	11	10
0		m_0	m_2	m_6	m_4
1		m_1	m_3	m_7	m_5

(b) Karnaugh map

		x_1	
		0	1
$x_2 x_3$	00	m_0	m_4
	01	m_1	m_5
	11	m_3	m_7
	10	m_2	m_6

There are 4 different versions!

x_1x_2		x_3			
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

x_2x_3		x_1			
		00	01	11	10
x_1	0	m_0	m_1	m_3	m_2
	1	m_4	m_5	m_7	m_6

x_1x_2	x_3	
	0	1
00	m_0	m_1
01	m_2	m_3
11	m_6	m_7
10	m_4	m_5

x_2x_3	x_1	
	0	1
00	m_0	m_4
01	m_1	m_5
11	m_3	m_7
10	m_2	m_6

Gray Code

- Sequence of binary codes
- Neighboring lines vary by only 1 bit

	000
	001
00	011
01	010
11	110
10	111
	101
	100

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_I$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These two neighbors
differ only in the LAST bit

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These two neighbors
differ only in the LAST bit

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These two neighbors
differ only in the FIRST bit

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These two neighbors
differ only in the FIRST bit

Adjacency Rules

$x_2 \backslash s x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

adjacent
columns

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These four neighbors
differ in the FIRST and LAST bit

They are similar in their MIDDLE bit

Gray Code & K-map

	s	x_1	x_2
m_0	0	0	0
m_1	0	0	1
m_2	0	1	0
m_3	0	1	1
m_4	1	0	0
m_5	1	0	1
m_6	1	1	0
m_7	1	1	1

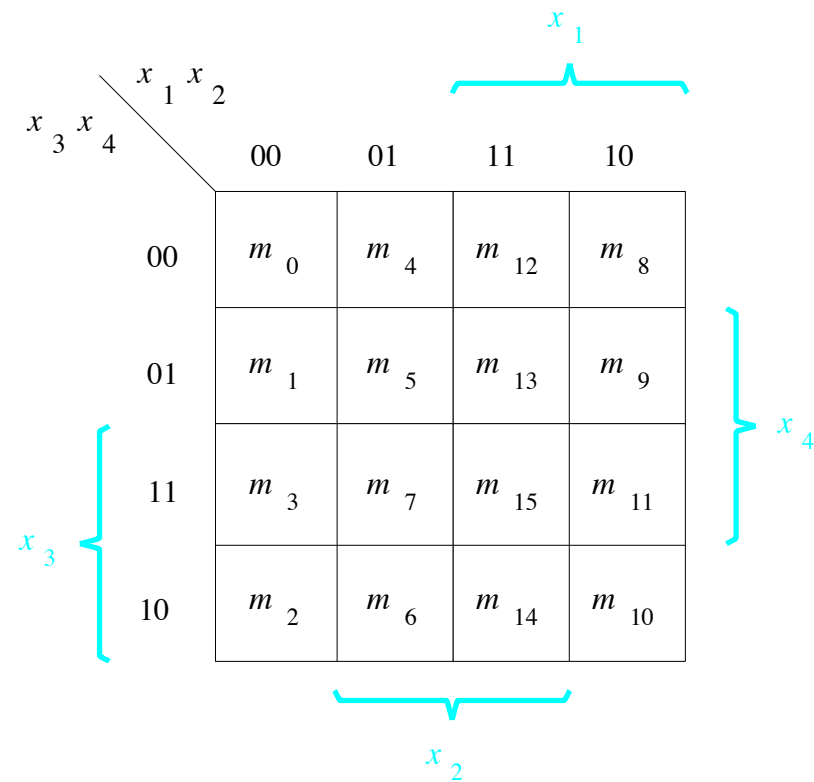
		$s \ x_1$			
		00	01	11	10
x_2	0	000	010	110	100
	1	001	011	111	101

These four neighbors
differ in the FIRST and LAST bit

They are similar in their MIDDLE bit

Four-Variable K-Map

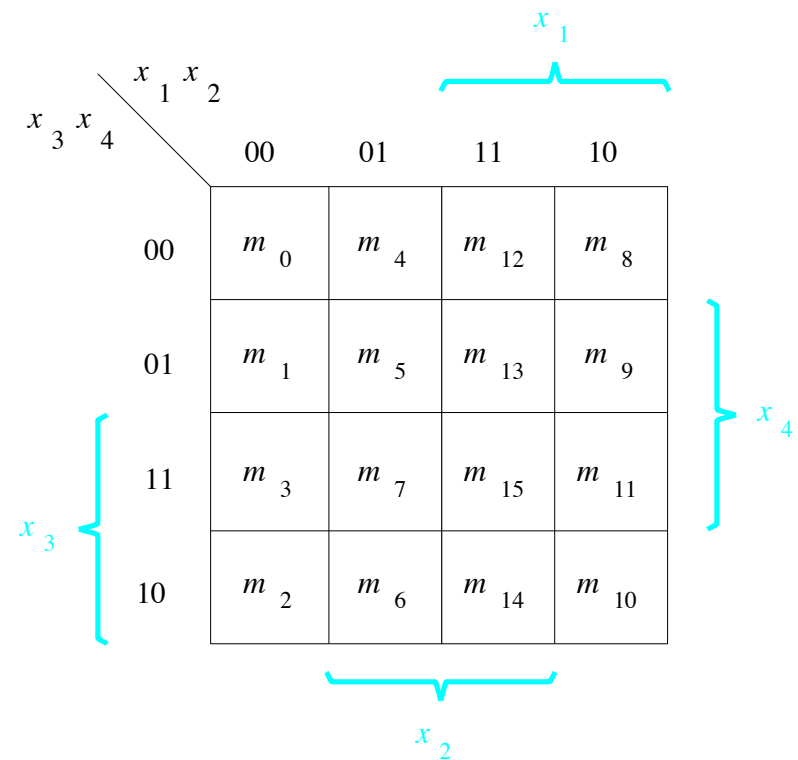
A four-variable Karnaugh map



[Figure 2.53 from the textbook]

A four-variable Karnaugh map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15



Adjacency Rules

		x_1x_2			
		00	01	11	10
x_3	0	m_0	m_2	m_6	m_4
	1	m_1	m_3	m_7	m_5

adjacent
columns



		x_1x_2			
		00	01	11	10
x_3x_4	00	m_0	m_4	m_{12}	m_8
	01	m_1	m_5	m_{13}	m_9
	11	m_3	m_7	m_{15}	m_{11}
	10	m_2	m_6	m_{14}	m_{10}

adjacent
columns

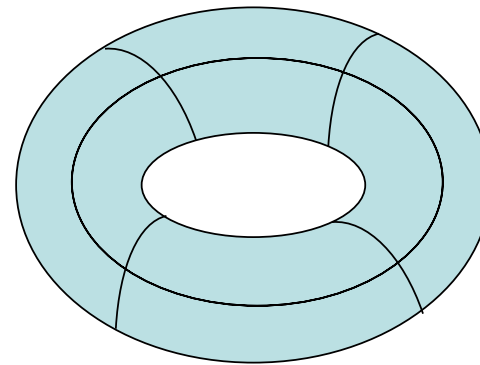


adjacent
rows



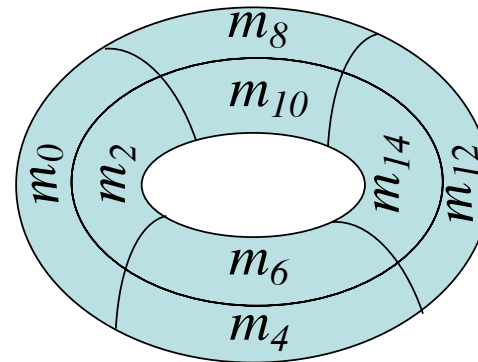
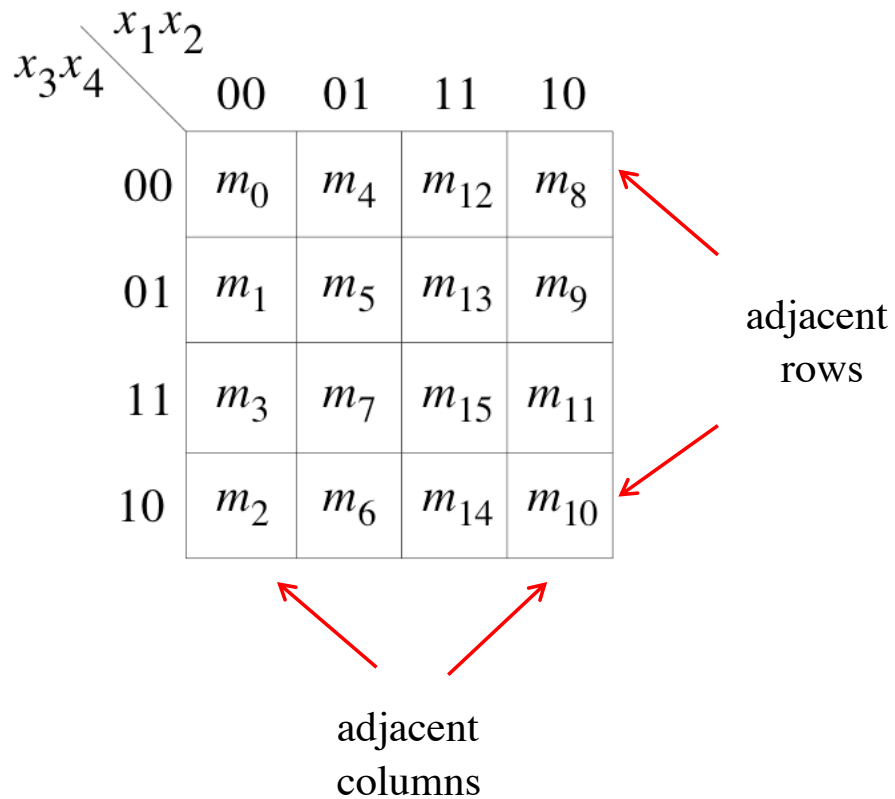
Adjacency Rules

x_1x_2					
x_3x_4	00	01	11	10	
	m_0	m_4	m_{12}	m_8	adjacent rows
	m_1	m_5	m_{13}	m_9	
	m_3	m_7	m_{15}	m_{11}	
	m_2	m_6	m_{14}	m_{10}	
					adjacent columns



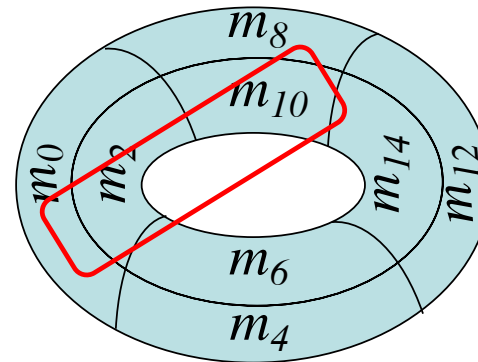
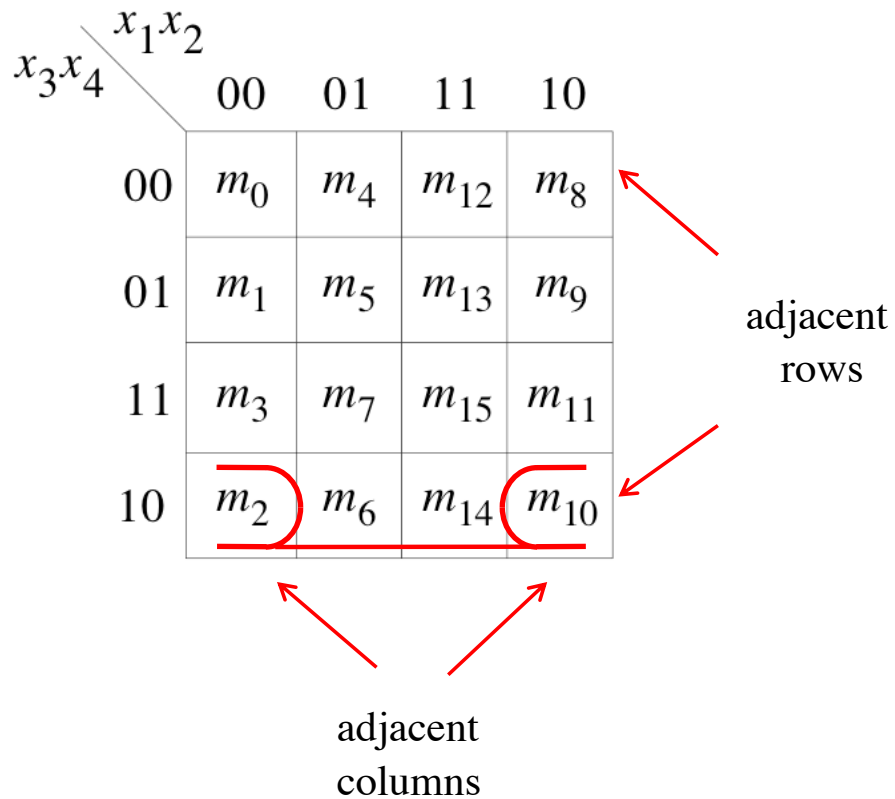
As if the K-map were
drawn on a torus

Adjacency Rules



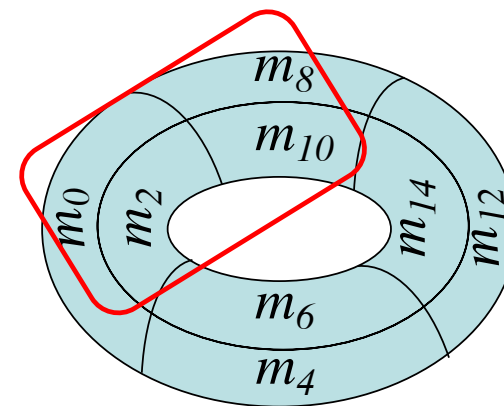
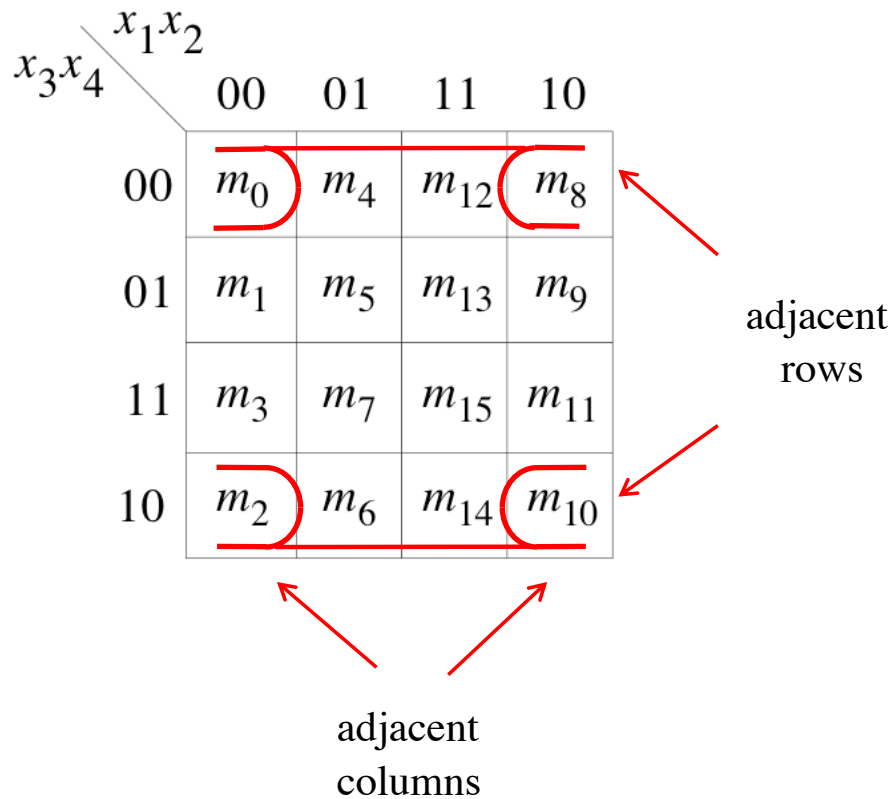
As if the K-map were drawn on a torus

Adjacency Rules



As if the K-map were drawn on a torus

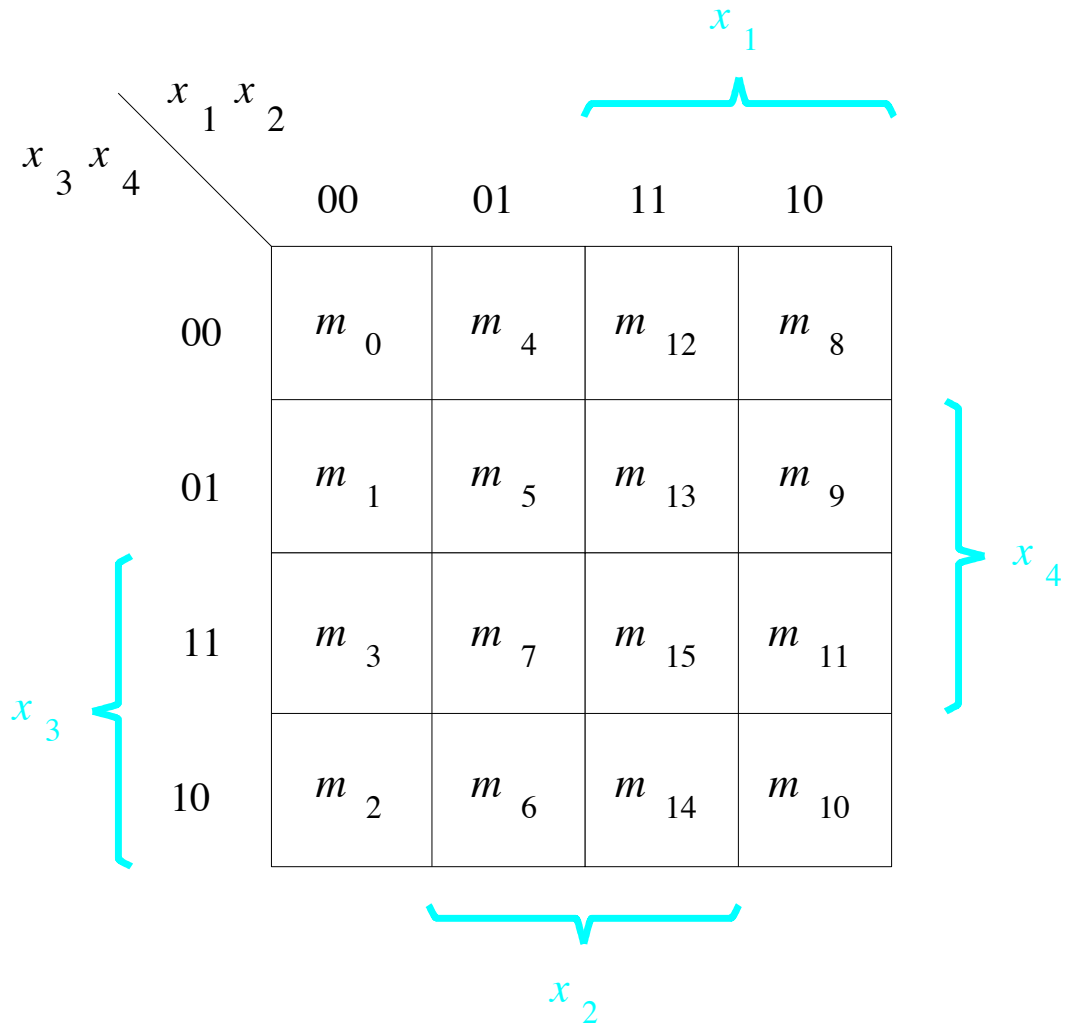
Adjacency Rules



As if the K-map were drawn on a torus

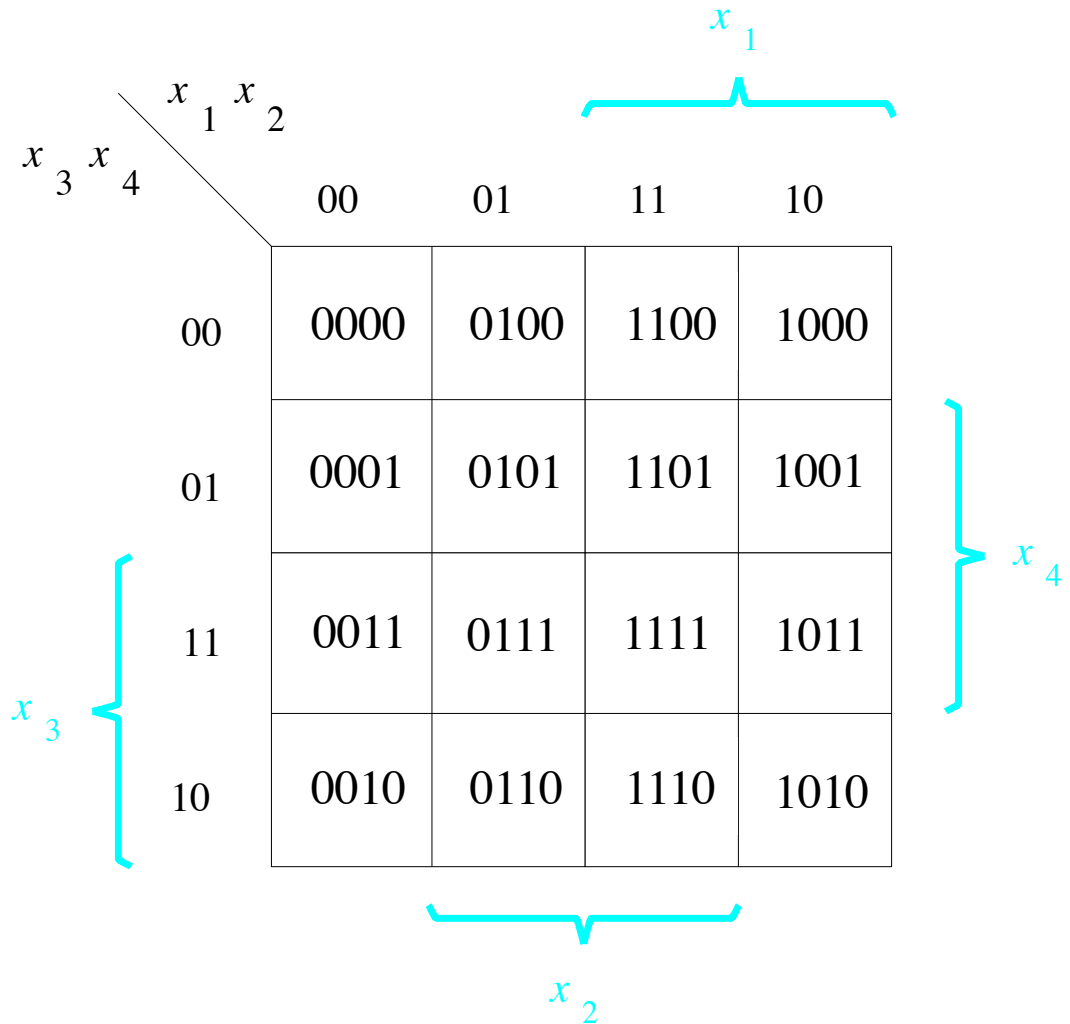
Gray Code & K-map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

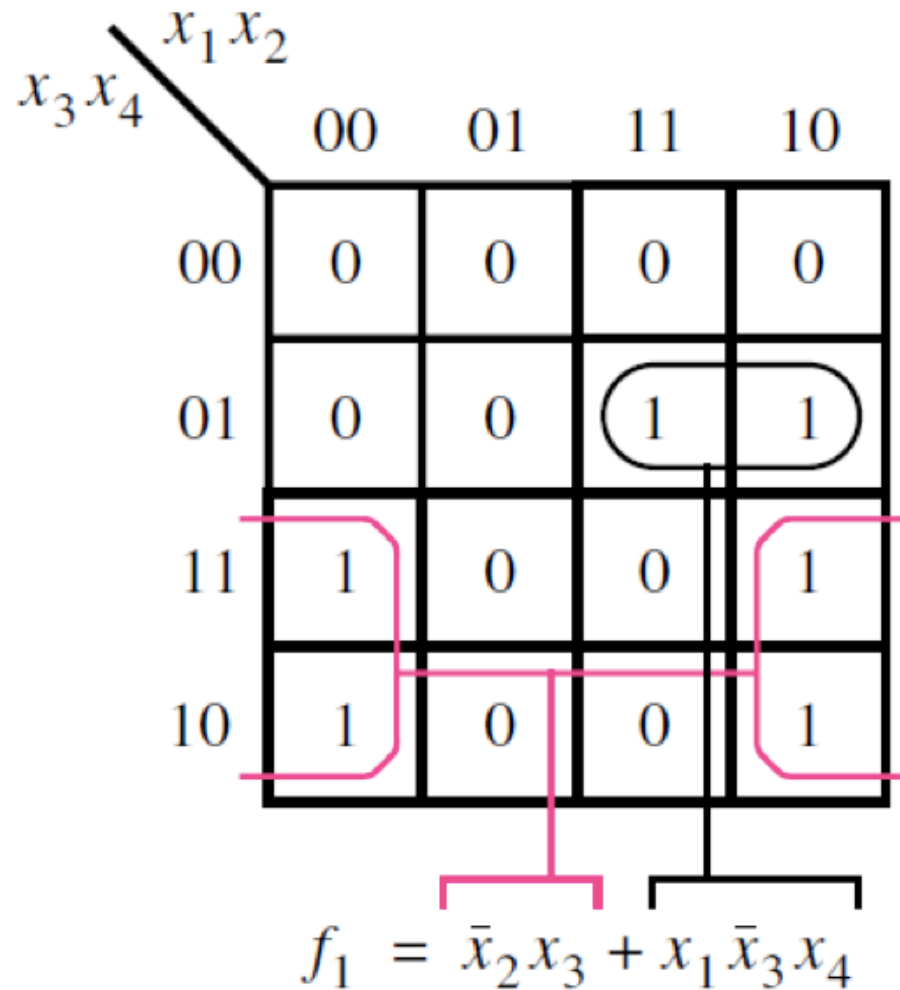


Gray Code & K-map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

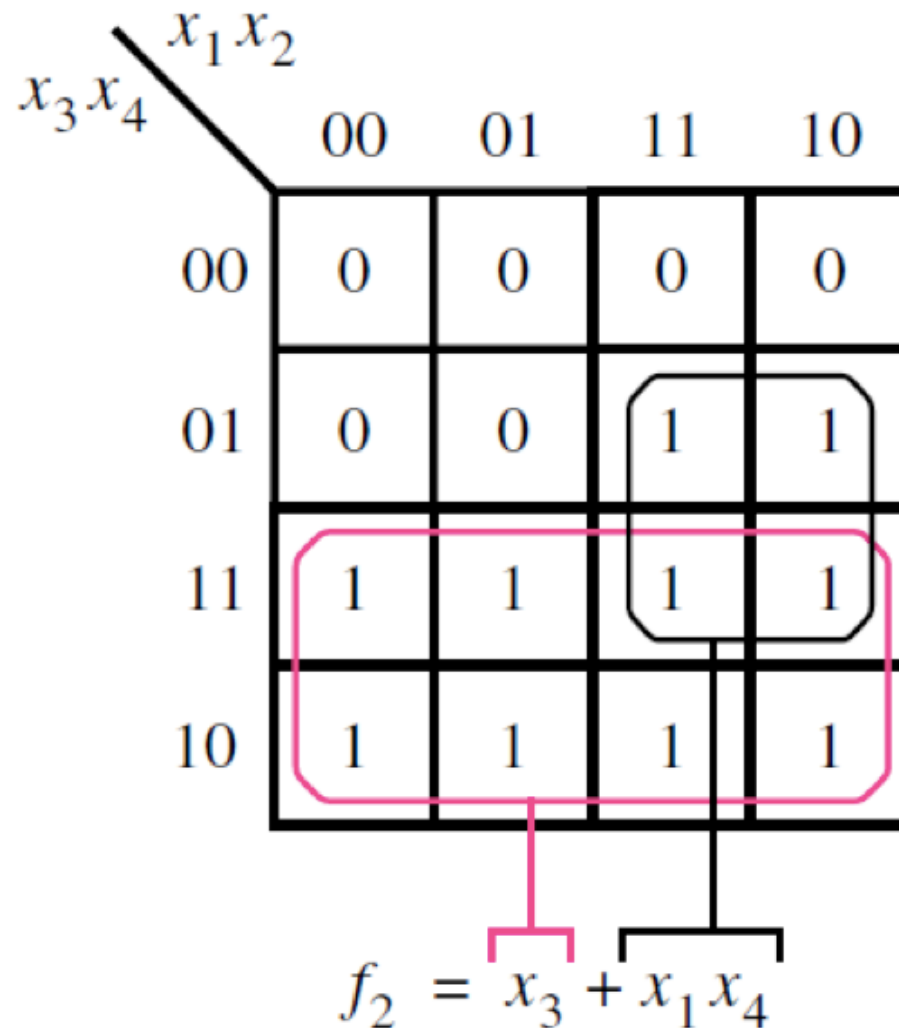


Example of a four-variable Karnaugh map



[Figure 2.54 from the textbook]

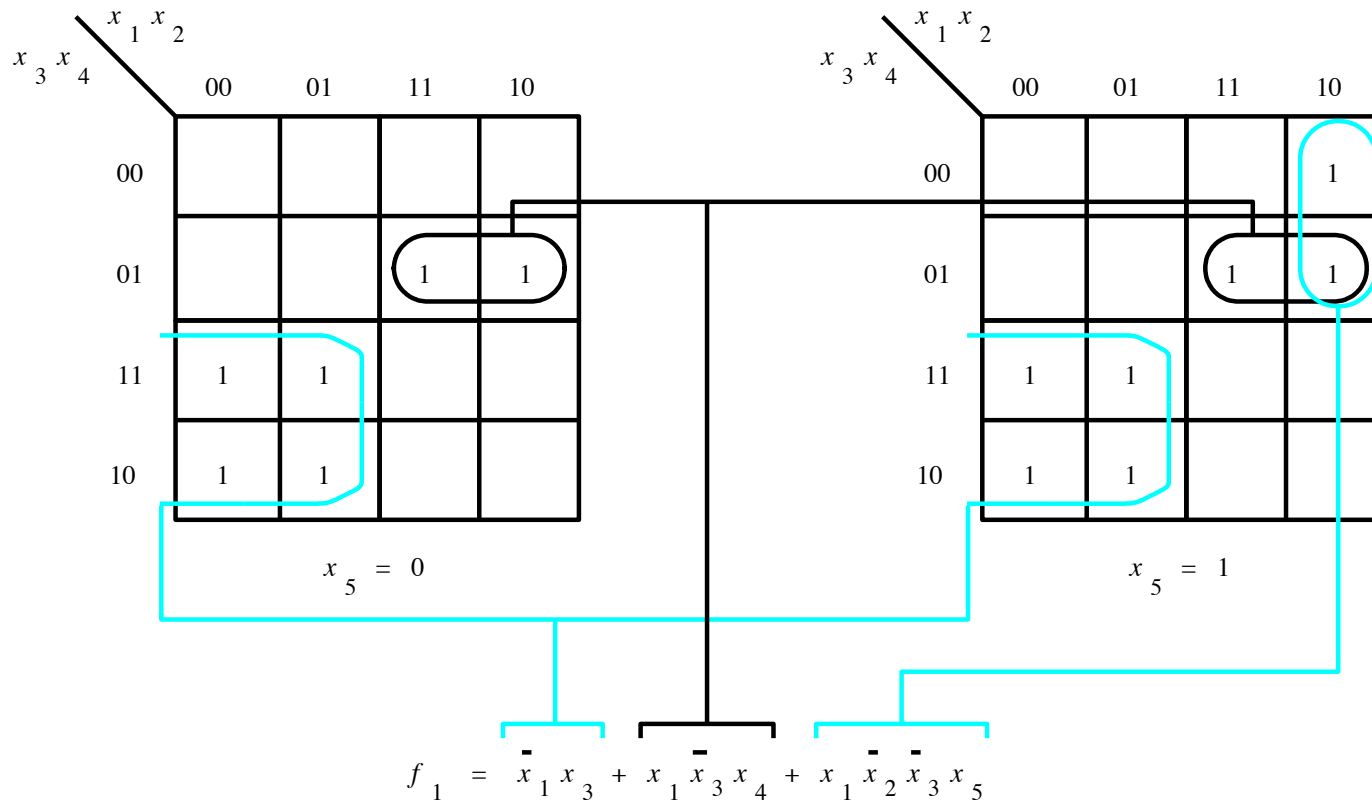
Example of a four-variable Karnaugh map



[Figure 2.54 from the textbook]

Five-Variable K-Map

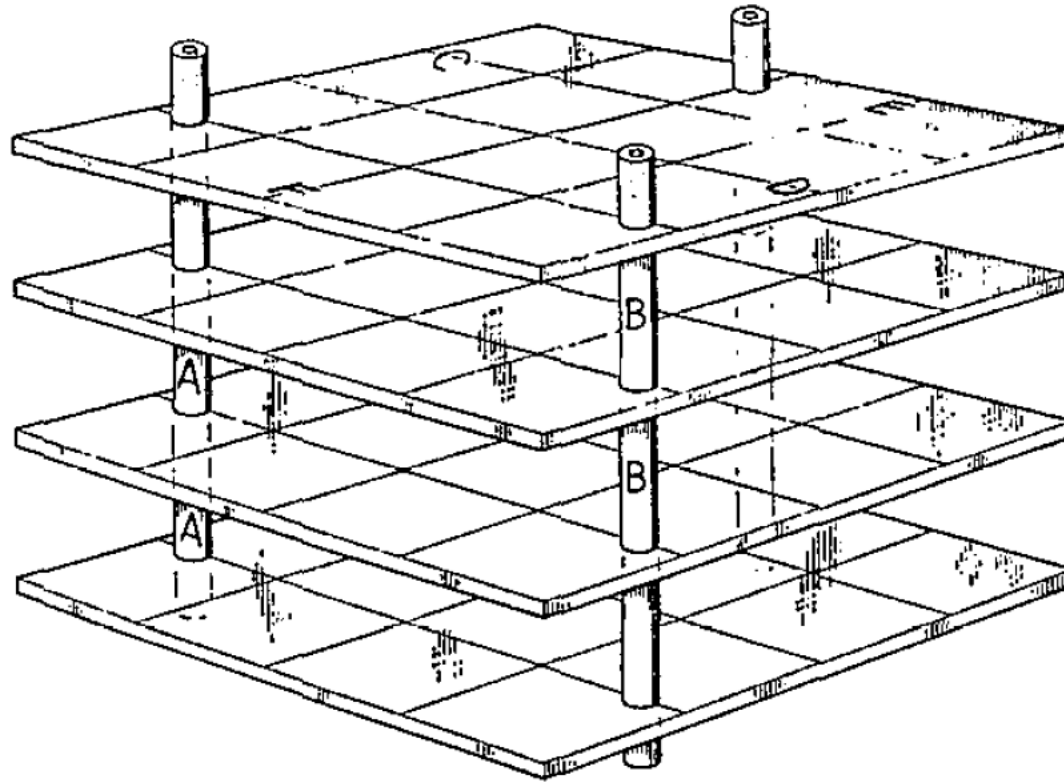
A five-variable Karnaugh map



[Figure 2.55 from the textbook]

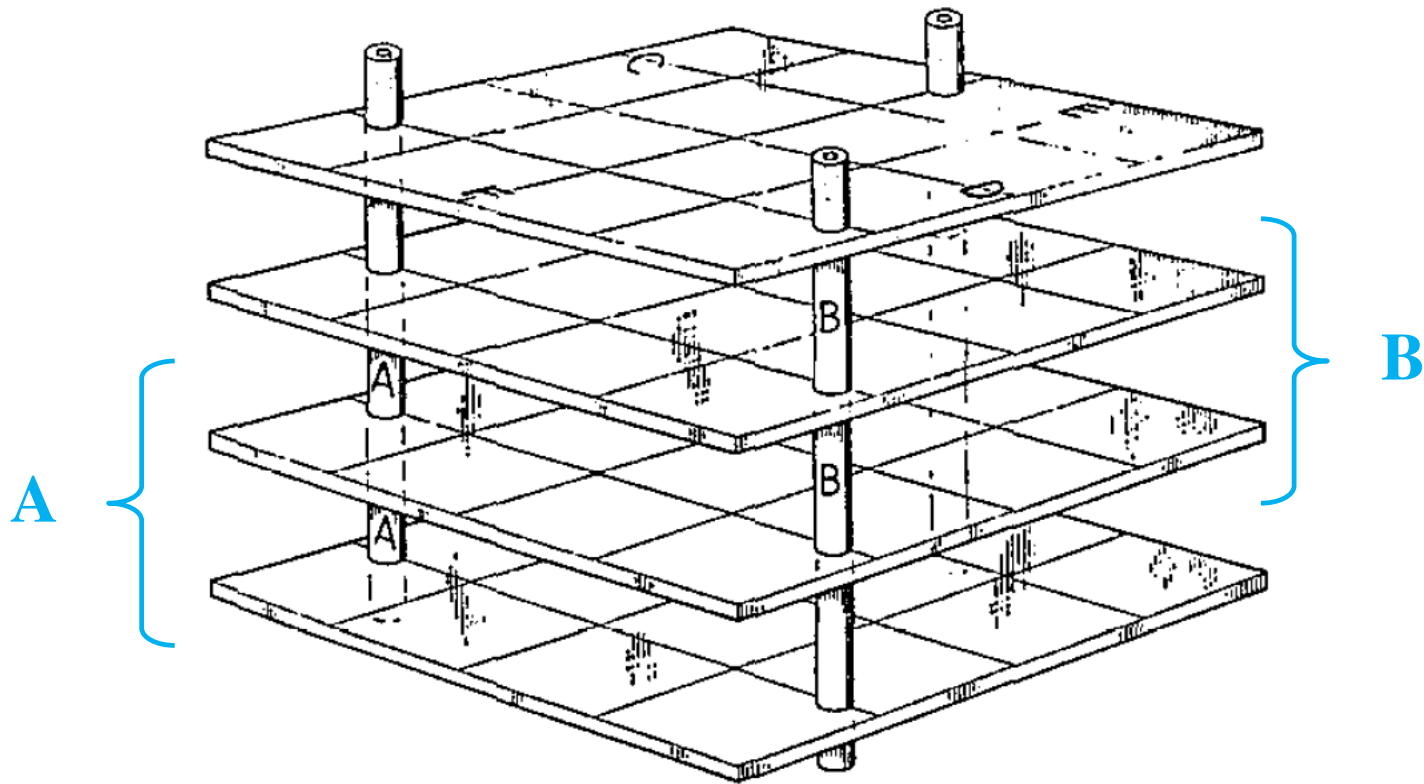
Six-Variable K-Map

A six-variable Karnaugh map



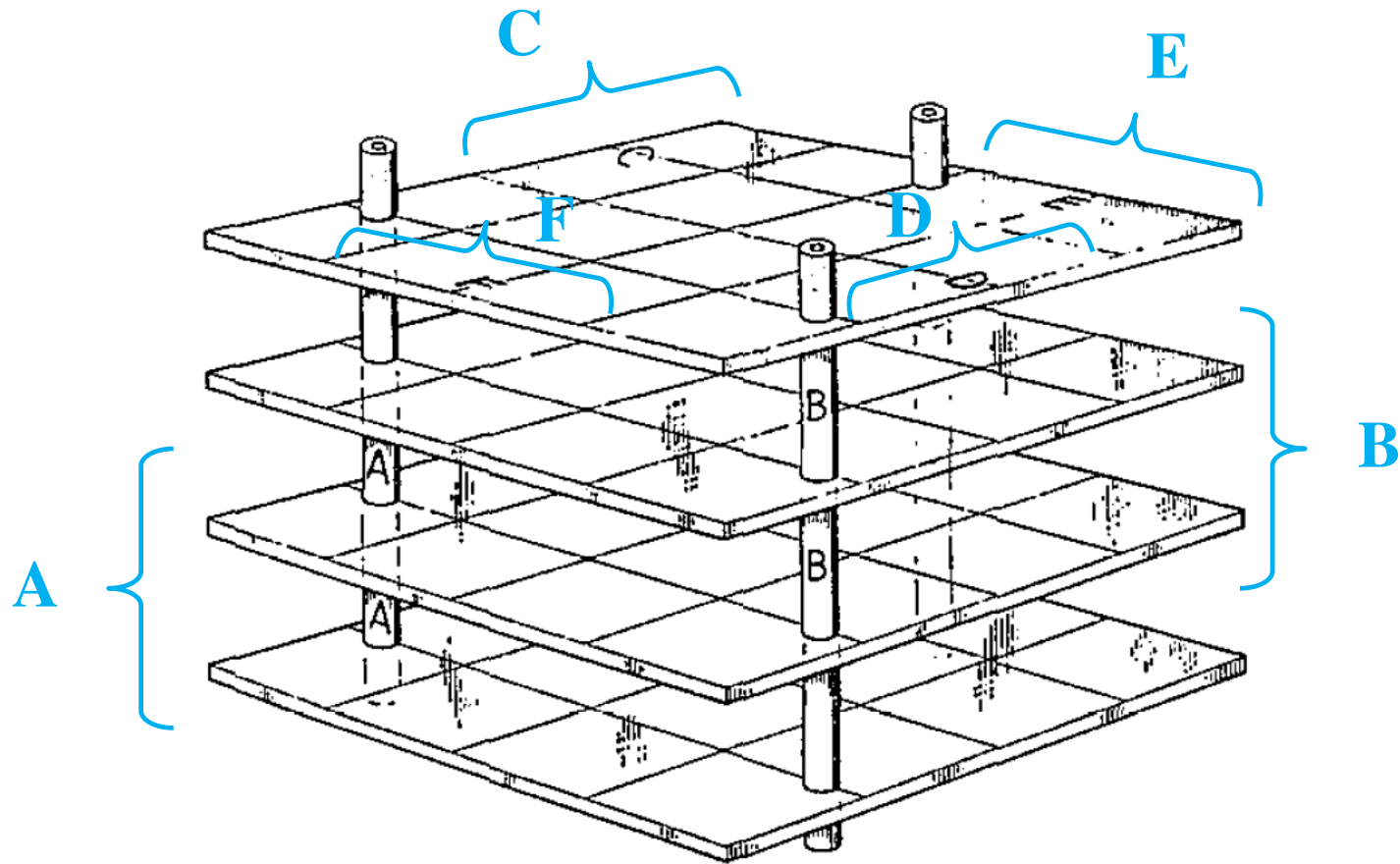
[Figure 16, in Karnaugh 1953]

A six-variable Karnaugh map



[Figure 16, in Karnaugh 1953]

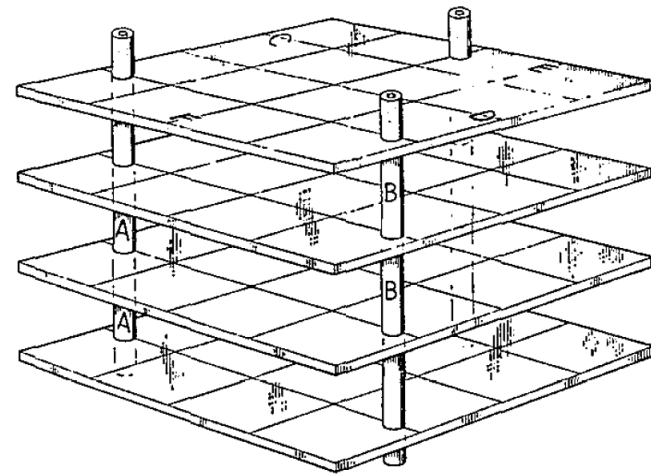
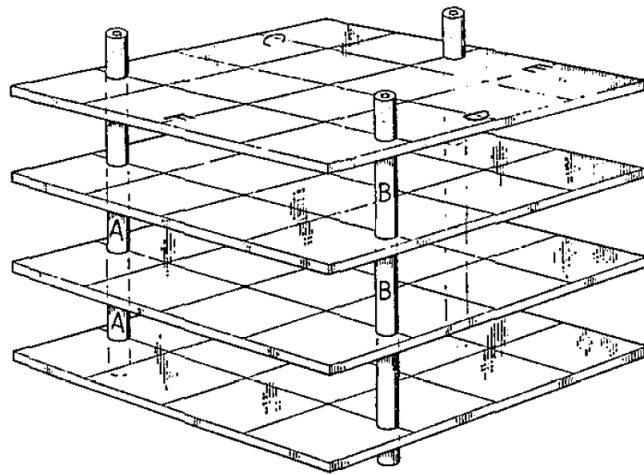
A six-variable Karnaugh map



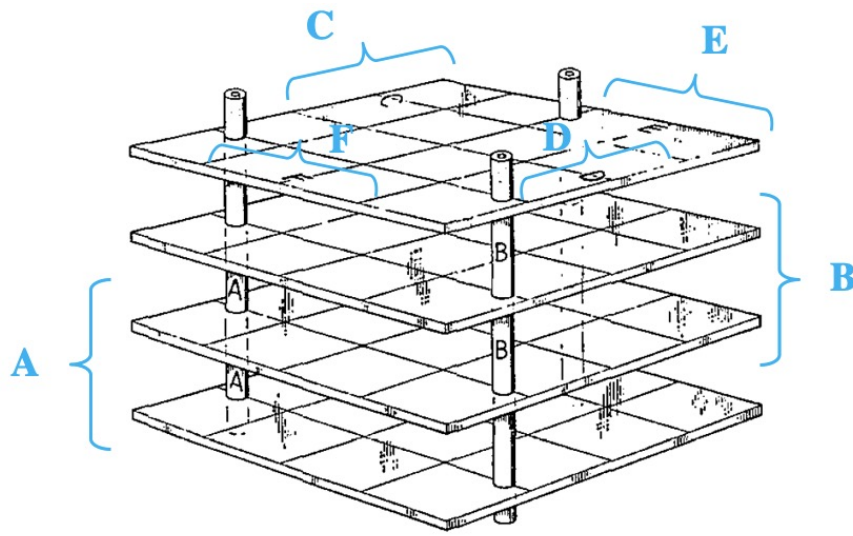
[Figure 16, in Karnaugh 1953]

Seven-Variable K-Map

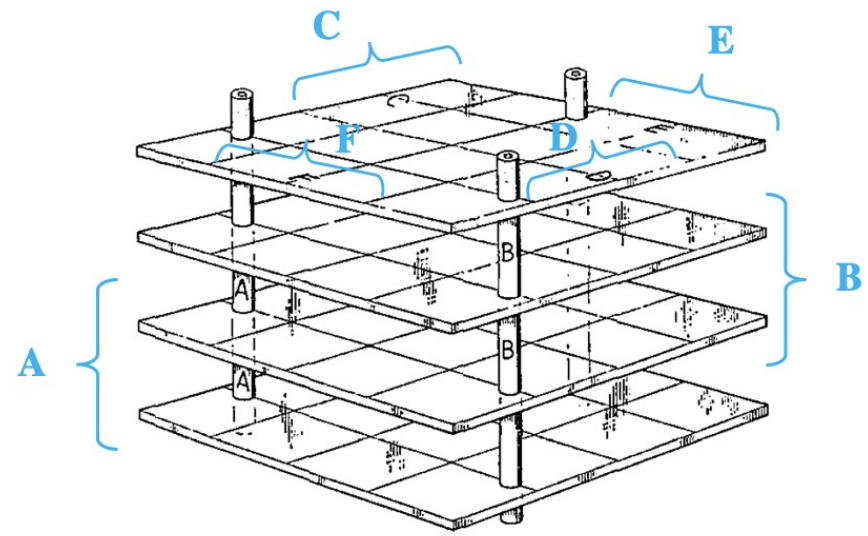
A seven-variable Karnaugh map



A seven-variable Karnaugh map



G = 0



G = 1

[Suggested in Karnaugh 1953]

Strategy For Minimization

Grouping Rules

- **Group “1”s with rectangles**
- **Both sides a power of 2:**
 - 1x1, 1x2, 2x1, 2x2, 1x4, 4x1, 2x4, 4x2, 4x4
- **Can use the same minterm more than once**
- **Can wrap around the edges of the map**
- **Some rules in selecting groups:**
 - Try to use as few groups as possible to cover all “1”s.
 - For each group, try to make it as large as you can (i.e., if you can use a 2x2, don’t use a 2x1 even if that is enough).

Terminology

Literal: a variable, complemented or uncomplemented

Some Examples:

- $\bar{X}_1$
- X_2

Terminology

- **Implicant:** product term that indicates the input combinations for which the function output is 1
- **Example**
 - $\bar{x}_1$ - indicates that $\bar{x}_1\bar{x}_2$ and $\bar{x}_1x_2$ yield output of 1

		x_1	
		0	1
x_2	0	1	0
	1	1	0

Terminology

- **Prime Implicant**
 - Implicant that cannot be combined into another implicant with fewer literals
 - Some Examples

$x_1 x_2$					
x_3		00	01	11	10
0	0		1	1	1
1	1	1	1	1	0

Not prime

$x_1 x_2$					
x_3		00	01	11	10
0	0		1	1	1
1	1		1	1	0

Prime

Terminology

- **Essential Prime Implicant**
 - Prime implicant that includes a minterm not covered by any other prime implicant
 - Some Examples

$x_1 x_2$					
x_3		00	01	11	10
0	0	0	1	1	1
1	1	1	1	0	0

Terminology

- **Cover**
 - Collection of implicants that account for all possible input valuations where output is 1
 - Ex. $x_1' x_2 x_3 + x_1 x_2 x_3' + x_1 x_2' x_3'$
 - Ex. $x_1' x_2 x_3 + x_1 x_3'$

$x_1 x_2$					
x_3		00	01	11	10
0		0	0	1	1
1		0	1	0	0

Example

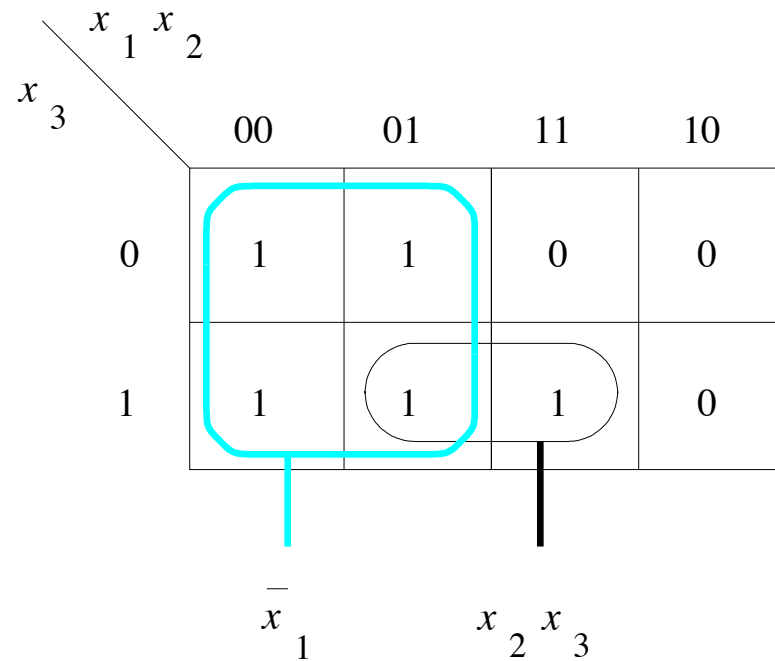
- Give the Number of
 - Implicants?
 - Prime Implicants?
 - Essential Prime Implicants?

x_3	$x_1 x_2$			
	00	01	11	10
0	1	1	0	0
1	1	1	1	0

Why concerned with minimization?

- **Simplified function**
- **Reduce the cost of the circuit**
 - **Cost: Gates + Inputs**
 - **Transistors**

Three-variable function $f(x_1, x_2, x_3) = \sum m(0, 1, 2, 3, 7)$

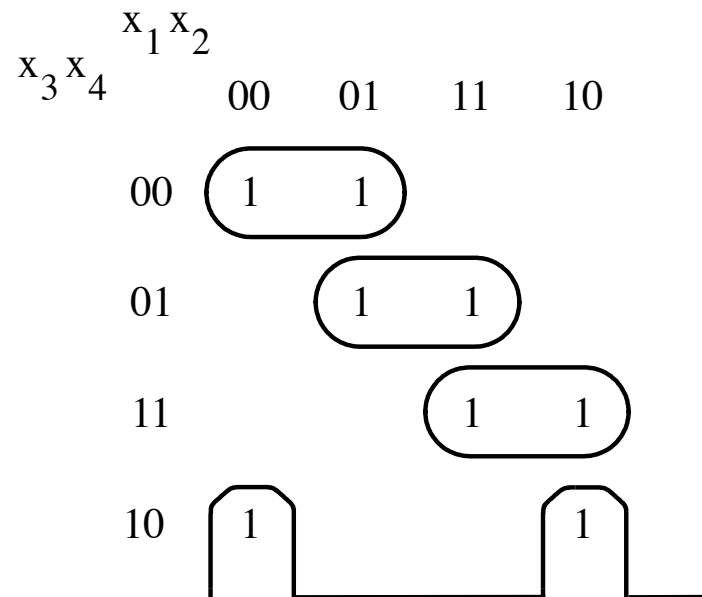


[Figure 2.56 from the textbook]

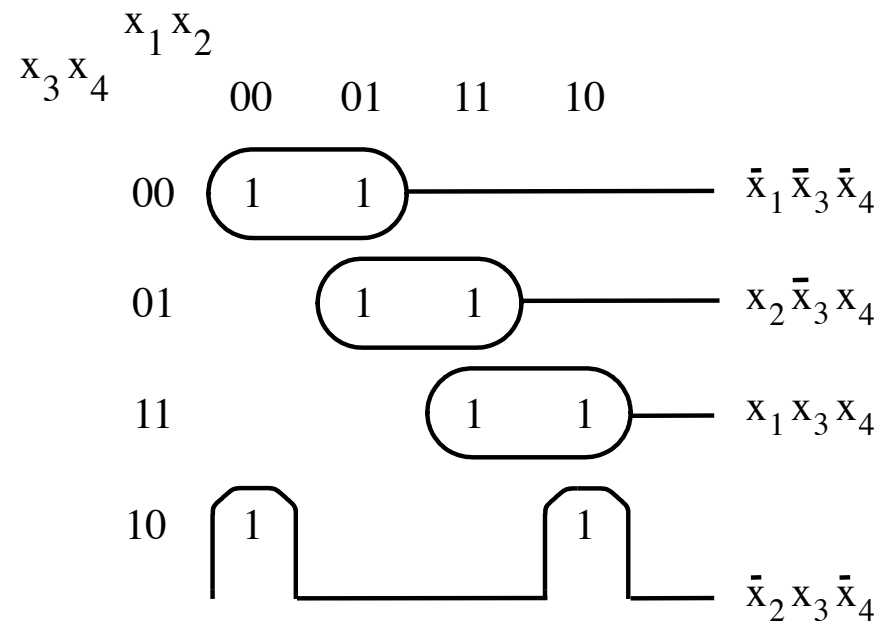
Example

$x_3 x_4$	$x_1 x_2$	00	01	11	10
00	1	1			
01		1	1		
11				1	1
10	1				1

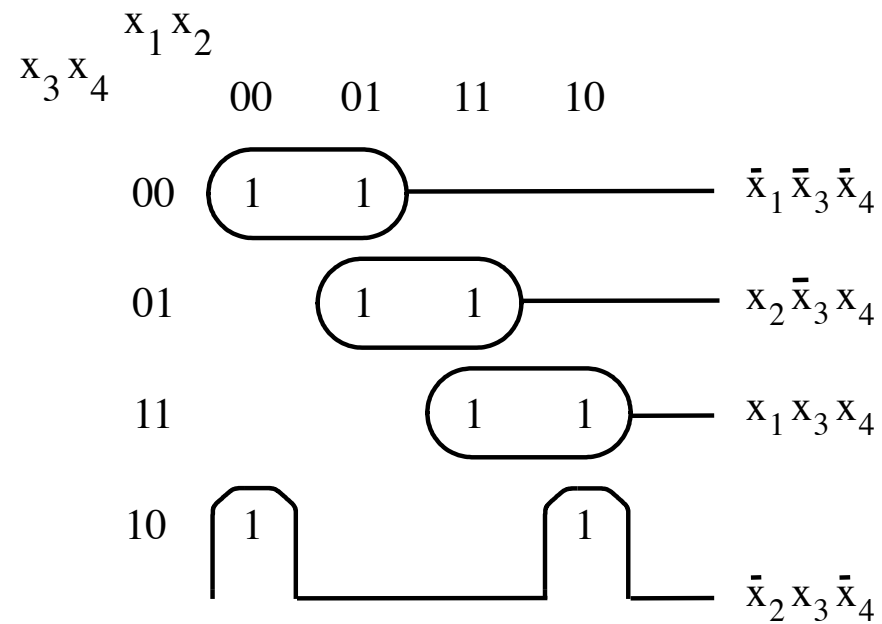
Example



Example



Example

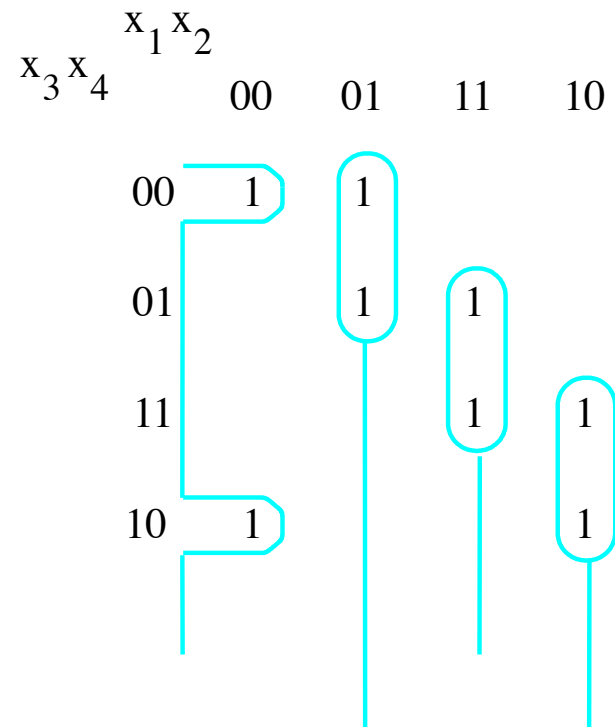


$$f = \bar{x}_1 \bar{x}_3 \bar{x}_4 + x_2 \bar{x}_3 x_4 + x_1 x_3 x_4 + \bar{x}_2 x_3 \bar{x}_4$$

Example: Another Solution

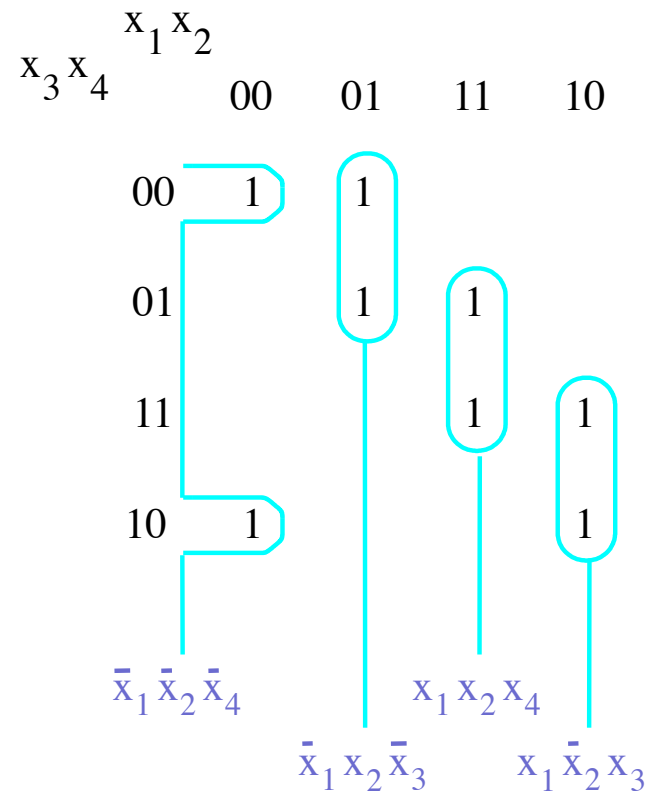
$x_3 x_4$		$x_1 x_2$			
		00	01	11	10
00	1	1			
01		1	1		
11				1	1
10	1				1

Example: Another Solution

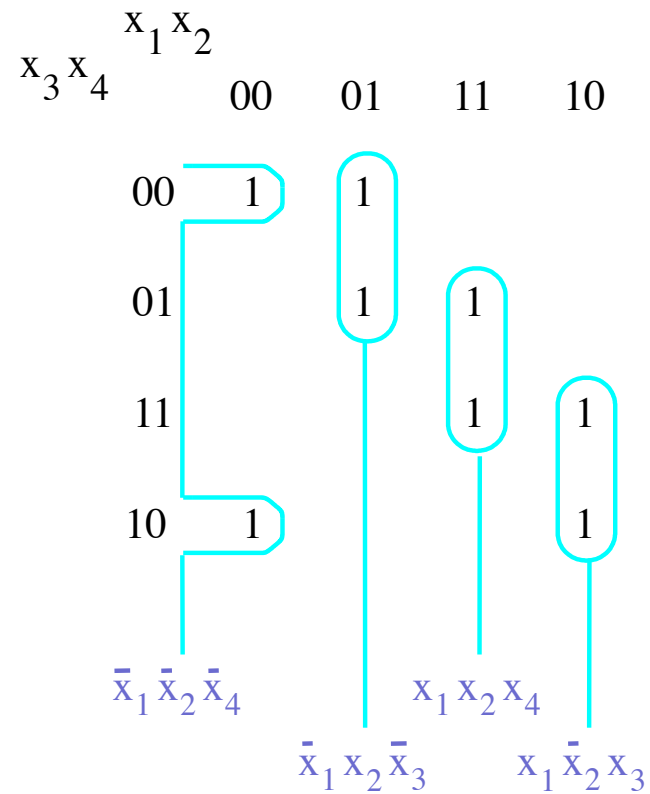


[Figure 2.59 from the textbook]

Example: Another Solution

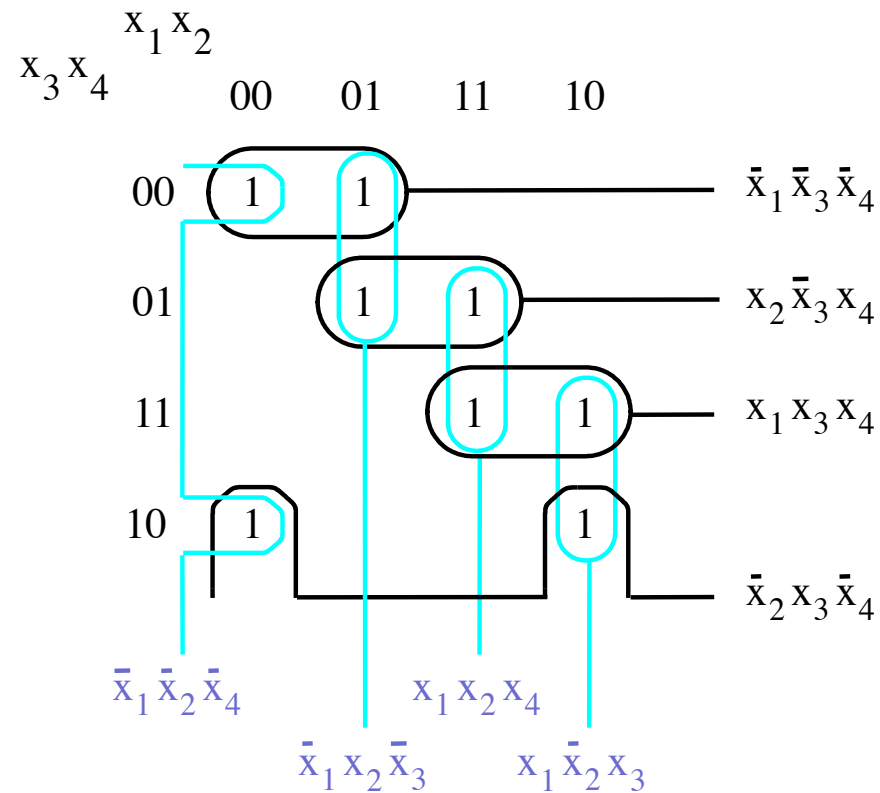


Example: Another Solution



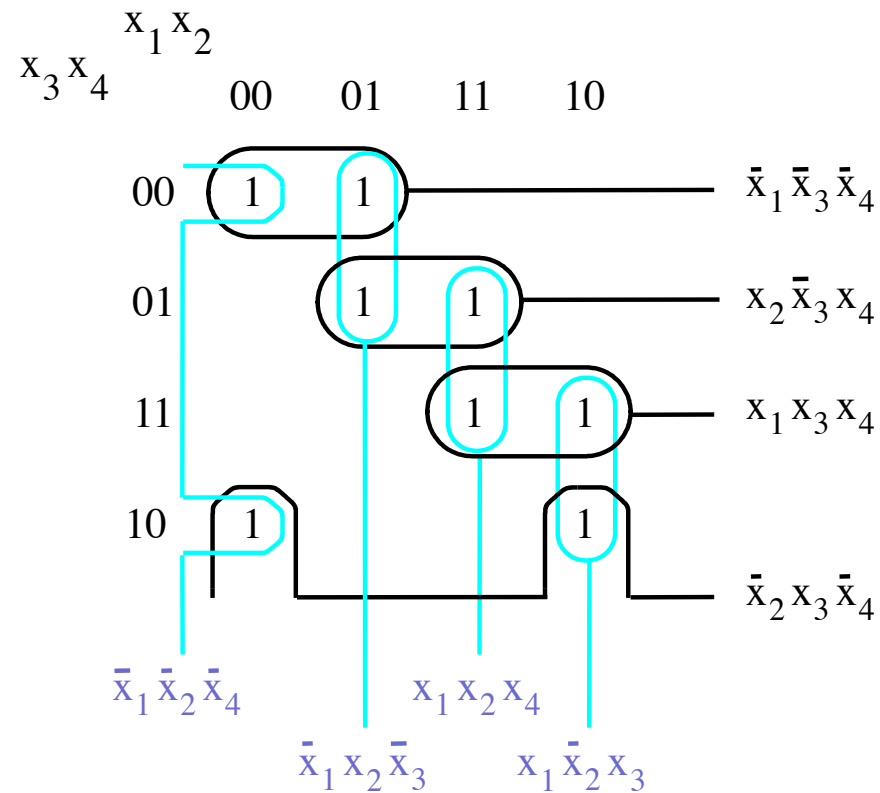
$$f = \bar{x}_1 \bar{x}_2 \bar{x}_4 + \bar{x}_1 x_2 \bar{x}_3 + x_1 x_2 x_4 + x_1 \bar{x}_2 x_3$$

Example: Both Are Valid Solutions



[Figure 2.59 from the textbook]

Example: Both Are Valid Solutions



$$f = \bar{x}_1 \bar{x}_3 \bar{x}_4 + x_2 \bar{x}_3 x_4 + x_1 x_3 x_4 + \bar{x}_2 x_3 \bar{x}_4$$

$$f = \bar{x}_1 \bar{x}_2 \bar{x}_4 + \bar{x}_1 x_2 \bar{x}_3 + x_1 x_2 x_4 + x_1 \bar{x}_2 x_3$$

Minimization of Product-of-Sums Forms

Do You Still Remember This Boolean Algebra Theorem?

14a. $x \cdot y + x \cdot \bar{y} = x$

14b. $(x + y) \cdot (x + \bar{y}) = x$

Combining

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$
0	0	
0	1	
1	0	
1	1	

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$
0	0	0
0	1	1
1	0	1
1	1	1

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$	
0	0	0	1
0	1	1	0
1	0	1	1
1	1	1	1

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$		
0	0	0	0	1
0	1	1	0	0
1	0	1	1	1
1	1	1	1	1

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$			
0	0	0	0	1	0
0	1	1	0	0	0
1	0	1	1	1	1
1	1	1	1	1	1

Let's prove 14.b

x	y	$(\mathbf{x} + \mathbf{y}) \bullet (\mathbf{x} + \overline{\mathbf{y}}) = \mathbf{x}$				
0	0	0	0	1	0	
0	1	1	0	0	0	
1	0	1	1	1	1	
1	1	1	1	1	1	

They are equal.

Grouping Example

		x_1	
		0	1
x_2	0	0	1
	1	1	1

M_0

		x_1	
		0	1
x_2	0	1	0
	1	1	1

M_2

Grouping Example

		x_1	
		0	1
x_2	0	0	1
	1	1	1

M_0

*

		x_1	
		0	1
x_2	0	1	0
	1	1	1

M_2

=

		x_1	
		0	1
x_2	0	0	0
	1	1	1

$M_0 * M_2$

Grouping Example

x_2	x_1	
	0	1
0	0	1
1	1	1

M_0

*

x_2	x_1	
	0	1
0	1	0
1	1	1

M_2

=

x_2	x_1	
	0	1
0	0	0
1	1	1

=

$M_0 * M_2$

Grouping Example

x_2	x_1	
	0	1
0	0	1
1	1	1

M_0

*

x_2	x_1	
	0	1
0	1	0
1	1	1

M_2

=

x_2	x_1	
	0	1
0	0	0
1	1	1

$M_0 * M_2$

Grouping Example

		x_1	
		0	1
x_2	0	0	1
	1	1	1

M_0

$(x_1 + x_2)$

*

*

*

		x_1	
		0	1
x_2	0	1	0
	1	1	1

M_2

$(\bar{x}_1 + x_2)$

=

=

=

		x_1	
		0	1
x_2	0	0	0
	1	1	1

$M_0 * M_2$

x_2

Property 14b (Combining)

Expressions with three variables (for three-variable K-maps)

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	1
	1	1	1	1	1

$$(x_1 + x_2 + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	0	1	1
	1	1	1	1	1

$$(x_1 + \overline{x_2} + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	1
	1	1	1	1	1

$$(\overline{x_1} + \overline{x_2} + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	0
	1	1	1	1	1

$$(\overline{x_1} + x_2 + x_3)$$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	0	1	1	1

$$(x_1 + x_2 + \bar{x}_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	1	0	1	1

$$(x_1 + \bar{x}_2 + \bar{x}_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	1	1	0	1

$$(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	1	1	1	0

$$(\bar{x}_1 + x_2 + \bar{x}_3)$$

**Expressions with two variables
(for three-variable K-maps)**

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	1
	1	0	1	1	1

$$(x_1 + x_2)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	0	1	1
	1	1	0	1	1

$$(x_1 + \overline{x_2})$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	1
	1	1	1	0	1

$$(\overline{x_1} + \overline{x_2})$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	0
	1	1	1	1	0

$$(\overline{x_1} + x_2)$$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	1
	1	1	1	1	1

$$(x_1 + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	1
	1	1	1	1	1

$$(\bar{x}_2 + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	0
	1	1	1	1	1

$$(\bar{x}_1 + x_3)$$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	0
	1	1	1	1	1

$$(x_2 + x_3)$$

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	0	0	1	1

$$(\overline{x_1} + \overline{x_3})$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	1	0	0	1

$$(\overline{x_2} + \overline{x_3})$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	1	1	0	0

$$(\overline{x_1} + \overline{x_3})$$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	1	1
	1	0	1	1	0

$$(\overline{x_2} + \overline{x_3})$$

**Expressions with one variable
(for three-variable K-maps)**

Groupings and Expressions

		x_1x_2			
		00	01	11	10
x_3	0	0	0	1	1
	1	0	0	1	1

(x_1)

		x_1x_2			
		00	01	11	10
x_3	0	1	0	0	1
	1	1	0	0	1

$(\bar{x}_2)$

		x_1x_2			
		00	01	11	10
x_3	0	1	1	0	0
	1	1	1	0	0

$(\bar{x}_1)$

		x_1x_2			
		00	01	11	10
x_3	0	0	1	1	0
	1	0	1	1	0

(x_2)

Groupings and Expressions

x_1x_2		00	01	11	10
x_3	0	0	0	0	0
	1	1	1	1	1

(x_3)

x_1x_2		00	01	11	10
x_3	0	1	1	1	1
	1	0	0	0	0

$(\overline{x_3})$

**Expressions with zero variables
(for three-variable K-maps)**

Groupings and Expressions

$x_3 \backslash x_1x_2$		00	01	11	10
		0	1	1	0
0	0	0	0	0	0
1	0	0	0	0	0

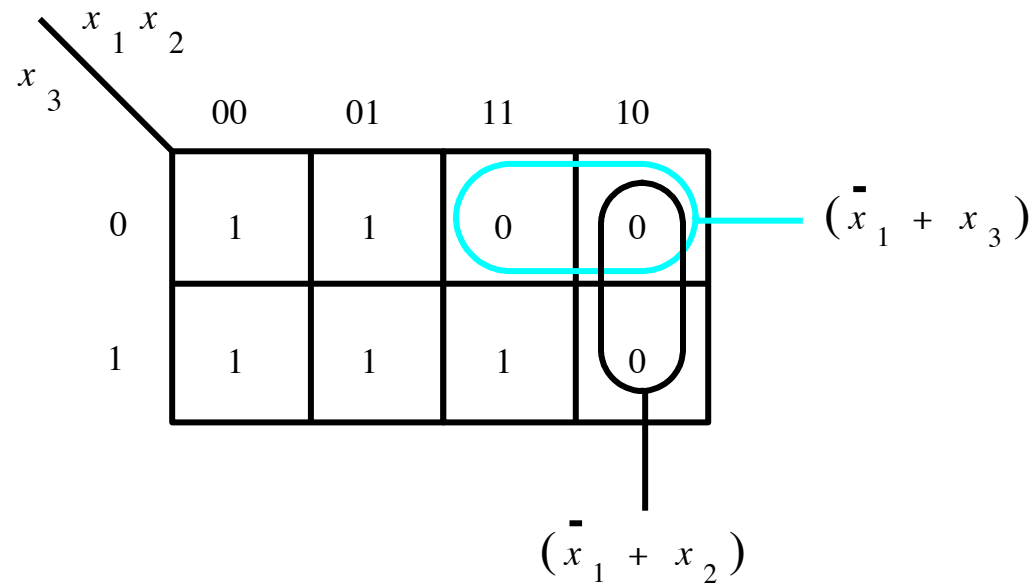
0

$x_3 \backslash x_1x_2$		00	01	11	10
		0	1	1	0
0	1	1	1	1	1
1	1	1	1	1	1

1

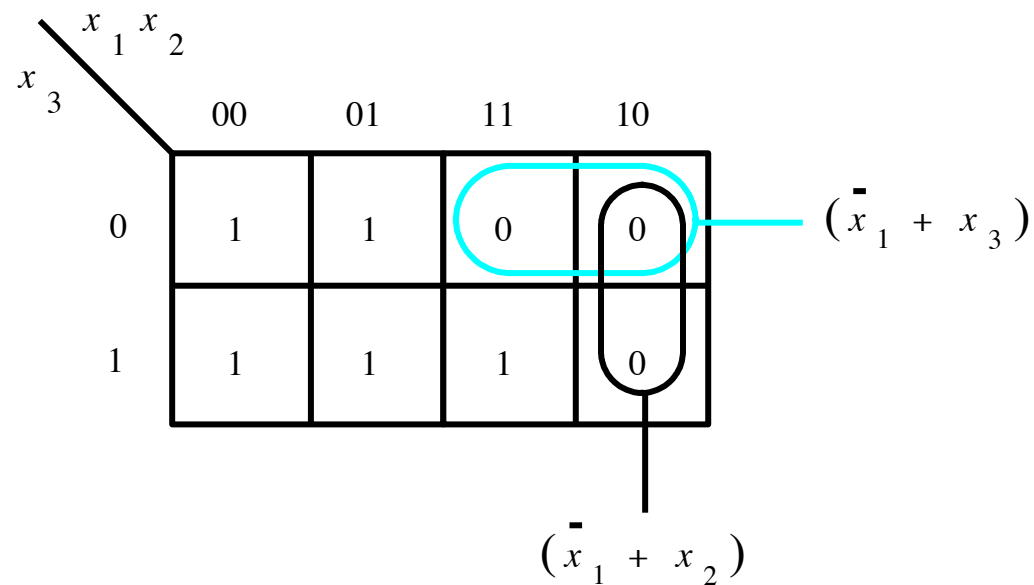
Some Examples

POS minimization of $f(x_1, x_2, x_3) = \prod M(4, 5, 6)$



[Figure 2.60 from the textbook]

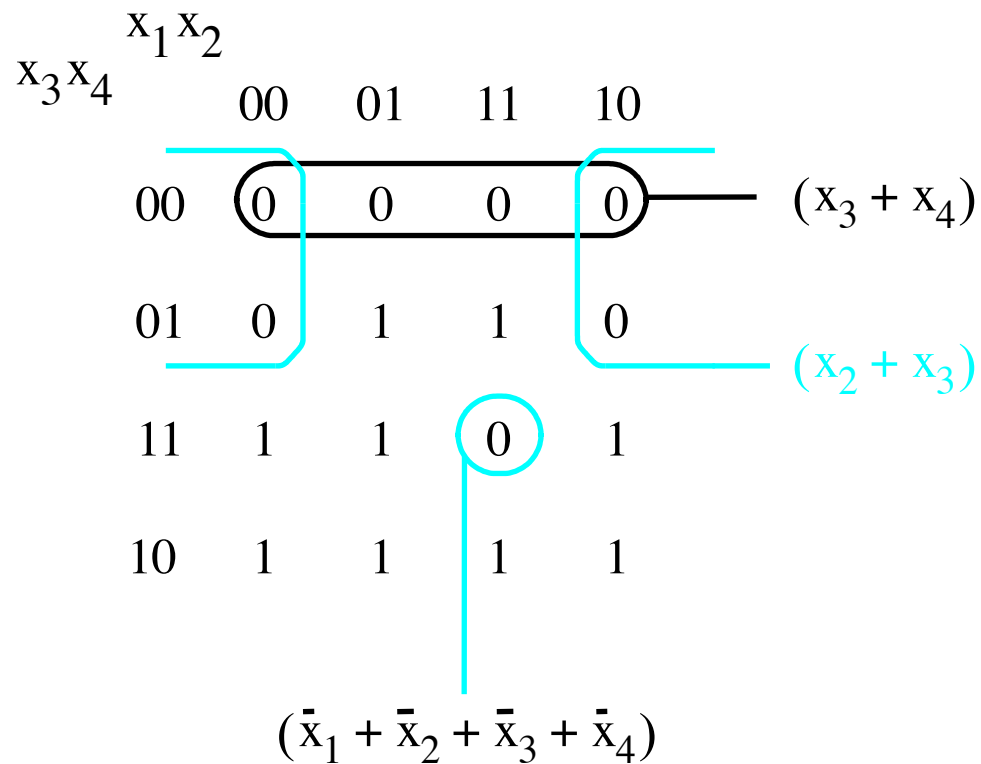
POS minimization of $f(x_1, x_2, x_3) = \prod M(4, 5, 6)$



$$f(x_1, x_2, x_3) = (\bar{x}_1 + x_3)(\bar{x}_1 + x_2)$$

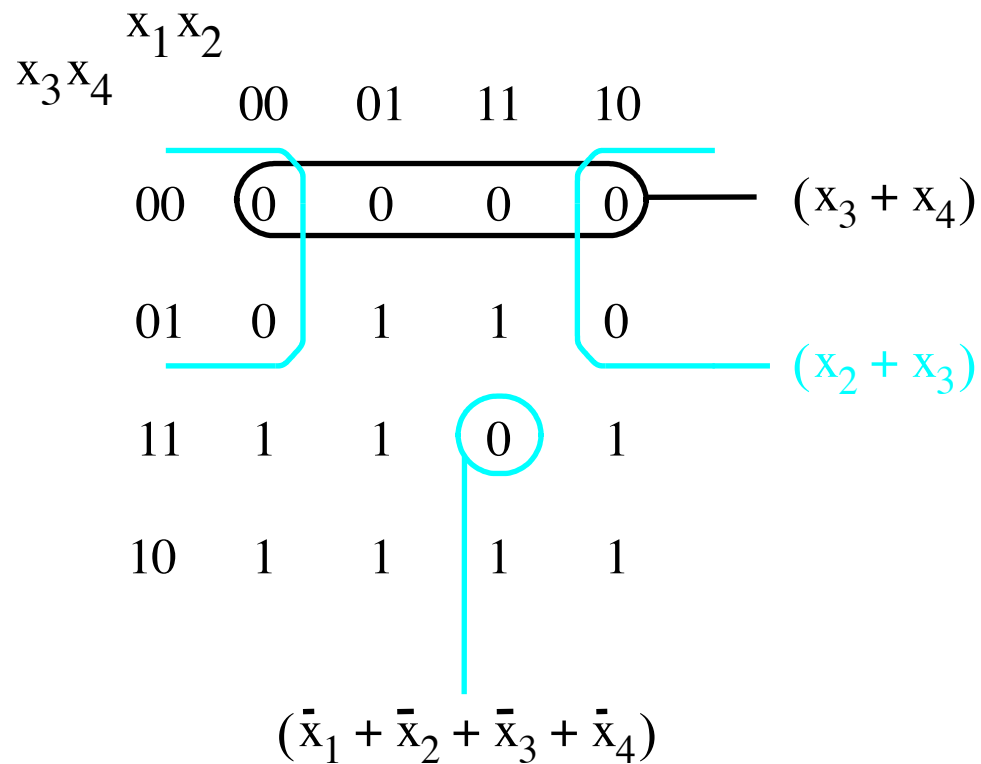
[Figure 2.60 from the textbook]

POS minimization of $f(x_1, \dots, x_4) = \prod M(0, 1, 4, 8, 9, 12, 15)$



[Figure 2.61 from the textbook]

POS minimization of $f(x_1, \dots, x_4) = \prod M(0, 1, 4, 8, 9, 12, 15)$



$$f(x_1, x_2, x_3, x_4) = (x_3 + x_4)(x_2 + x_3)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4)$$

[Figure 2.61 from the textbook]

Questions?

THE END