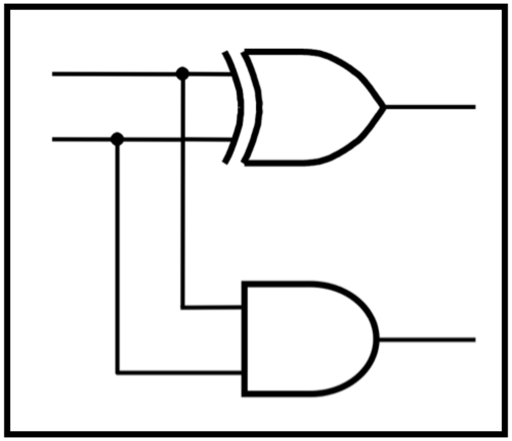


CprE 2810: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Signed Numbers

CprE 281: Digital Logic
Iowa State University, Ames, IA

Signed Integer Numbers

CprE 281: Digital Logic
Iowa State University, Ames, IA

Today's Lecture is About Addition and Subtraction of Signed Numbers

Quick Review

Signed v.s. Unsigned Numbers

Signed v.s. Unsigned Numbers

positive
and
negative
integers

only
positive
integers

Signed v.s. Unsigned Numbers

positive
and
negative
integers

and zero

only
positive
integers

and zero

Two Different Types of Binary Numbers

Unsigned numbers

- All bits jointly represent a positive integer.
- Negative numbers cannot be represented this way.

Signed numbers

- The left-most bit represents the sign of the number.
- If that bit is 0, then the number is positive.
- If that bit is 1, then the number is negative.
- The magnitude of the largest number that can be represented in this way is twice smaller than the largest number in the unsigned representation.

Two Different Types of Binary Numbers

Unsigned numbers

- All bits jointly represent a positive integer.
- Negative numbers cannot be represented this way.

Signed numbers

There are 3 different ways to represent signed numbers. They will be introduced today. But only the last method will be used later.

- The left-most bit represents the sign of the number.
- If that bit is 0, then the number is positive.
- If that bit is 1, then the number is negative.
- The magnitude of the largest number that can be represented in this way is twice smaller than the largest number in the unsigned representation.

Important Clarificaiton

Important Clarification: There are two types of addition

- **Addition of Boolean variables, e.g.,**

  $x + y$ where $x, y \in \{0, 1\}$

- **Addition of n-bit Binary numbers, e.g.,**

  $x_4x_3x_2x_1x_0 + y_4y_3y_2y_1y_0$ where each $x_k, y_k \in \{0, 1\}$

Important Clarification: There are two types of addition

- **Addition of Boolean variables, e.g.,**

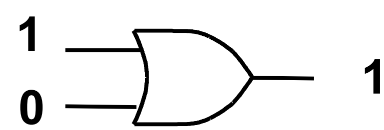
  1 + 0 = 1

- **Addition of n-bit Binary numbers, e.g.,**

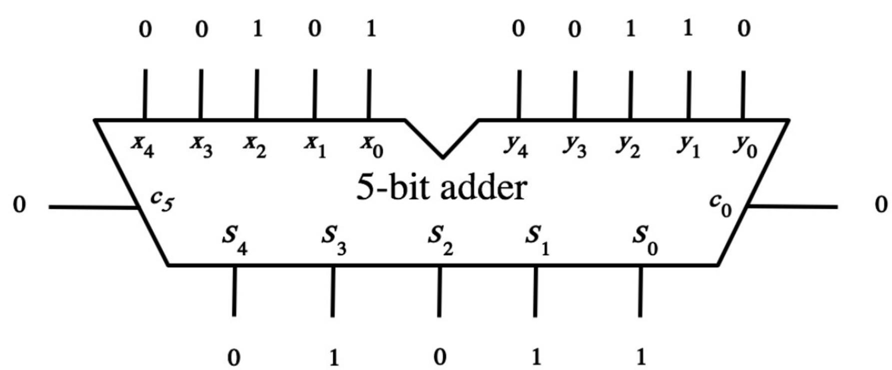
  00101 + 00110 = 01011

Important Clarification: There are two types of addition

- **Addition of two Boolean variables, e.g.,**

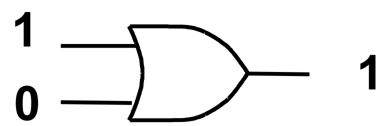


- **Addition of two 5-bit Binary numbers, e.g.,**

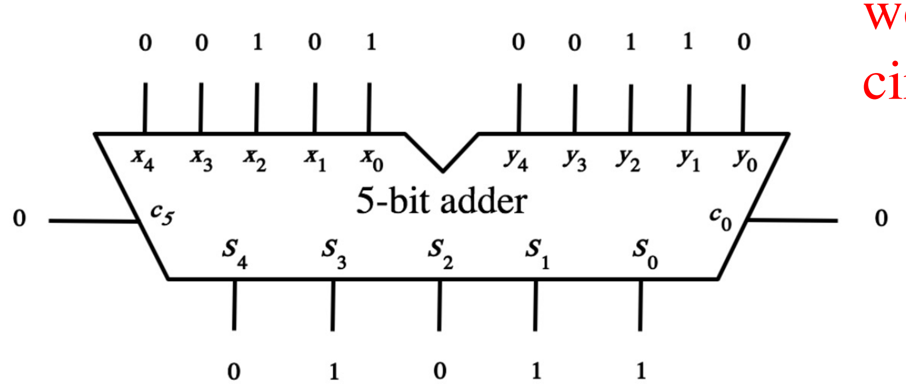


Important Clarification: There are two types of addition

- **Addition of two Boolean variables, e.g.,**



- **Addition of two 5-bit Binary numbers, e.g.,**



we derived this circuit last time

Important Clarification: There are two types of addition

- Addition of two Boolean variables, e.g.,

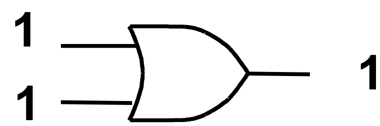
  1 + 1 = 1 (according to the rules of Boolean algebra)

- Addition of two 1-bit Binary numbers, e.g.,

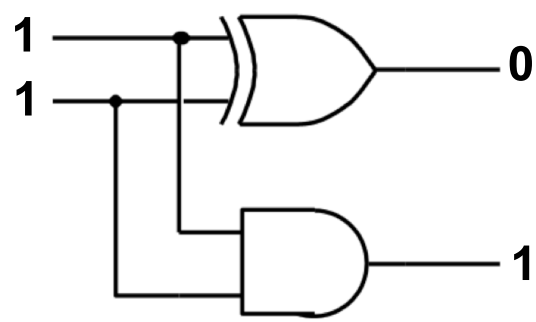
  1 + 1 = 10 (because in decimal 1 + 1 = 2)

Important Clarification: There are two types of addition

- **Addition of two Boolean variables, e.g.,**



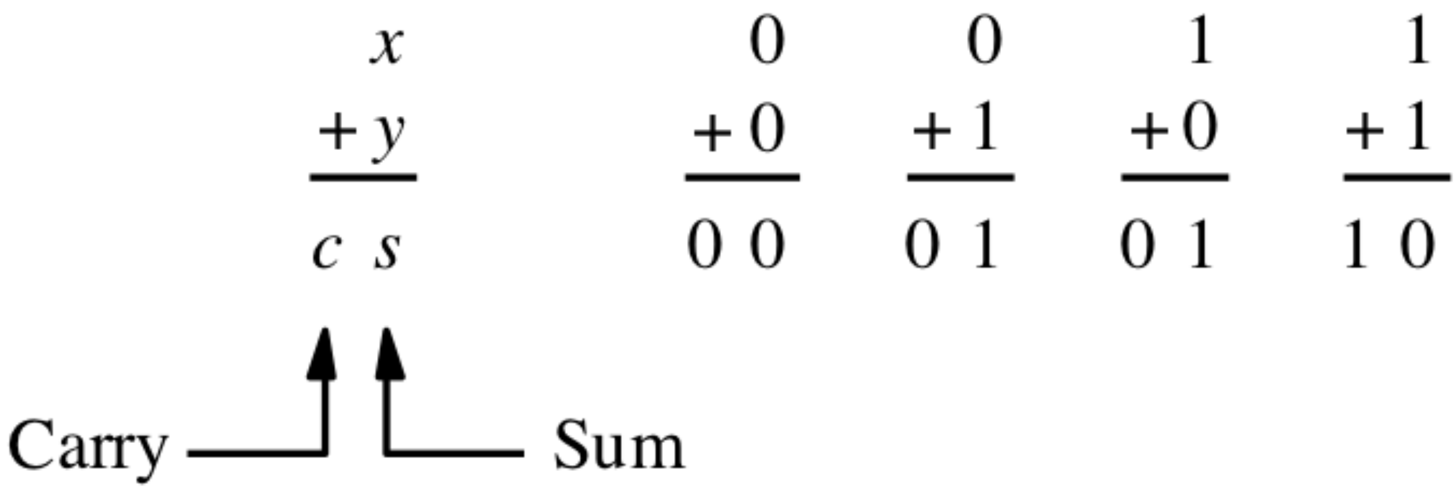
- **Addition of two 1-bit Binary numbers, e.g.,**



In this case, the adder circuit simplifies to the half-adder.

Addition of 1-bit Unsigned Numbers

Addition of two 1-bit numbers
(there are four possible cases)



[Figure 3.1a from the textbook]

Addition of two 1-bit numbers
(there are four possible cases)

$$\begin{array}{r} x \\ +\,y \\ \hline c\ \ s \end{array} \qquad\qquad \begin{array}{r} 0 \\ +\,0 \\ \hline \underbrace{0\ \ 0}_{0_{10}} \end{array} \qquad \begin{array}{r} 0 \\ +\,1 \\ \hline \underbrace{0\ \ 1}_{1_{10}} \end{array} \qquad \begin{array}{r} 1 \\ +\,0 \\ \hline \underbrace{0\ \ 1}_{1_{10}} \end{array} \qquad \begin{array}{r} 1 \\ +\,1 \\ \hline \underbrace{1\ \ 0}_{2_{10}} \end{array}$$

Carry Sum

[Figure 3.1a from the textbook]

Adding two bits
(the truth table)

x	y	Carry c	Sum s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

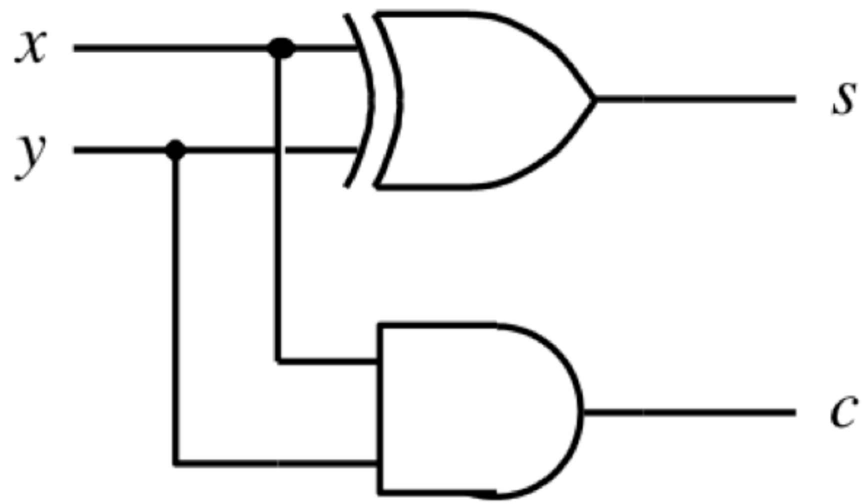
[Figure 3.1b from the textbook]

Adding two bits
(the truth table)

x		y		Carry c	Sum s	
0	+	0	=	0	0	$= 0_{10}$
0	+	1	=	0	1	$= 1_{10}$
1	+	0	=	0	1	$= 1_{10}$
1	+	1	=	1	0	$= 2_{10}$

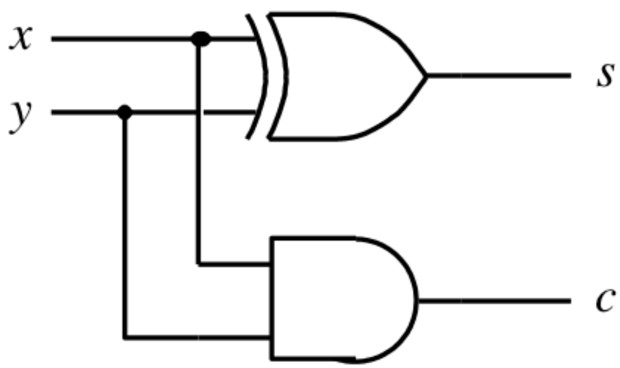
[Figure 3.1b from the textbook]

Adding two bits
(the logic circuit)

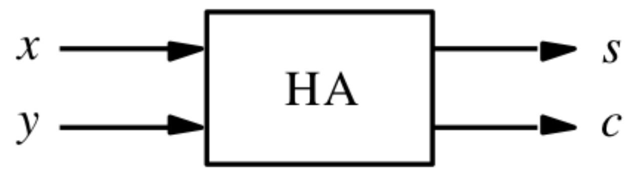


[Figure 3.1c from the textbook]

The Half-Adder



(c) Circuit



(d) Graphical symbol

[Figure 3.1c-d from the textbook]

Addition of Multibit Unsigned Numbers

Addition of multibit numbers

Generated carries $\longrightarrow$ 1 1 1 0

$$
\begin{array}{rcl}
X &=& x_4x_3x_2x_1x_0 \\
+\,Y &=& y_4y_3y_2y_1y_0 \\
\hline
S &=& s_4s_3s_2s_1s_0
\end{array}
\qquad
\begin{array}{r}
0\,1\,1\,1\,1 \\
+\,0\,1\,0\,1\,0 \\
\hline
1\,1\,0\,0\,1
\end{array}
\qquad
\begin{array}{r}
(15)_{10} \\
+\,(10)_{10} \\
\hline
(25)_{10}
\end{array}
$$

$$
\begin{array}{ccccc}
\cdots & c_{i+1} & c_i & \cdots \\
\cdots & \cdots & x_i & \cdots \\
\cdots & \cdots & y_i & \cdots \\
\hline
\cdots & \cdots & s_i & \cdots
\end{array}
$$

Bit position *i*

[Figure 3.2 from the textbook]

Analogy with addition in base 10

$$\begin{array}{r} x_2 \; x_1 \; x_0 \\ + \quad y_2 \; y_1 \; y_0 \\ \hline s_2 \; s_1 \; s_0 \end{array}$$

Analogy with addition in base 10

$$\begin{array}{rccc} & 3 & 8 & 9 \\ + & 1 & 5 & 7 \\ \hline & 5 & 4 & 6 \end{array}$$

Analogy with addition in base 10

carry	0	1	1	0
		3	8	9
+		1	5	7
		5	4	6

Analogy with addition in base 10

$$
\begin{array}{rcccc}
 & c_3 & c_2 & c_1 & c_0 \\
 & & x_2 & x_1 & x_0 \\
+ & & y_2 & y_1 & y_0 \\
\hline
 & & s_2 & s_1 & s_0
\end{array}
$$

Another example in base 10

$$
\begin{array}{r}
9\ 3\ 8 \\
+\quad 2\ 1\ 4 \\
\hline
1\ 1\ 5\ 2
\end{array}
$$

Another example in base 10

carry	1	0	1	0
		9	3	8
+		2	1	4
		1	5	2

Example in base 2

$$
\begin{array}{r}
1\ 0\ 1 \\
+\quad 1\ 1\ 0 \\
\hline
\end{array}
$$

Example in base 2

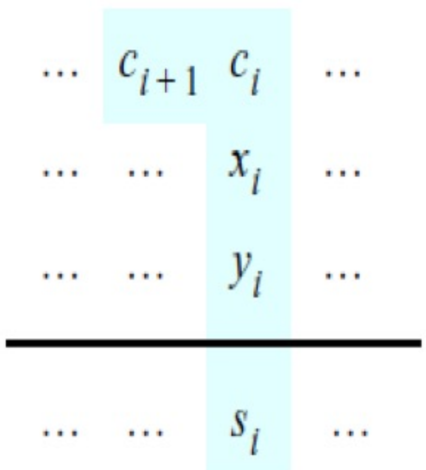
carry		1	0	0	0
			1	0	1
+			1	1	0
		1	0	1	1

Example in base 2

carry		1	0	0	0
			1	0	1
+			1	1	0
		1	0	1	1

$$
\begin{array}{r}
5_{10} \\
+\ 6_{10} \\
\hline
11_{10}
\end{array}
$$

Problem Statement and Truth Table

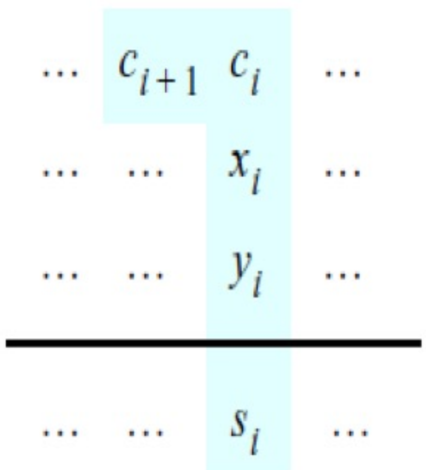


[Figure 3.2b from the textbook]

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

[Figure 3.3a from the textbook]

Problem Statement and Truth Table



[Figure 3.2b from the textbook]

c_i x_i y_i		c_{i+1}	s_i	
$0 + 0 + 0$	=	0	0	$= 0_{10}$
$0 + 0 + 1$	=	0	1	$= 1_{10}$
$0 + 1 + 0$	=	0	1	$= 1_{10}$
$0 + 1 + 1$	=	1	0	$= 2_{10}$
$1 + 0 + 0$	=	0	1	$= 1_{10}$
$1 + 0 + 1$	=	1	0	$= 2_{10}$
$1 + 1 + 0$	=	1	0	$= 2_{10}$
$1 + 1 + 1$	=	1	1	$= 3_{10}$

[Figure 3.3a from the textbook]

Let’s fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

c_i \ $x_i y_i$	00	01	11	10
0				
1				

$s_i =$

c_i \ $x_i y_i$	00	01	11	10
0				
1				

$c_{i+1} =$

[Figure 3.3a-b from the textbook]

Let's fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

c_i \ $x_i y_i$	00	01	11	10
0		1		1
1	1		1	

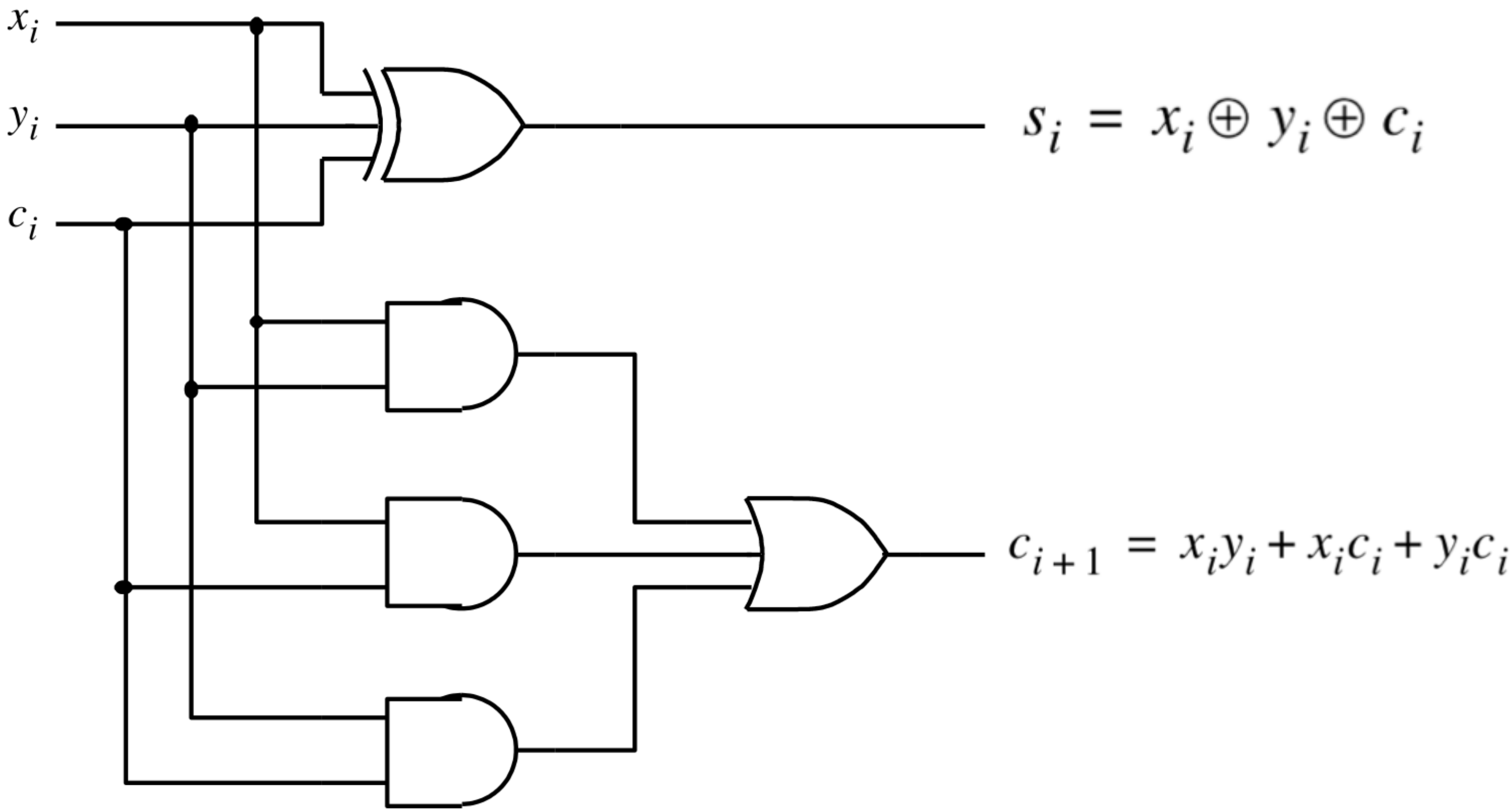
$$s_i = x_i \oplus y_i \oplus c_i$$

c_i \ $x_i y_i$	00	01	11	10
0			1	
1		1	1	1

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

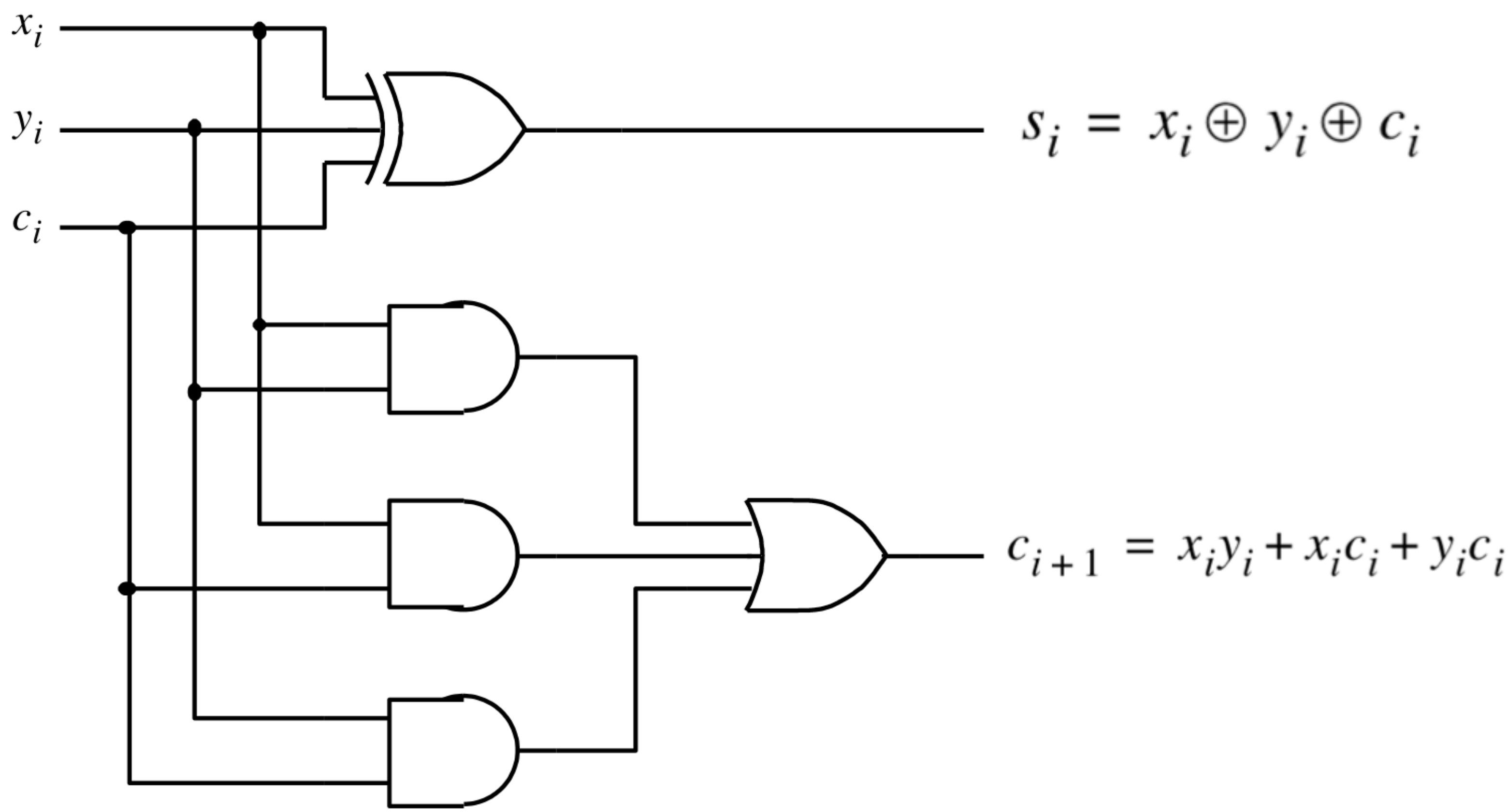
[Figure 3.3a-b from the textbook]

The circuit for the two expressions



[Figure 3.3c from the textbook]

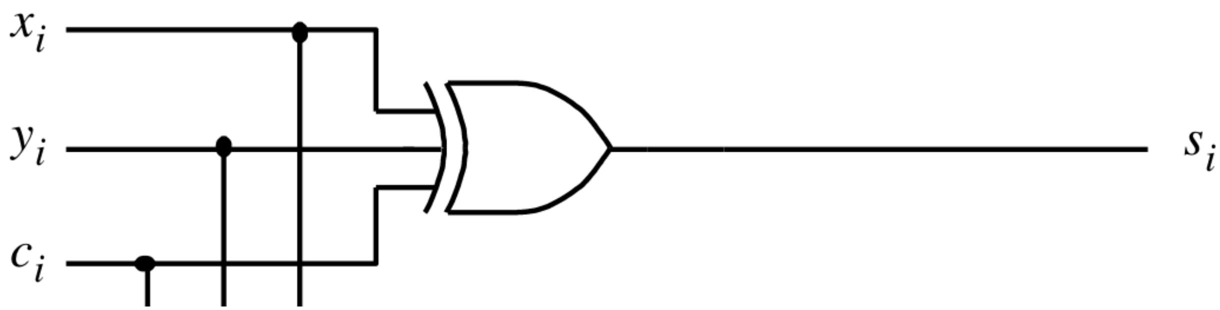
This is called the Full-Adder

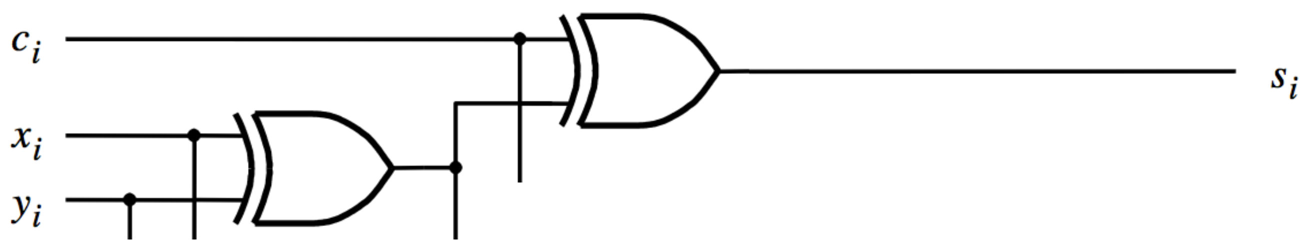


[Figure 3.3c from the textbook]

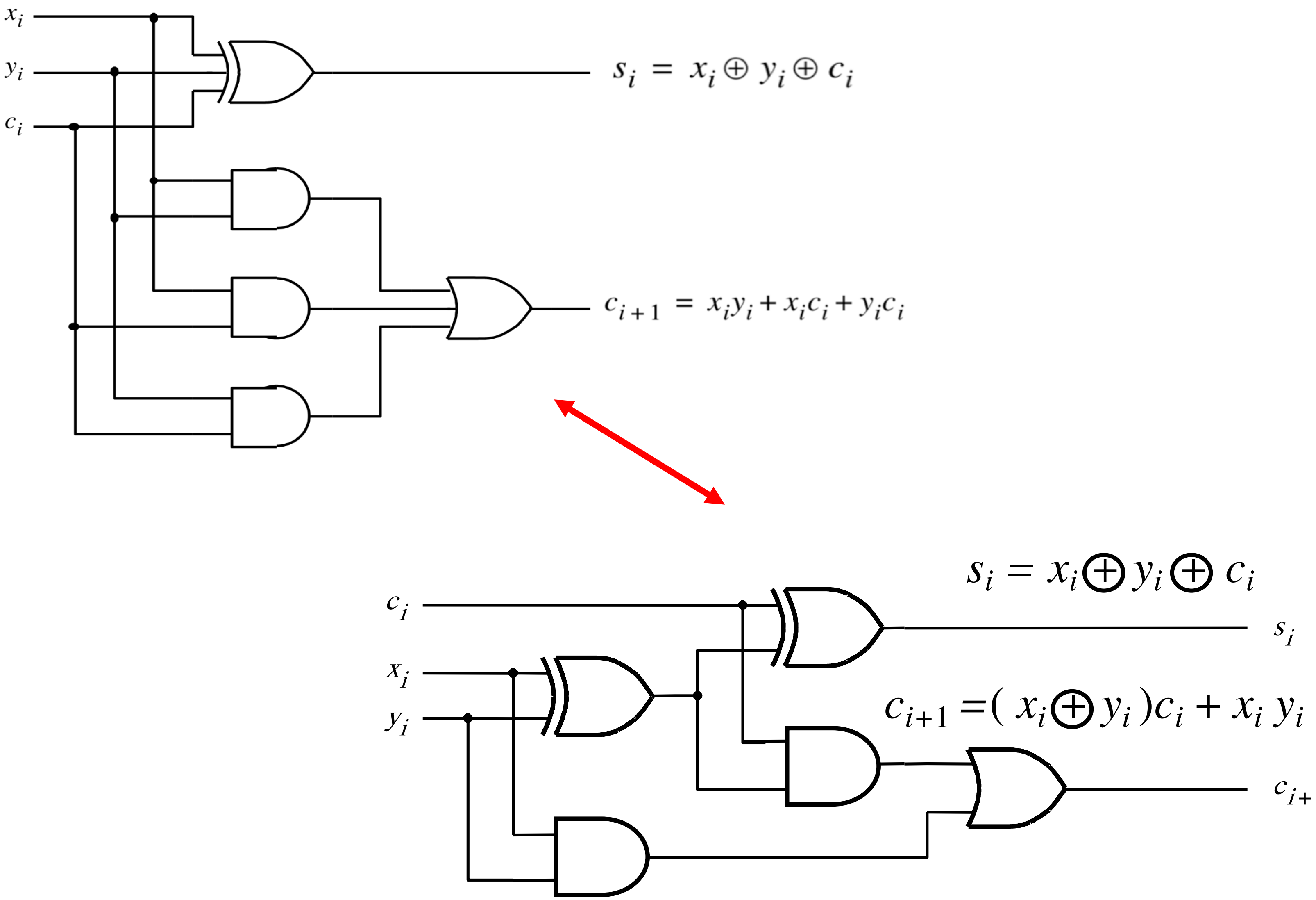
XOR Magic
(s_i can be implemented in two different ways)

$$s_i = x_i \oplus y_i \oplus c_i$$

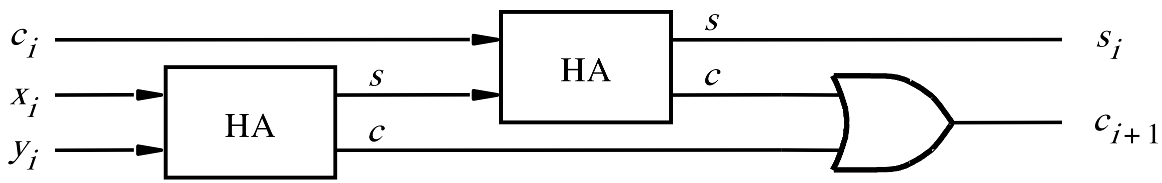




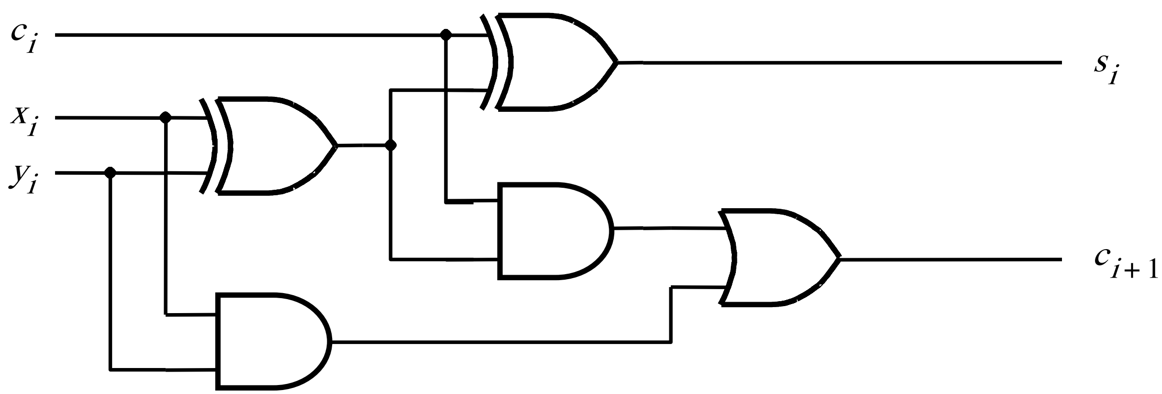
These two circuits are equivalent



A decomposed implementation of the full-adder circuit



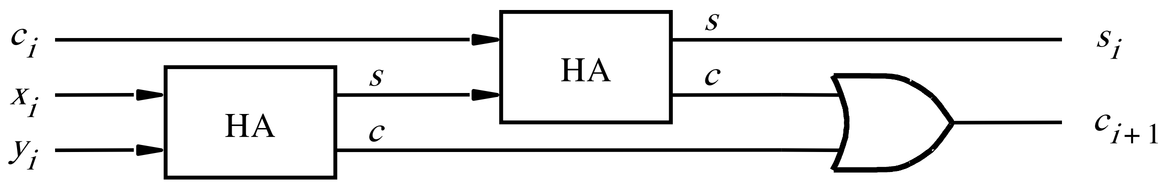
(a) Block diagram



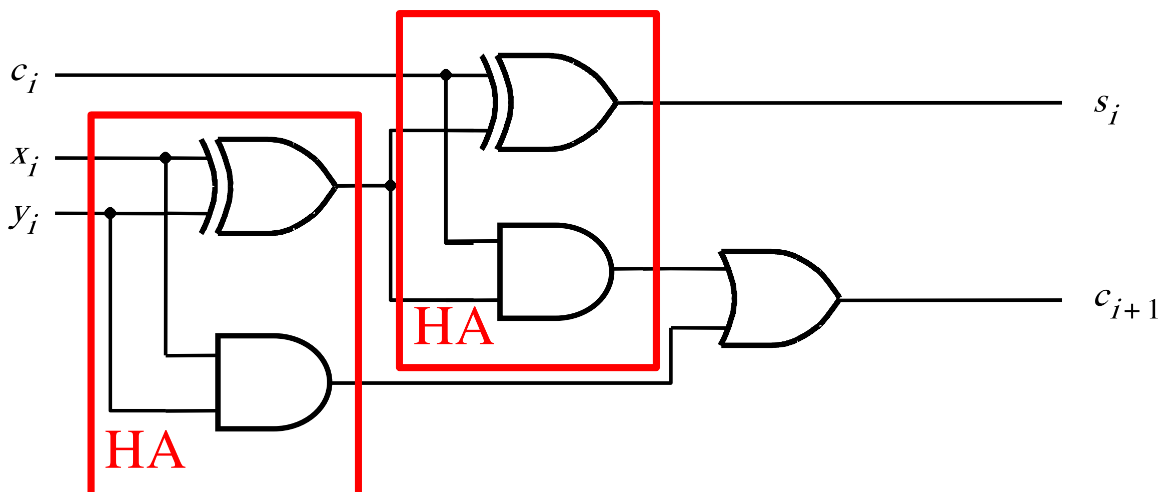
(b) Detailed diagram

[Figure 3.4 from the textbook]

A decomposed implementation of the full-adder circuit



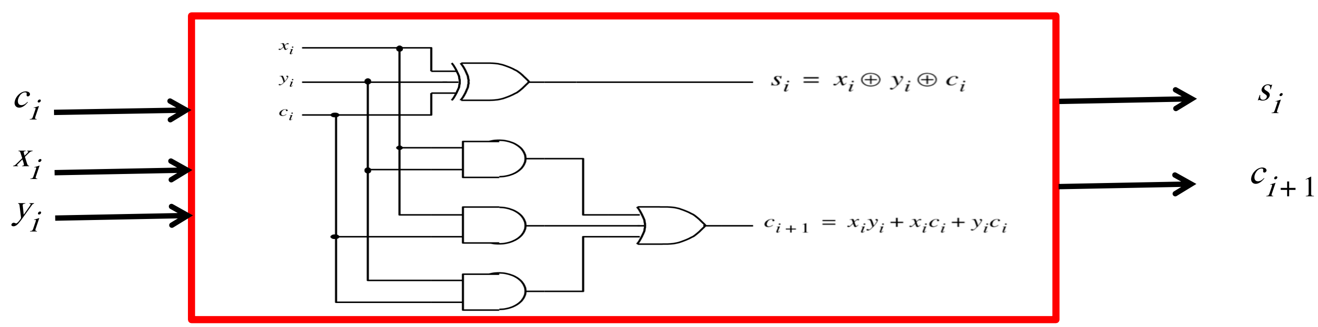
(a) Block diagram



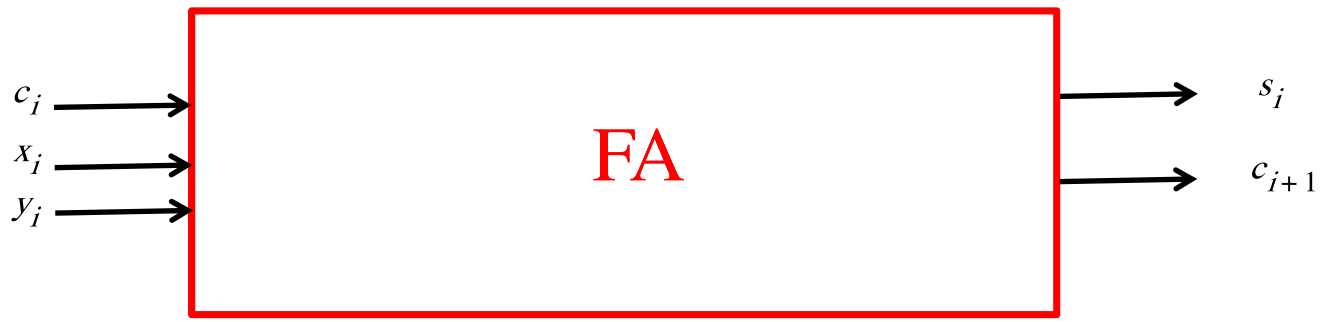
(b) Detailed diagram

[Figure 3.4 from the textbook]

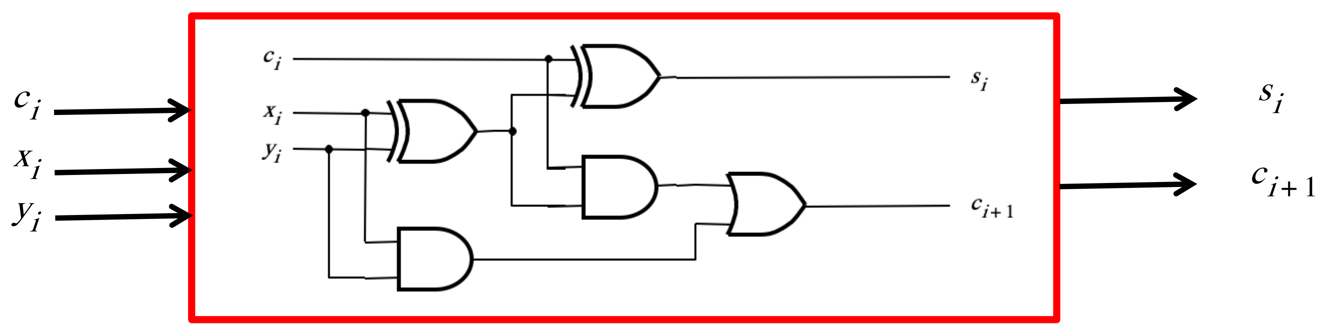
The Full-Adder Abstraction



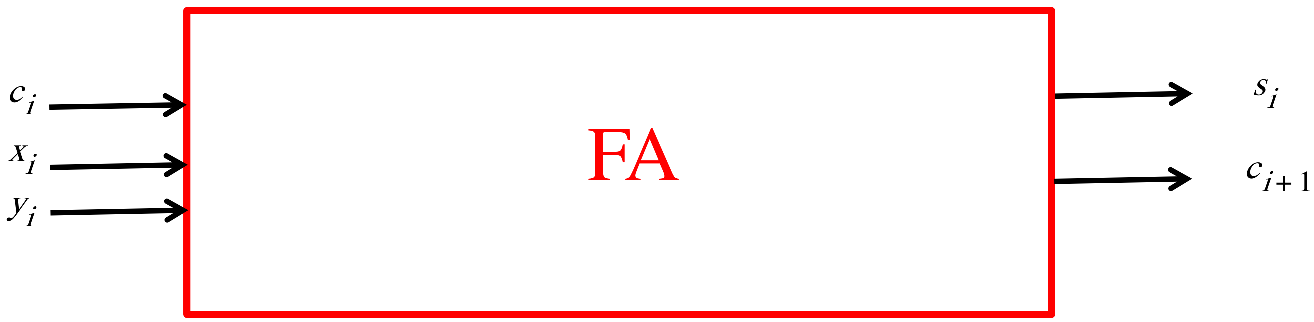
The Full-Adder Abstraction



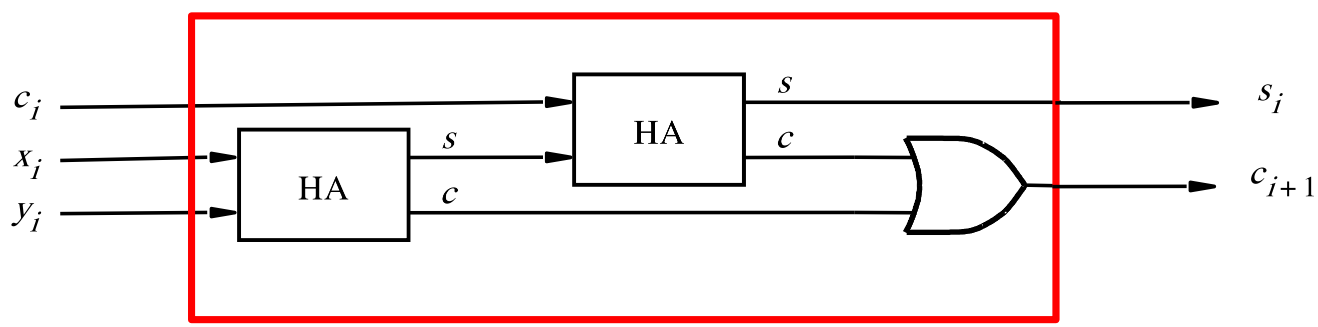
The Full-Adder Abstraction



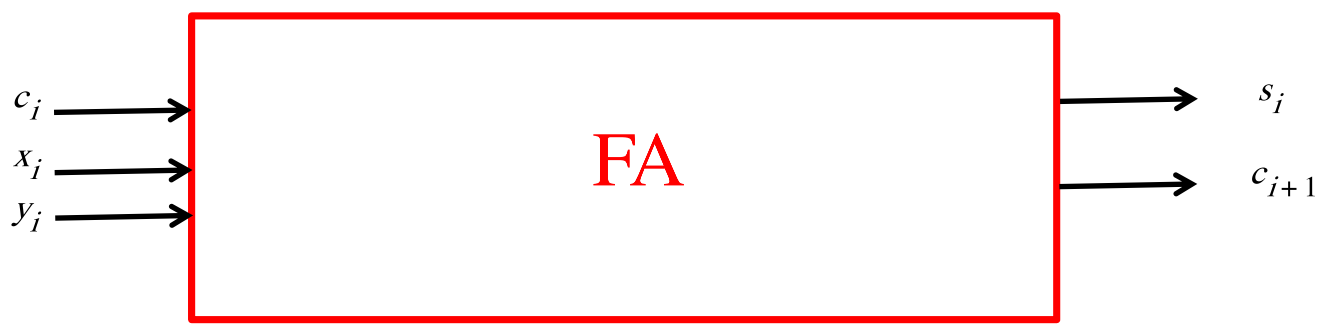
The Full-Adder Abstraction



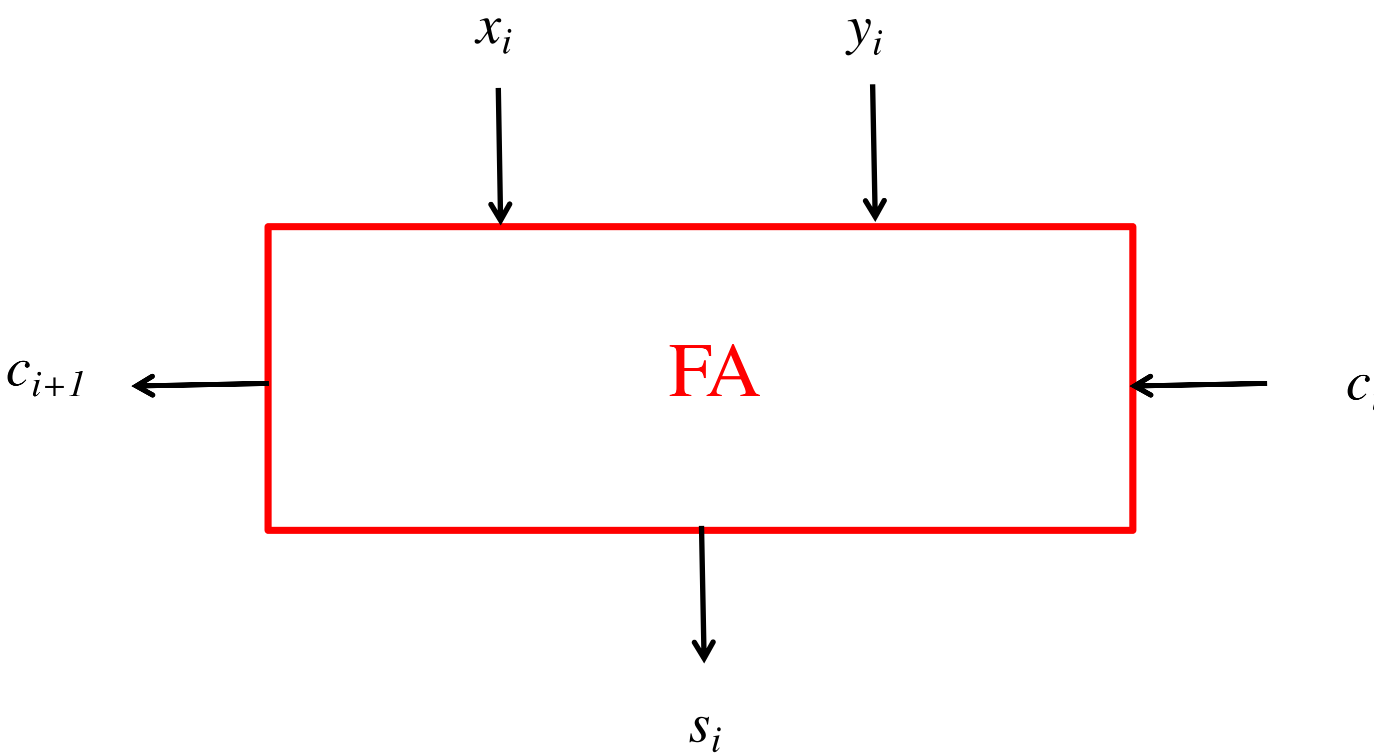
The Full-Adder Abstraction



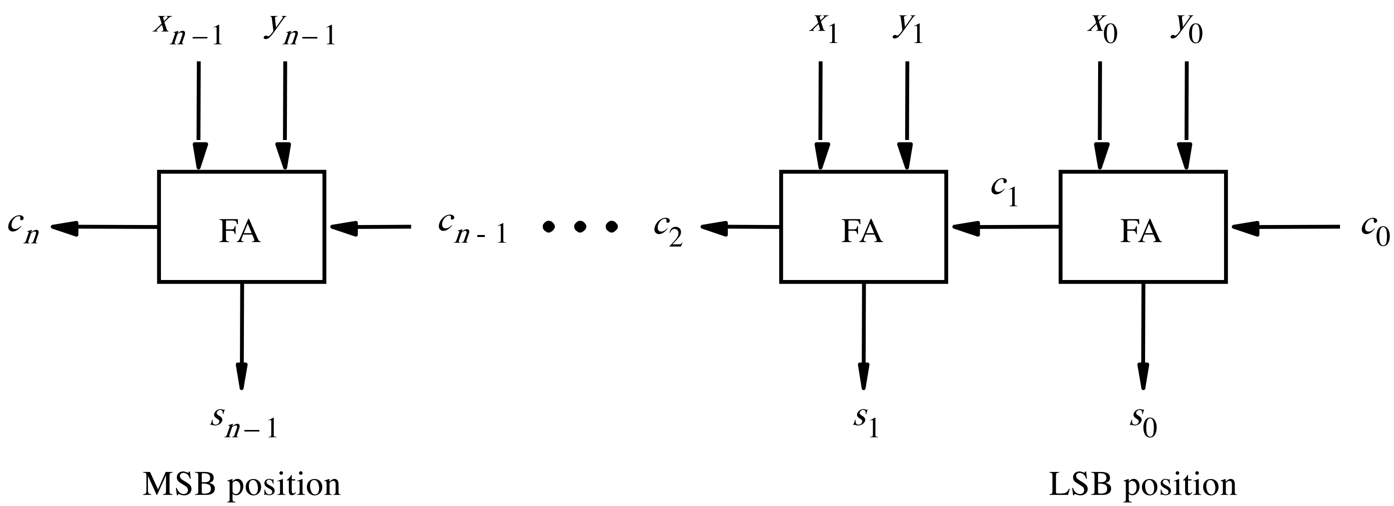
The Full-Adder Abstraction



We can place the arrows anywhere

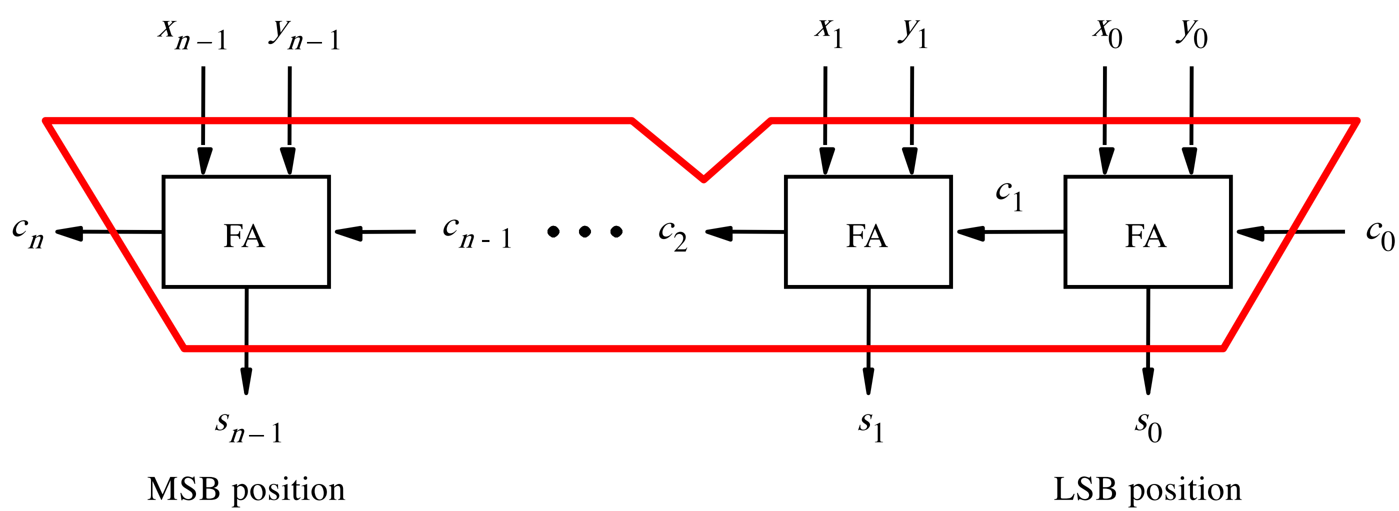


n-bit ripple-carry adder

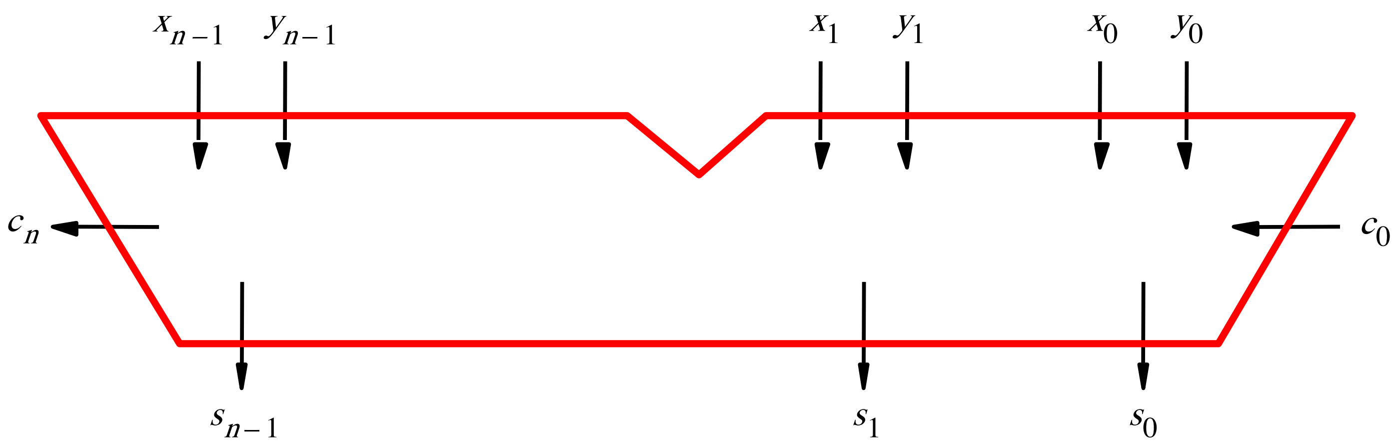


[Figure 3.5 from the textbook]

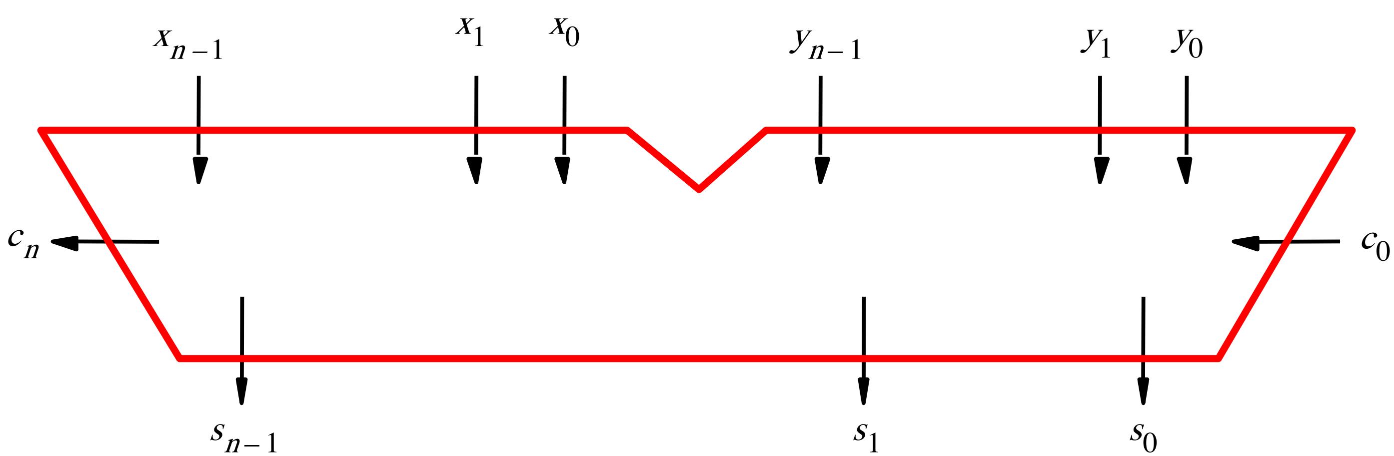
n-bit ripple-carry adder abstraction



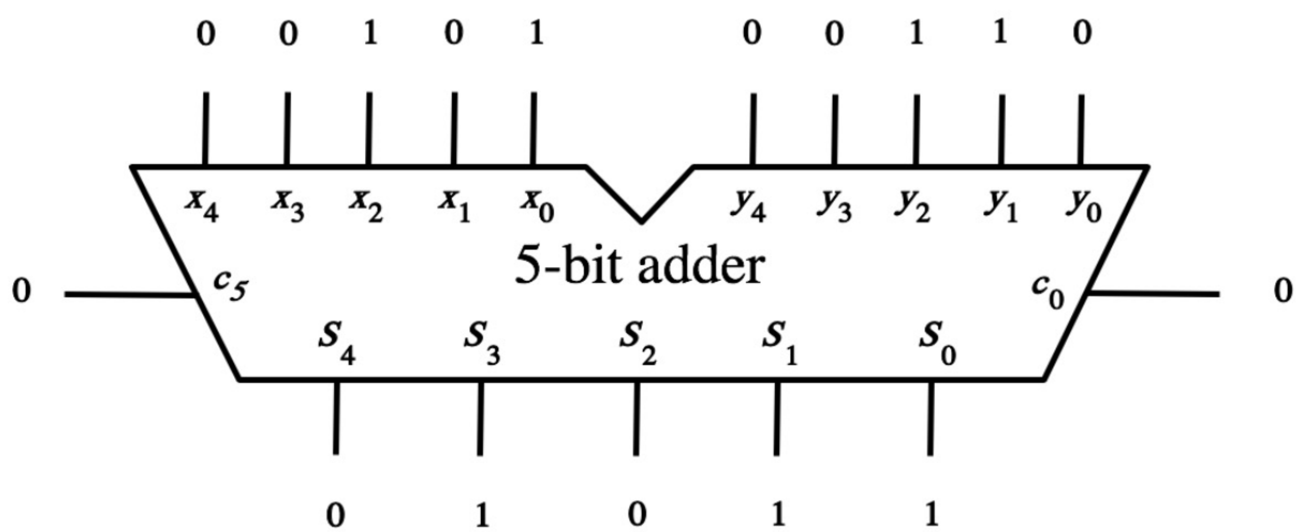
n-bit ripple-carry adder abstraction



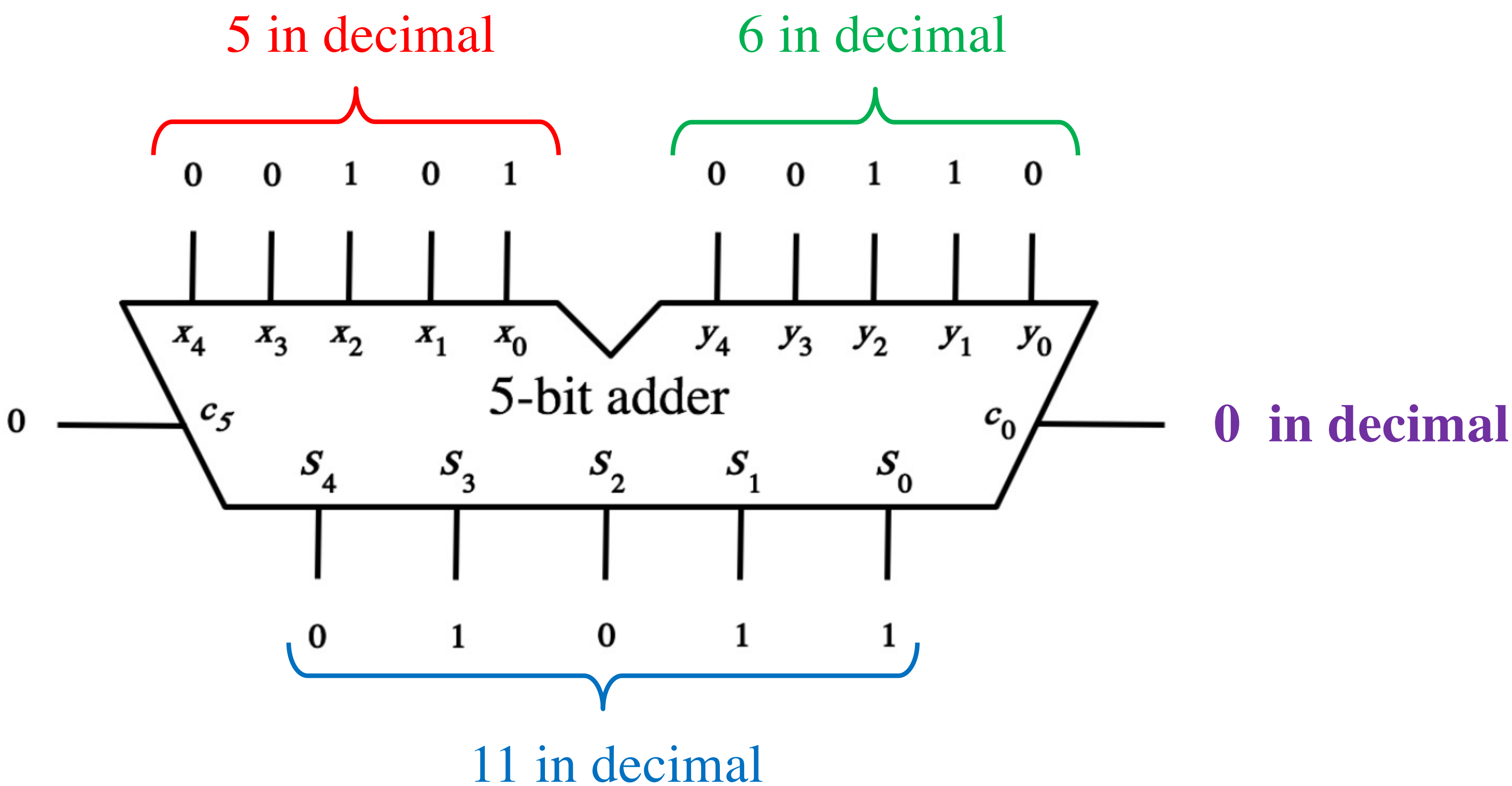
The x and y lines are typically grouped together for better visualization, but the underlying logic remains the same



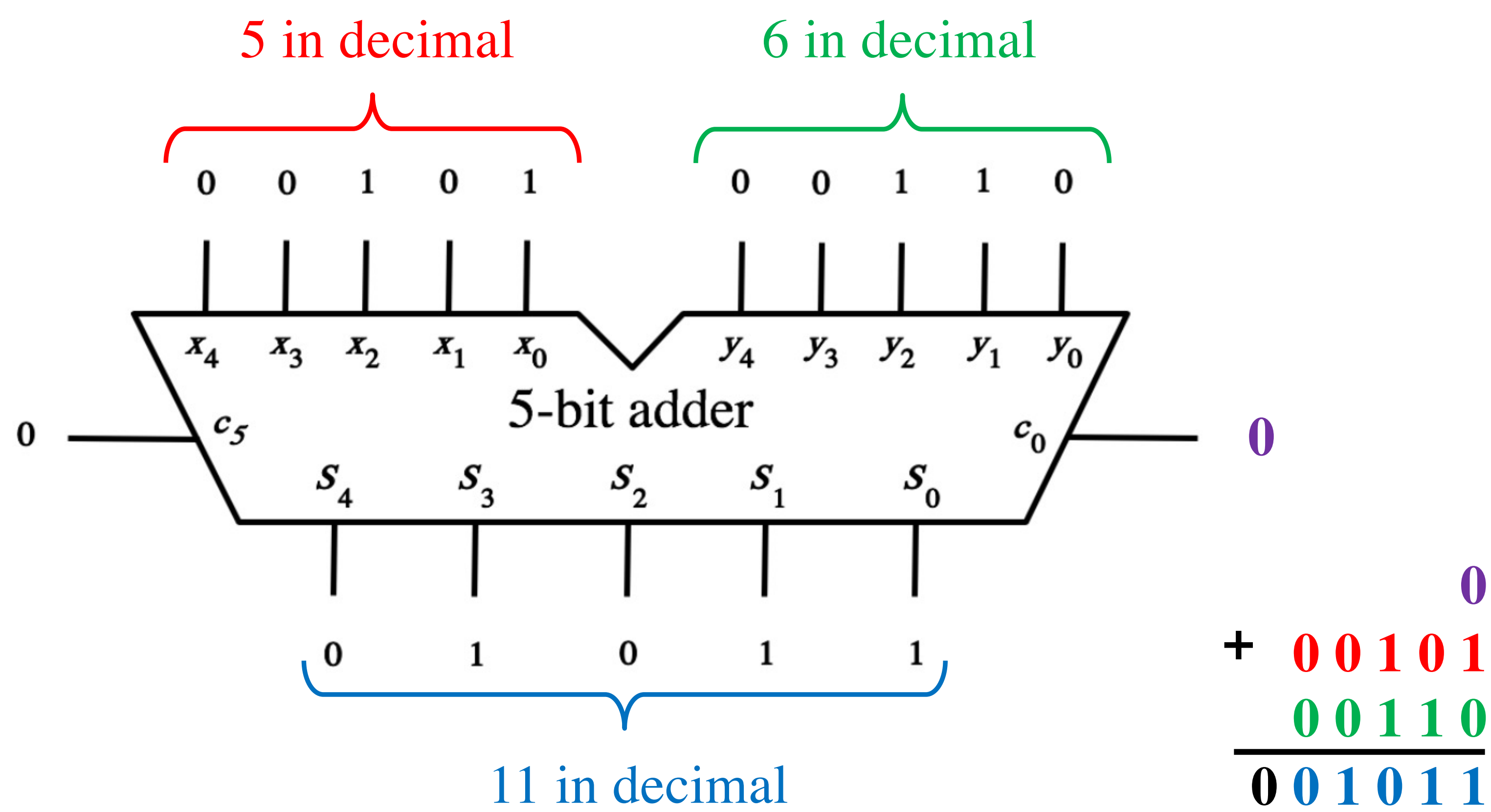
Example:
Computing 5+6 using a 5-bit adder



Example:
Computing 5+6 using a 5-bit adder



Example:
Computing 5+6 using a 5-bit adder



Math Review

Math Review: Subtraction

$$\begin{array}{r} 39 \\ -\ 15 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 39 \\ -\ 15 \\ \hline 24 \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline 21 \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline 22 \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline 21 \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline 18 \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline 19 \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline 19 \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline ?? \end{array}$$

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline ?? \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline 21 \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline 22 \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline 21 \end{array}$$

Why?

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline 18 \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline 19 \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline 19 \end{array}$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

Another Way to Do Subtraction

$$82 – 64 = 82 + 100 – 100 - 64$$

$$= 82 + (100 – 64) - 100$$

Another Way to Do Subtraction

$$
\begin{aligned}
82 - 64 &= 82 + 100 - 100 - 64 \\
&= 82 + (100 - 64) - 100 \\
&= 82 + (99 + 1 - 64) - 100
\end{aligned}
$$

Another Way to Do Subtraction

$$82 – 64 = 82 + 100 – 100 - 64$$

$$= 82 + (100 – 64) - 100$$

$$= 82 + (99 + 1 – 64) - 100$$

$$= 82 + (99 – 64) +1 - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

Does not require borrows

$$= 82 + (99 - 64) + 1 - 100$$

9’s Complement
(subtract each digit from 9)

$$\begin{array}{r} 99 \\ -\ 64 \\ \hline 35 \end{array}$$

10’s Complement

(subtract each digit from 9 and add 1 to the result)

$$
\begin{array}{r}
99 \\
-\ 64 \\
\hline
35
\end{array}
+ 1 = 36
$$

Another Way to Do Subtraction

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

$$= 118 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

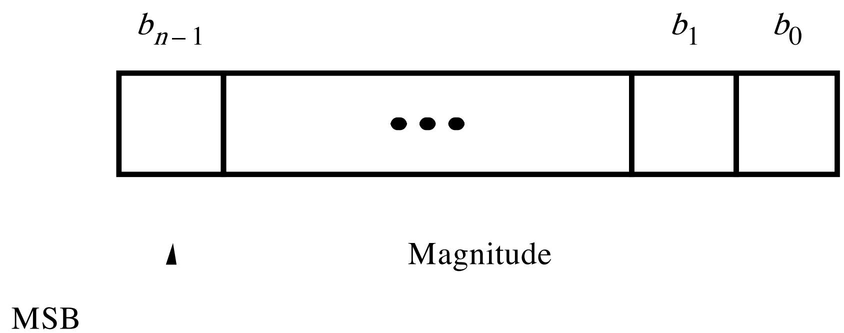
$$= 118 - 100$$ // Just delete the leading 1.

// No need to subtract 100.

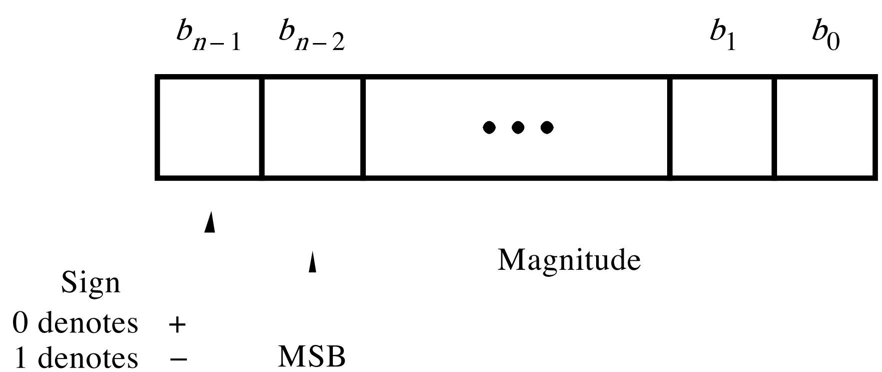
$$= 18$$

Ways to Represent Negative Integers

Formats for representation of integers



(a) Unsigned number



(b) Signed number

[Figure 3.7 from the textbook]

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

This represents + 44.

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

This represents + 172.

Three Different Ways to Represent Negative Integer Numbers

- Sign and magnitude
- 1's complement
- 2's complement

Three Different Ways to Represent Negative Integer Numbers

- **Sign and magnitude**
- **1's complement**
- **2's complement**

only this method is used in modern computers

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

[Table 3.2 from the textbook]

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

The top half is the same in all three representations.
It corresponds to the positive integers.

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

In all three representations the first bit represents the sign.
If that bit is 1, then the number is negative.

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

Notice that in this representation there are two zeros!

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

There are two zeros in this representation as well!

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

In this representation there is one more negative number.

Sign and Magnitude

Sign and Magnitude Representation
(using the left-most bit as the sign)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

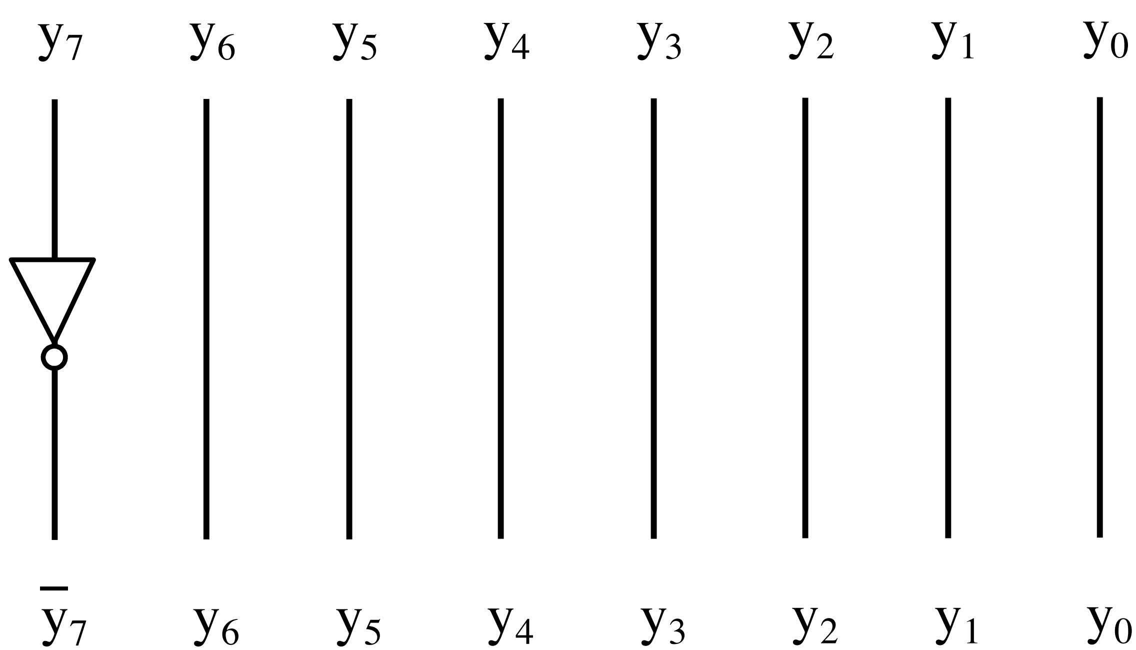
This represents + 44.

Sign and Magnitude Representation
(using the left-most bit as the sign)

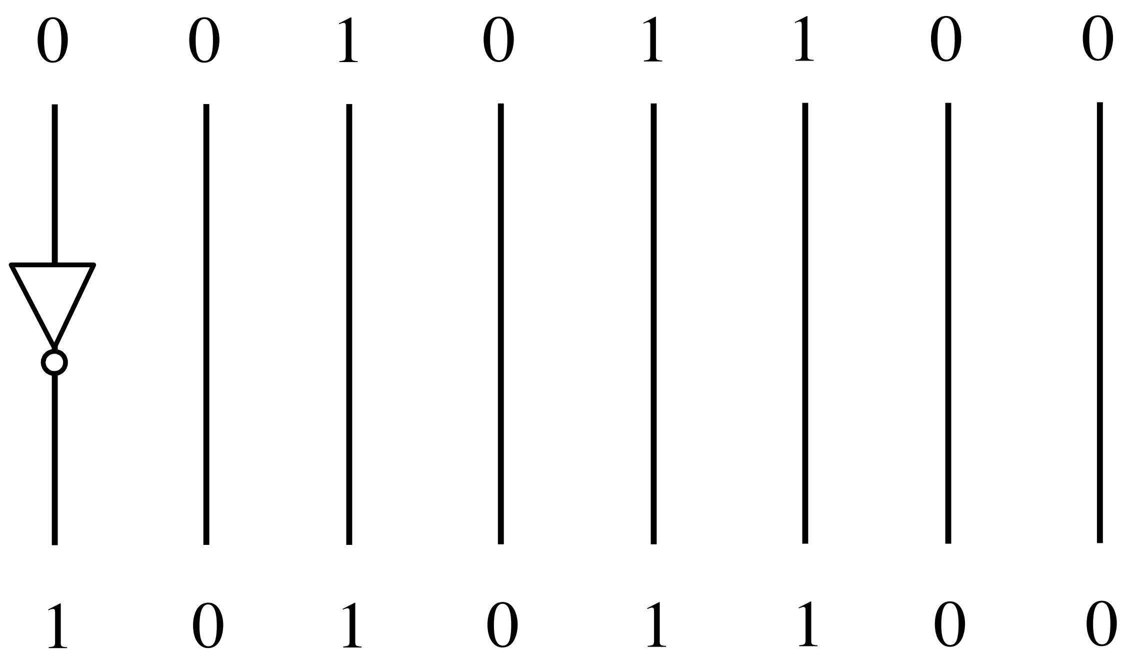
sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

This represents – 44.

Circuit for negating a number stored in sign and magnitude representation



Circuit for negating a number stored in sign and magnitude representation



1’s Complement

1's complement
(subtract each digit from 1)

Let K be the negative equivalent of an n-bit positive number P.

Then, in 1's complement representation K is obtained by subtracting P from $2^n - 1$, namely

$$K = (2^n - 1) - P$$

This means that K can be obtained by inverting all bits of P.

1's complement
(subtract each digit from 1)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 1's complement representation K is obtained by subtracting P from $2^8 - 1$, namely

$$K = (2^8 - 1) - P = 255 - P$$

This means that K can be obtained by inverting all bits of P.

Provided that P is between 0 and 127, because the most significant bit must be zero to indicate that it is positive.

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

1’s Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

$$2^5 + 2^3 + 2^2 = 44$$

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

+ 44 in 1's complement representation

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44
positive								

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44
negative								

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

$2^7 + 2^6 + 2^4 + 2^1 + 2^0 = 211$ (as unsigned)

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

211 = 255 – 44 (as unsigned)

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

$\underbrace{11010011}_{\text{- 44 in 1's complement representation}}$

1's complement
(subtract each digit from 1)

	1	1	1	1	1	1	1	1
−	0	0	1	0	1	1	0	0
	1	1	0	1	0	0	1	1

1's complement
(subtract each digit from 1)

No need to borrow!

$$
\begin{array}{cccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}
$$

1's complement
(subtract each digit from 1)

$$
\begin{array}{r l}
 & \boxed{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1} \quad 255 \\
- & \boxed{0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0} \quad 44 \\
\hline
 & \;\,1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1
\end{array}
$$

1's complement
(subtract each digit from 1)

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & & & & & & & & \\
 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\hline
 & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1}
\end{array}
\quad 211
$$

211 = 255 – 44 (as unsigned)

1’s complement
(subtract each digit from 1)

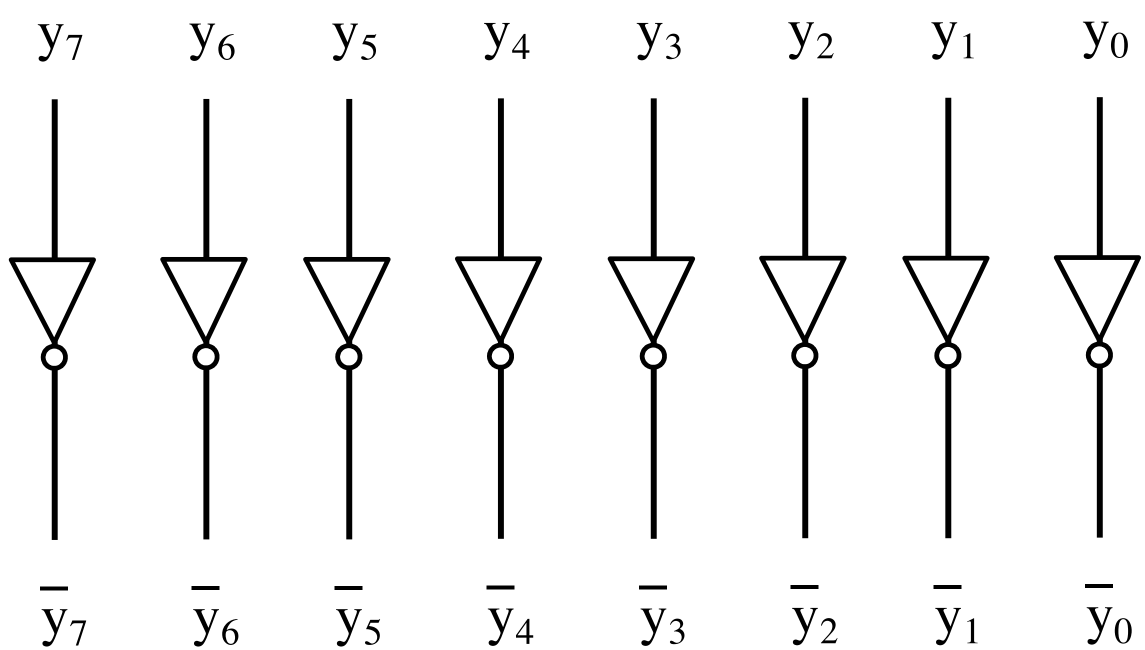
$$\begin{array}{r} 1\;1\;1\;1\;1\;1\;1\;1 \\ -\;0\;0\;1\;0\;1\;1\;0\;0 \\ \hline \boxed{1\;1\;0\;1\;0\;0\;1\;1} \end{array} \quad -44$$

211 = 255 – 44 (as unsigned)

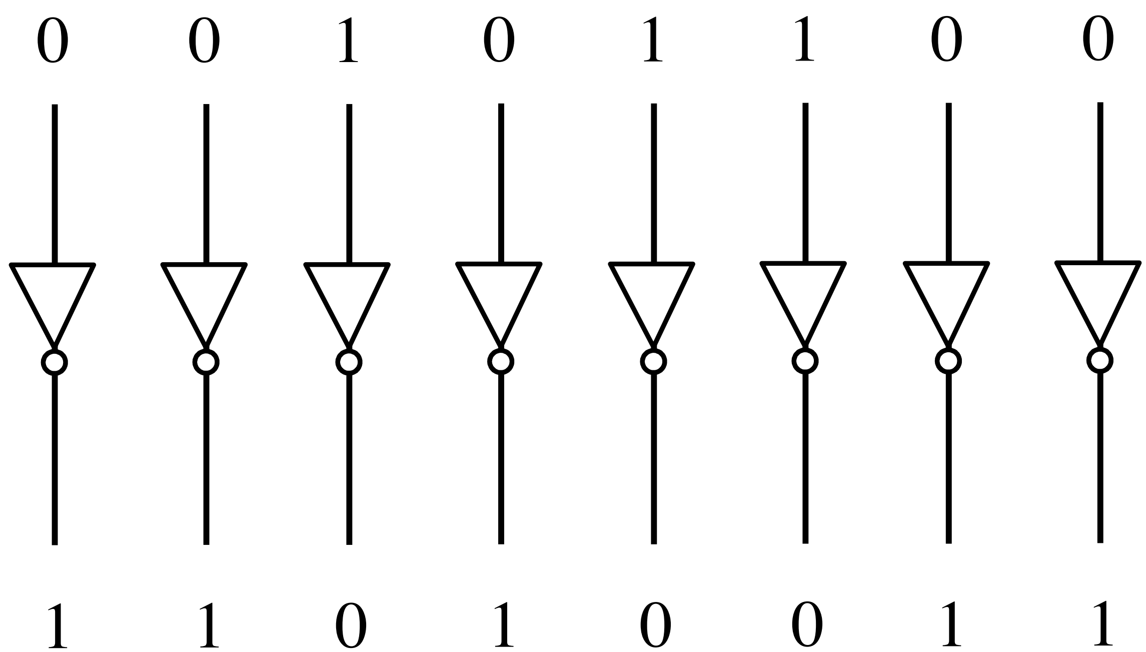
or

- 44 in 1’s complement representation

Circuit for negating a number stored in 1's complement representation



Circuit for negating a number stored in 1's complement representation



This works in reverse too
(from negative to positive)

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

211 (as unsigned)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

44 = 255 – 211 (as unsigned)

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

- 44 in 1's complement representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

+ 44 in 1's complement representation

Negate these numbers stored in 1's complement representation

0 1 0 1

1 0 1 1

1 1 1 0

0 1 1 1

Negate these numbers stored in 1's complement representation

0 1 0 1

1 0 1 0

1 0 1 1

0 1 0 0

1 1 1 0

0 0 0 1

0 1 1 1

1 0 0 0

Just flip 1's to 0's and vice versa.

Negate these numbers stored in 1's complement representation

0 1 0 1 = +5

1 0 1 0 = -5

1 0 1 1 = -4

0 1 0 0 = +4

1 1 1 0 = -1

0 0 0 1 = +1

0 1 1 1 = +7

1 0 0 0 = -7

Just flip 1's to 0's and vice versa.

Addition of two numbers stored in 1's complement representation

There are four cases to consider

- `(+5)   +   (+2)`
- `(–5)   +   (+2)`
- `(+5)   +   (–2)`
- `(–5)   +   (–2)`

There are four cases to consider

- `(+5)  +  (+2)      positive plus positive`
- `(–5)  +  (+2)      negative plus positive`
- `(+5)  +  (–2)      positive plus negative`
- `(–5)  +  (–2)      negative plus negative`

A) Example of 1's complement addition

(+5)	0 1 0 1
+ (+2)	+ 0 0 1 0
(+7)	0 1 1 1

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

A) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +\,(+2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

B) Example of 1's complement addition

$$\begin{array}{r} (-5) \\ +\,(+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,0 \\ +\,0\,0\,1\,0 \\ \hline 1\,1\,0\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

B) Example of 1's complement addition

$$\begin{array}{r} (-5) \\ +(+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,0 \\ +\,0\,0\,1\,0 \\ \hline 1\,1\,0\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$\begin{array}{rr} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$(+5)$	$0\,1\,0\,1$
$+(-2)$	$+\,1\,1\,0\,1$
$(+3)$	$1\,0\,0\,1\,0$

But this is 2!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$
\begin{array}{r}
(+5) \\
+(-2) \\
\hline
(+3)
\end{array}
\qquad
\begin{array}{r}
0\,1\,0\,1 \\
+\,1\,1\,0\,1 \\
\hline
1\,0\,0\,1\,0 \\
1 \\
\hline
0\,0\,1\,1
\end{array}
$$

We need to perform one
more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \\ 1 \\ \hline 0\,0\,1\,1 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$\begin{array}{rr} (-5) & 1\,0\,1\,0 \\ +\,(-2) & +\,1\,1\,0\,1 \\ \hline (-7) & 1\,0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

D) Example of 1's complement addition

$$\begin{array}{r} (-5) \\ +\,(-2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,0 \\ +\,1\,1\,0\,1 \\ \hline 1\;0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$
\begin{array}{rr}
 & (-5) \\
+ & (-2) \\
\hline
 & (-7)
\end{array}
\qquad
\begin{array}{rr}
 & 1\,0\,1\,0 \\
+ & 1\,1\,0\,1 \\
\hline
1 & 0\,1\,1\,1
\end{array}
$$

But this is +7!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$
\begin{array}{r}
(-5) \\
+\,(-2) \\
\hline
(-7)
\end{array}
\qquad
\begin{array}{r}
1\,0\,1\,0 \\
+\;1\,1\,0\,1 \\
\hline
1\,0\,1\,1\,1 \\
1 \\
\hline
1\,0\,0\,0
\end{array}
$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$\begin{array}{rr} & (-5) \\ + & (-2) \\ \hline & (-7) \end{array} \qquad \begin{array}{rr} & 1\,0\,1\,0 \\ + & 1\,1\,0\,1 \\ \hline 1 & 0\,1\,1\,1 \\ & 1 \\ \hline & 1\,0\,0\,0 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

Implications for arithmetic operations in 1’s complement representation

- We could do addition in 1’s complement, but the circuit will need to handle these exceptions.
- In some cases, it will run faster that others, thus creating uncertainties in the timing.
- Therefore, 1’s complement is not used in practice to do arithmetic operations.
- But it may show up as an intermediary step in doing 2’s complement operations.

2's Complement

2's complement
(subtract each digit from 1 and add 1 to the result)

Let K be the negative equivalent of an n-bit positive number P.

Then, in 2's complement representation K is obtained by subtracting P from 2^n , namely

$$K = 2^n - P$$

2's complement
(subtract each digit from 1 and add 1 to the result)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 2's complement representation K is obtained by subtracting P from 2^8 , namely

$$K = 2^8 - P = 256 - P$$

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

+ 44

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44
positive								

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44
negative								

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44

$212 = 256 - 44$

Deriving 2's complement

For a positive n-bit number P, let K_1 and K_2 denote its 1's and 2's complements, respectively.

$$K_1 = (2^n - 1) - P$$

$$K_2 = 2^n - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2's complement can be computed by inverting all bits of P and then adding 1 to the resulting 1's-complement number.

Deriving 2's complement

For a positive 8-bit number P, let K_1 and K_2 denote its 1's and 2's complements, respectively.

$$K_1 = (2^n - 1) - P = 255 - P$$

$$K_2 = 2^n - P = 256 - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2's complement can be computed by inverting all bits of P and then adding 1 to the resulting 1's-complement number.

Negate these numbers stored in 2's complement representation

0 1 0 1

1 1 1 0

1 1 0 0

0 1 1 1

Negate these numbers stored in 2's complement representation

0 1 0 1	1 1 1 0
1 0 1 0	0 0 0 1
1 1 0 0	0 1 1 1
0 0 1 1	1 0 0 0

Invert all bits…

Negate these numbers stored in 2's complement representation

$$\begin{array}{r} 0\,1\,0\,1 \\ 1\,0\,1\,0 \\ +\quad\quad 1 \\ \hline 1\,0\,1\,1 \end{array} \qquad \begin{array}{r} 1\,1\,1\,0 \\ 0\,0\,0\,1 \\ +\quad\quad 1 \\ \hline 0\,0\,1\,0 \end{array}$$

$$\begin{array}{r} 1\,1\,0\,0 \\ 0\,0\,1\,1 \\ +\quad\quad 1 \\ \hline 0\,1\,0\,0 \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ 1\,0\,0\,0 \\ +\quad\quad 1 \\ \hline 1\,0\,0\,1 \end{array}$$

.. then add 1.

Negate these numbers stored in 2's complement representation

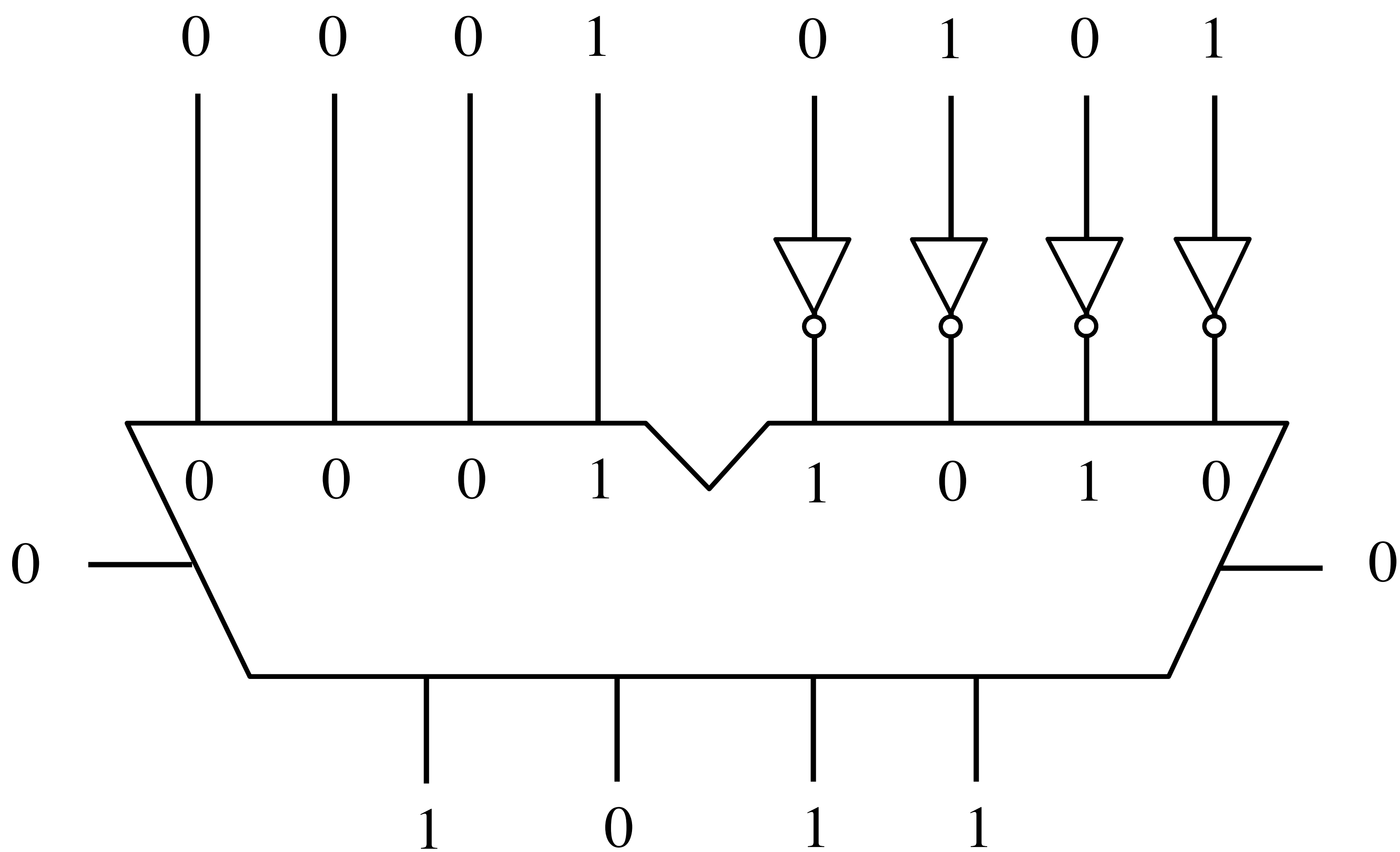
$$\begin{array}{rl} & 0\,1\,0\,1 \quad = +5 \\ & 1\,0\,1\,0 \\ + & 1 \\ \hline & 1\,0\,1\,1 \quad = -5 \end{array}$$

$$\begin{array}{rl} & 1\,1\,1\,0 \quad = -2 \\ & 0\,0\,0\,1 \\ + & 1 \\ \hline & 0\,0\,1\,0 \quad = +2 \end{array}$$

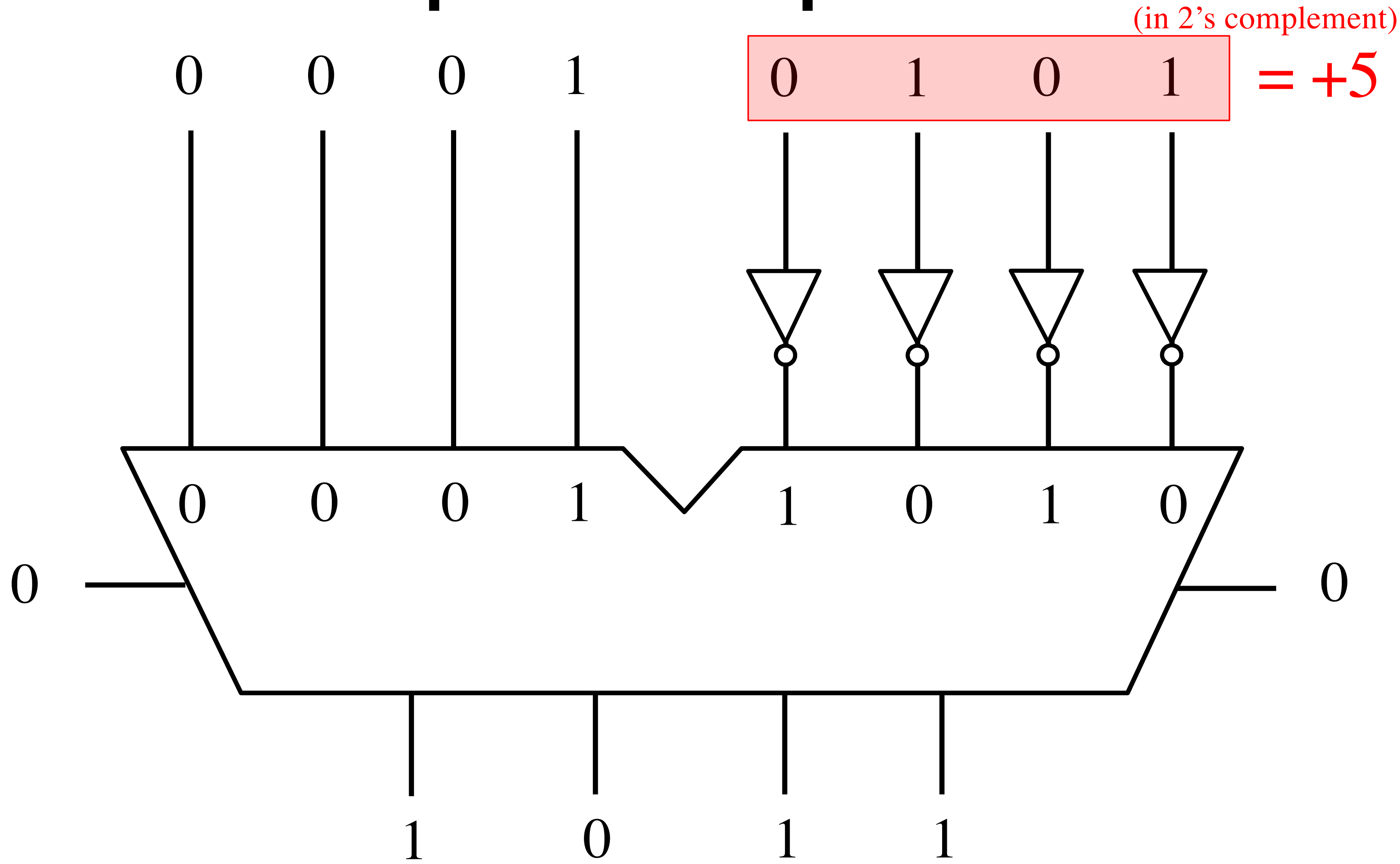
$$\begin{array}{rl} & 1\,1\,0\,0 \quad = -4 \\ & 0\,0\,1\,1 \\ + & 1 \\ \hline & 0\,1\,0\,0 \quad = +4 \end{array}$$

$$\begin{array}{rl} & 0\,1\,1\,1 \quad = +7 \\ & 1\,0\,0\,0 \\ + & 1 \\ \hline & 1\,0\,0\,1 \quad = -7 \end{array}$$

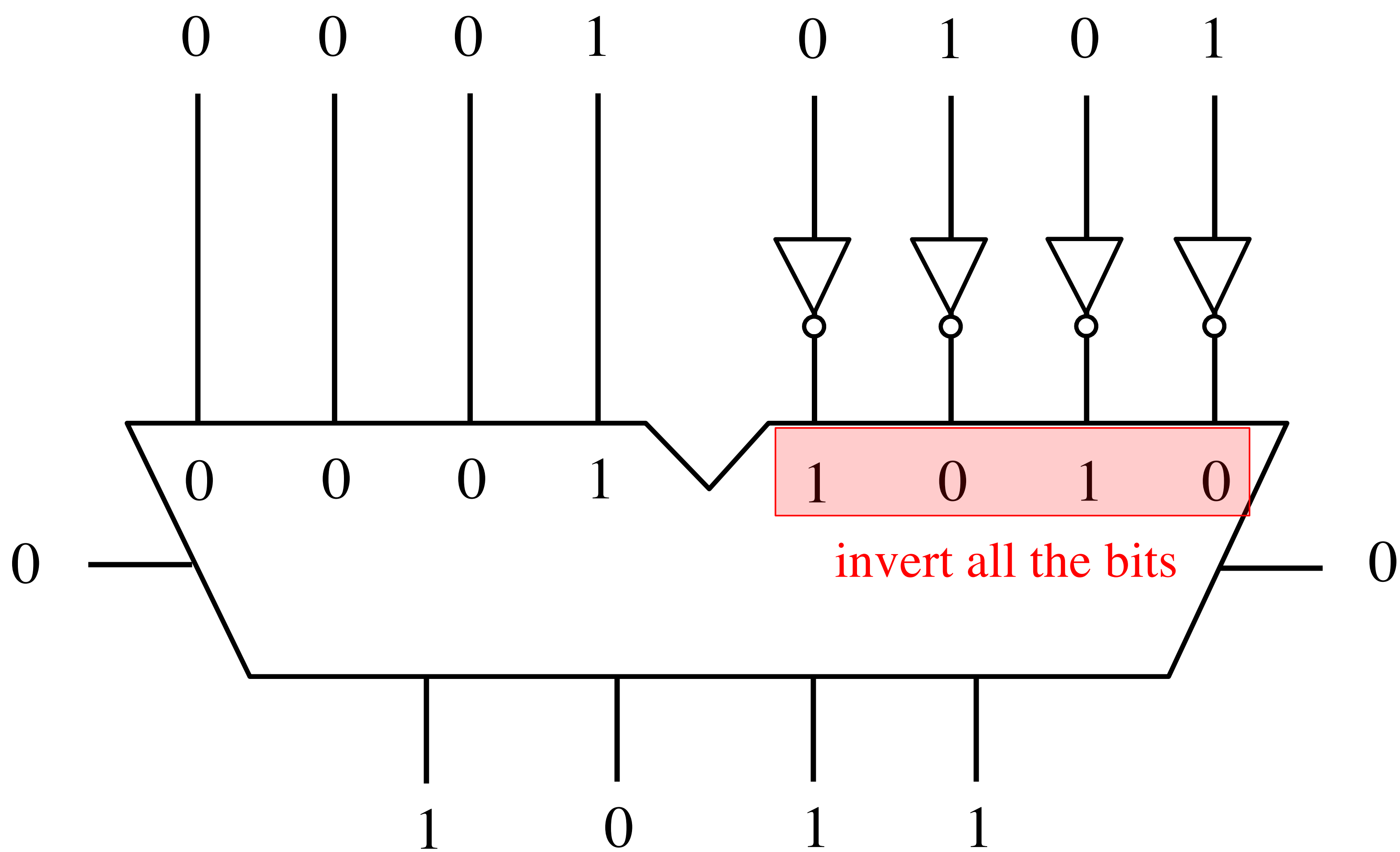
Circuit #1 for negating a number stored in 2's complement representation



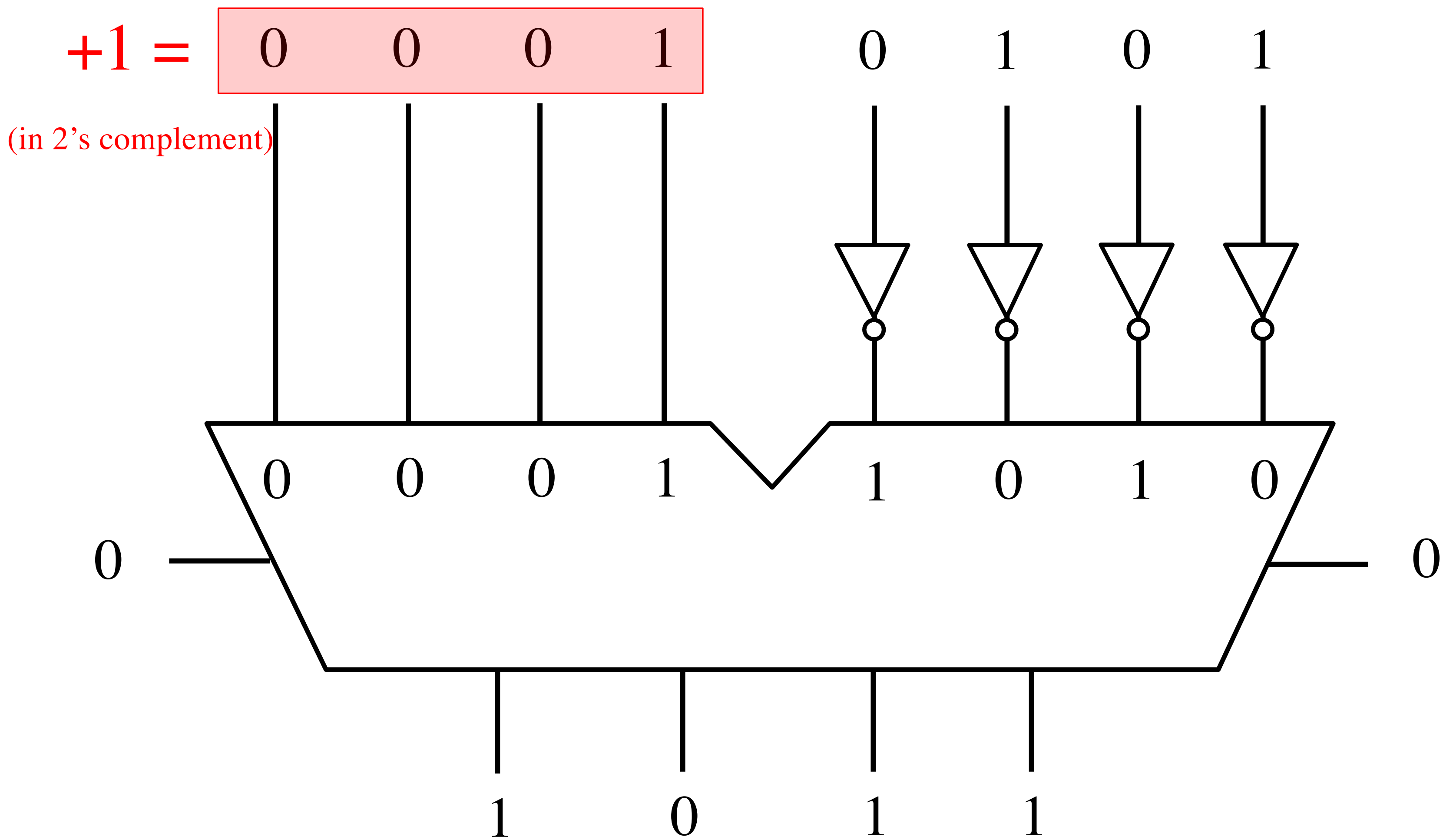
Circuit #1 for negating a number stored in 2's complement representation



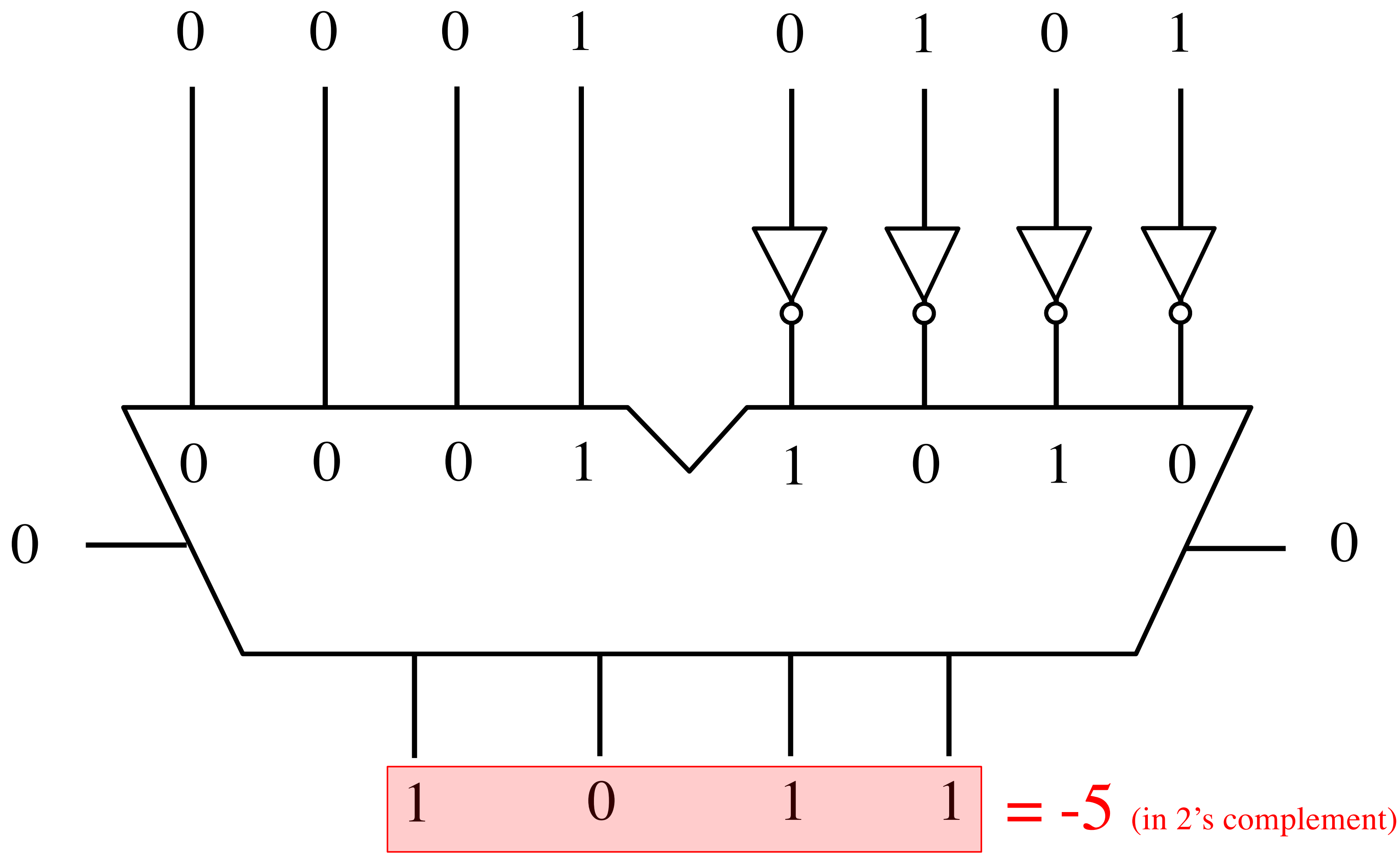
Circuit #1 for negating a number stored in 2's complement representation



Circuit #1 for negating a number stored in 2's complement representation

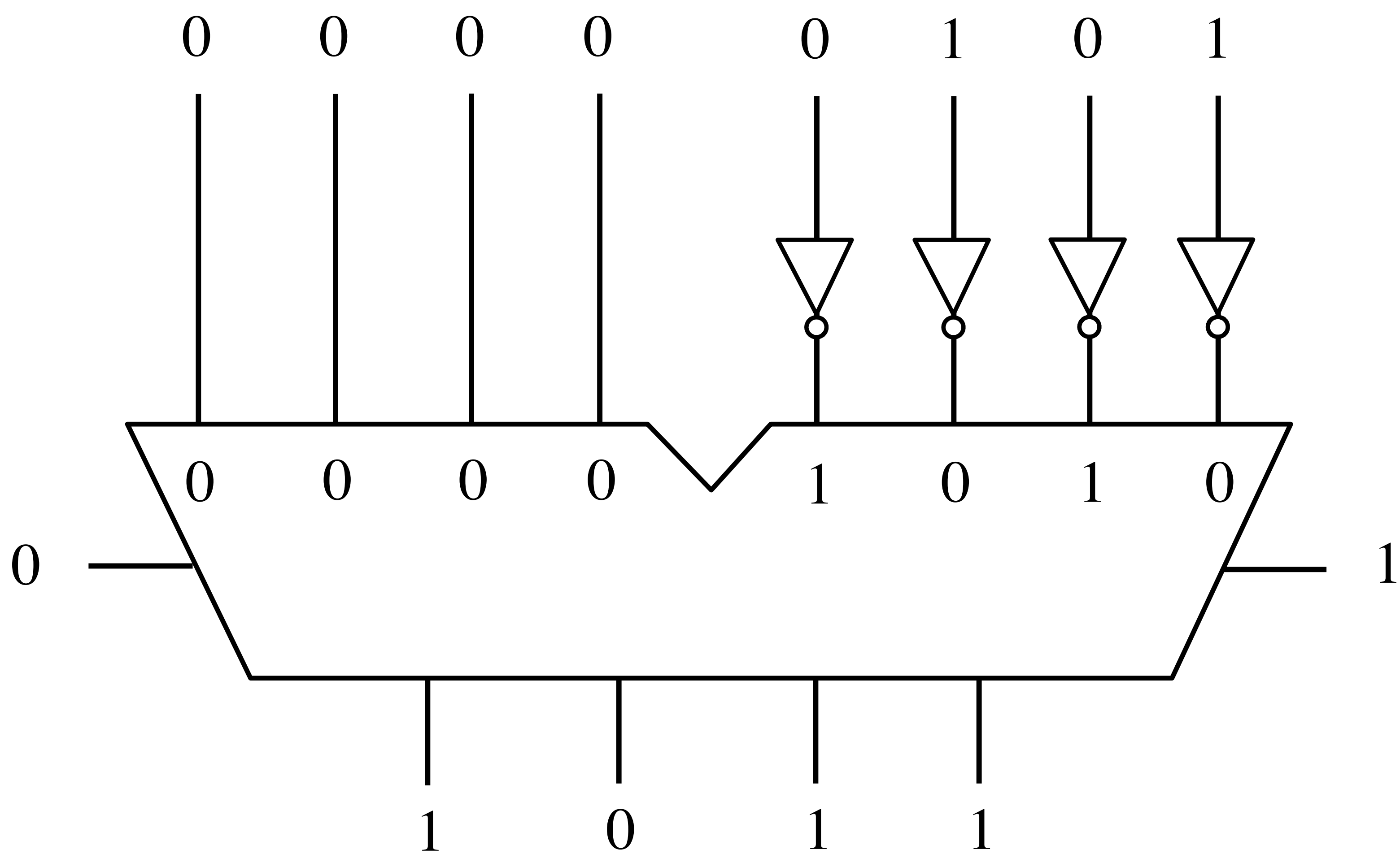


Circuit #1 for negating a number stored in 2's complement representation

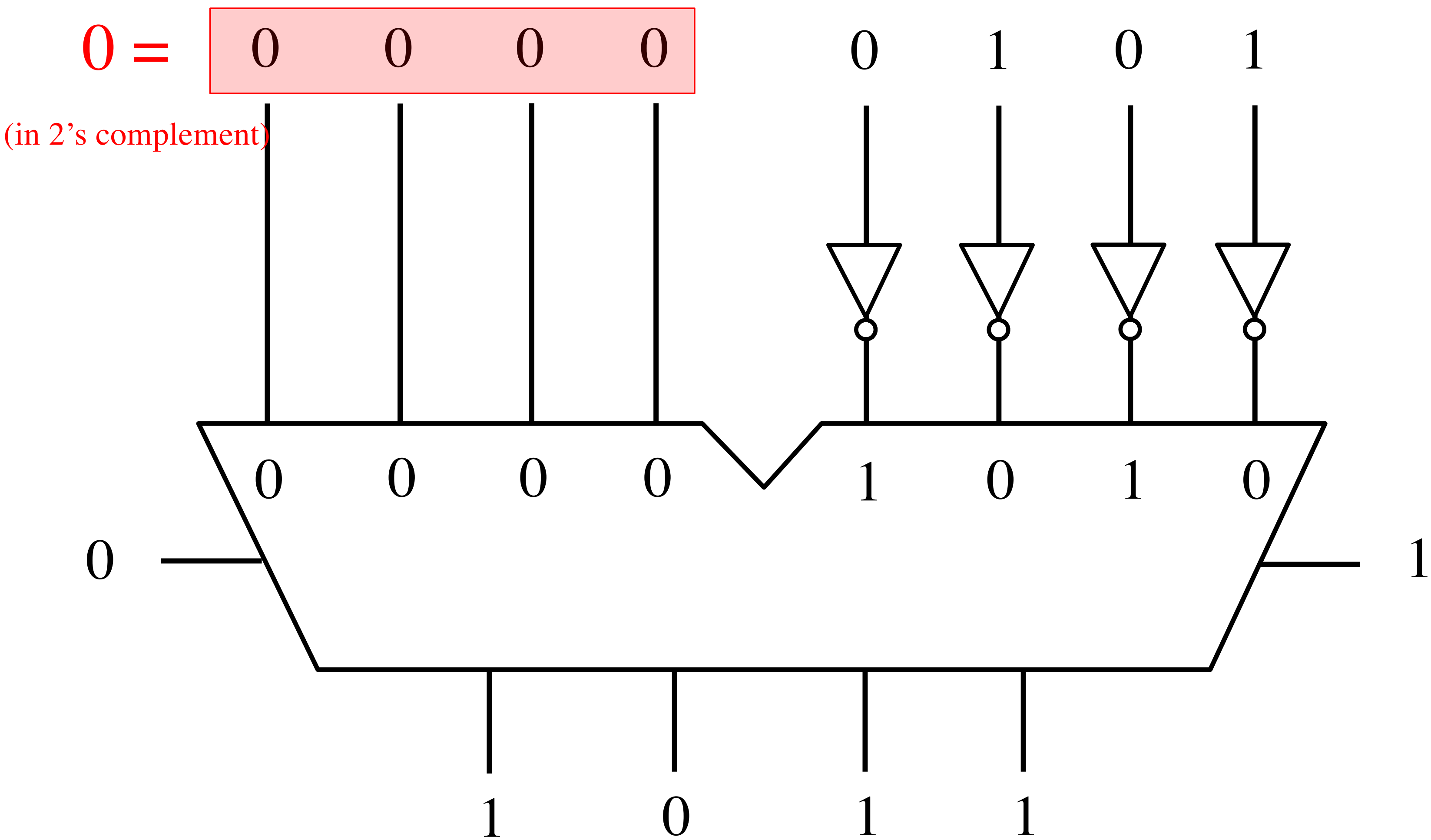


Alternative Circuit

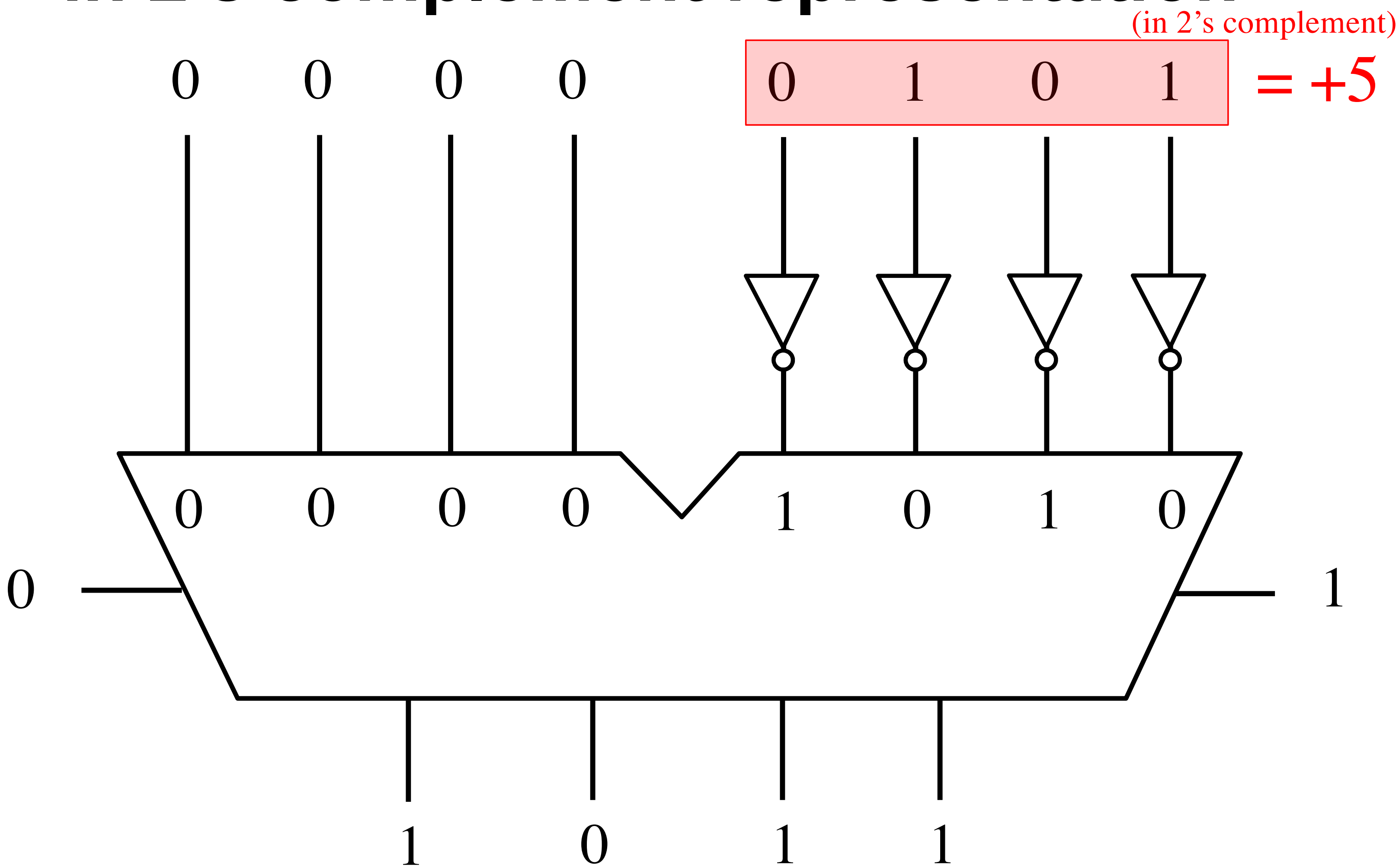
Circuit #2 for negating a number stored in 2's complement representation



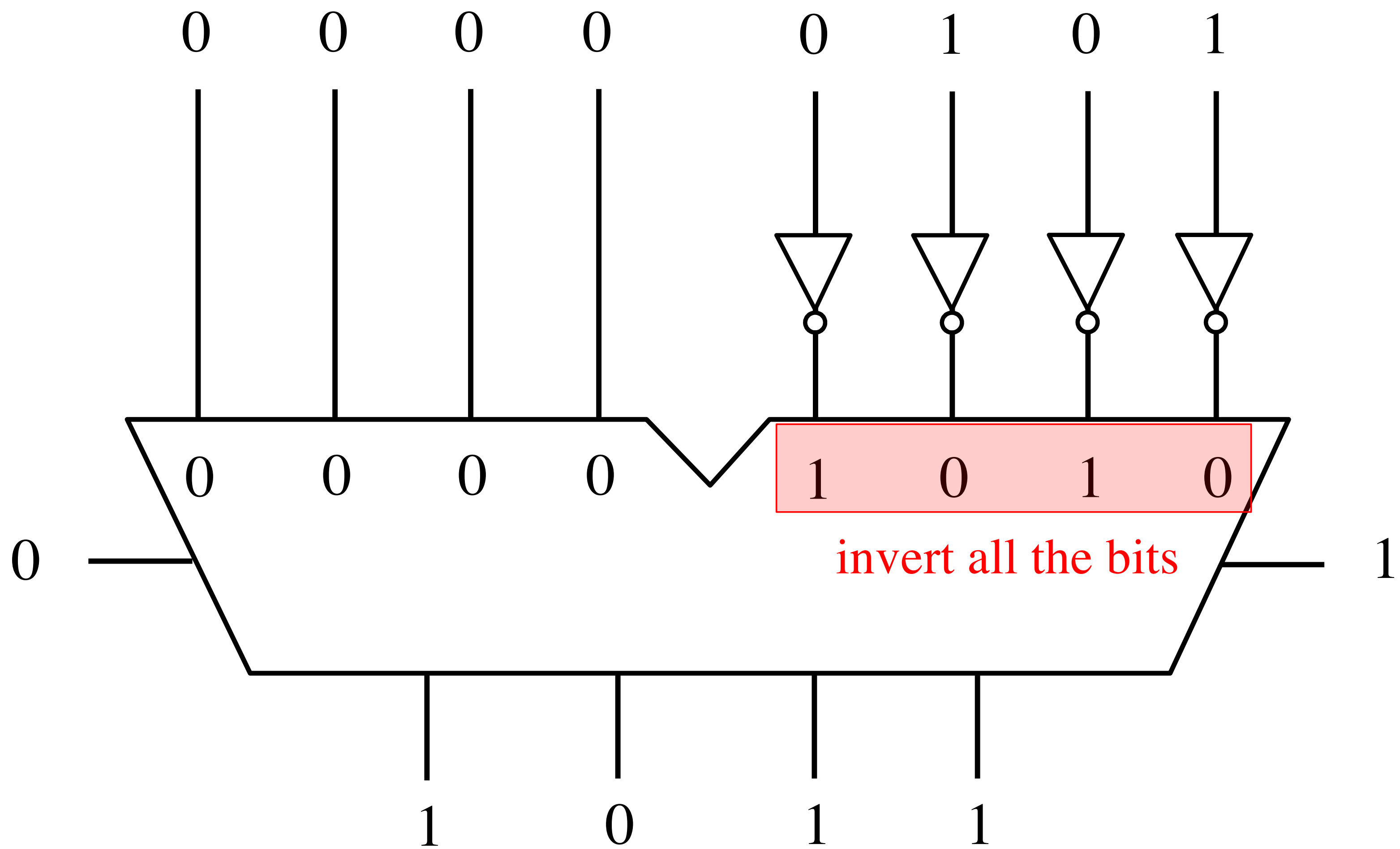
Circuit #2 for negating a number stored in 2's complement representation



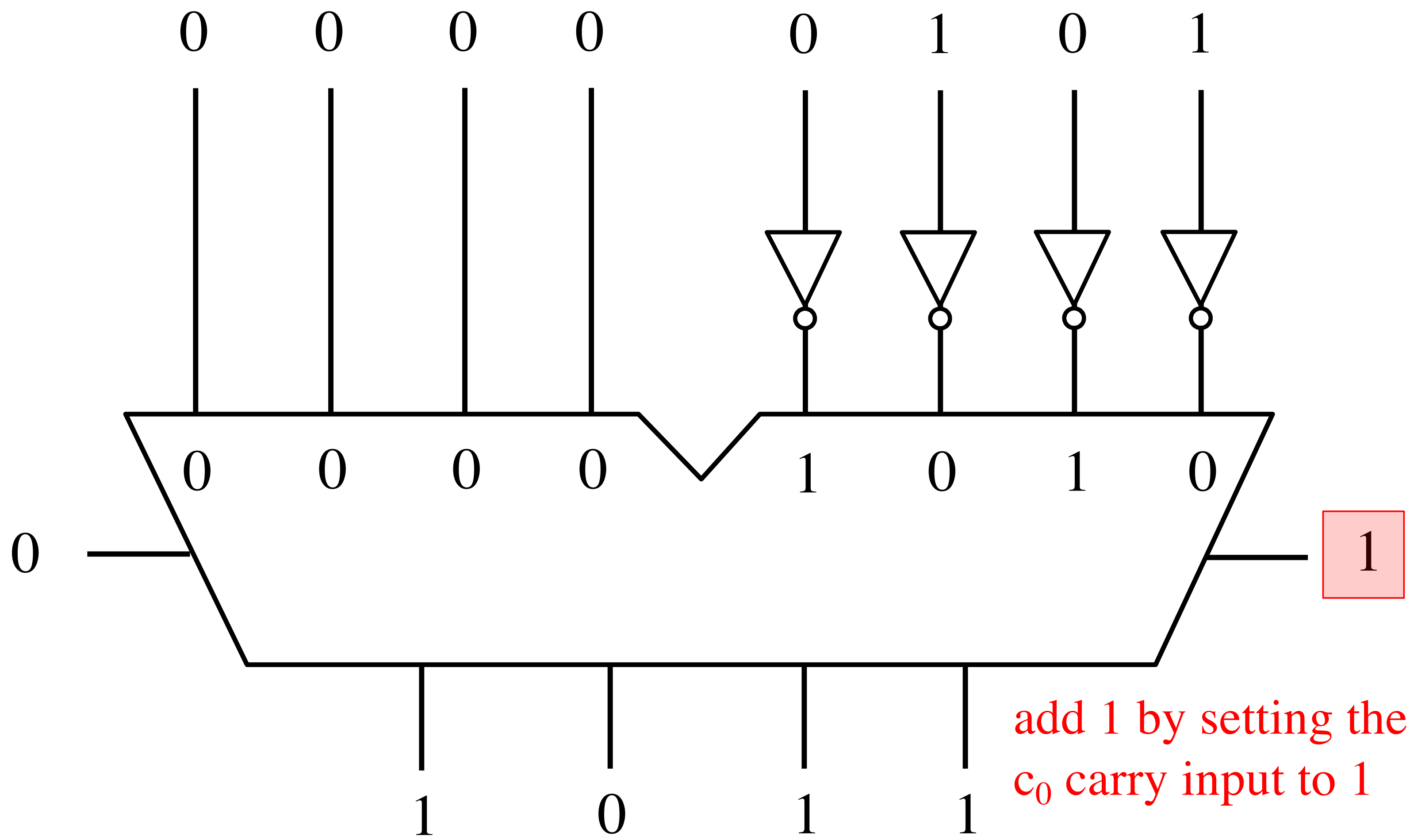
Circuit #2 for negating a number stored in 2's complement representation



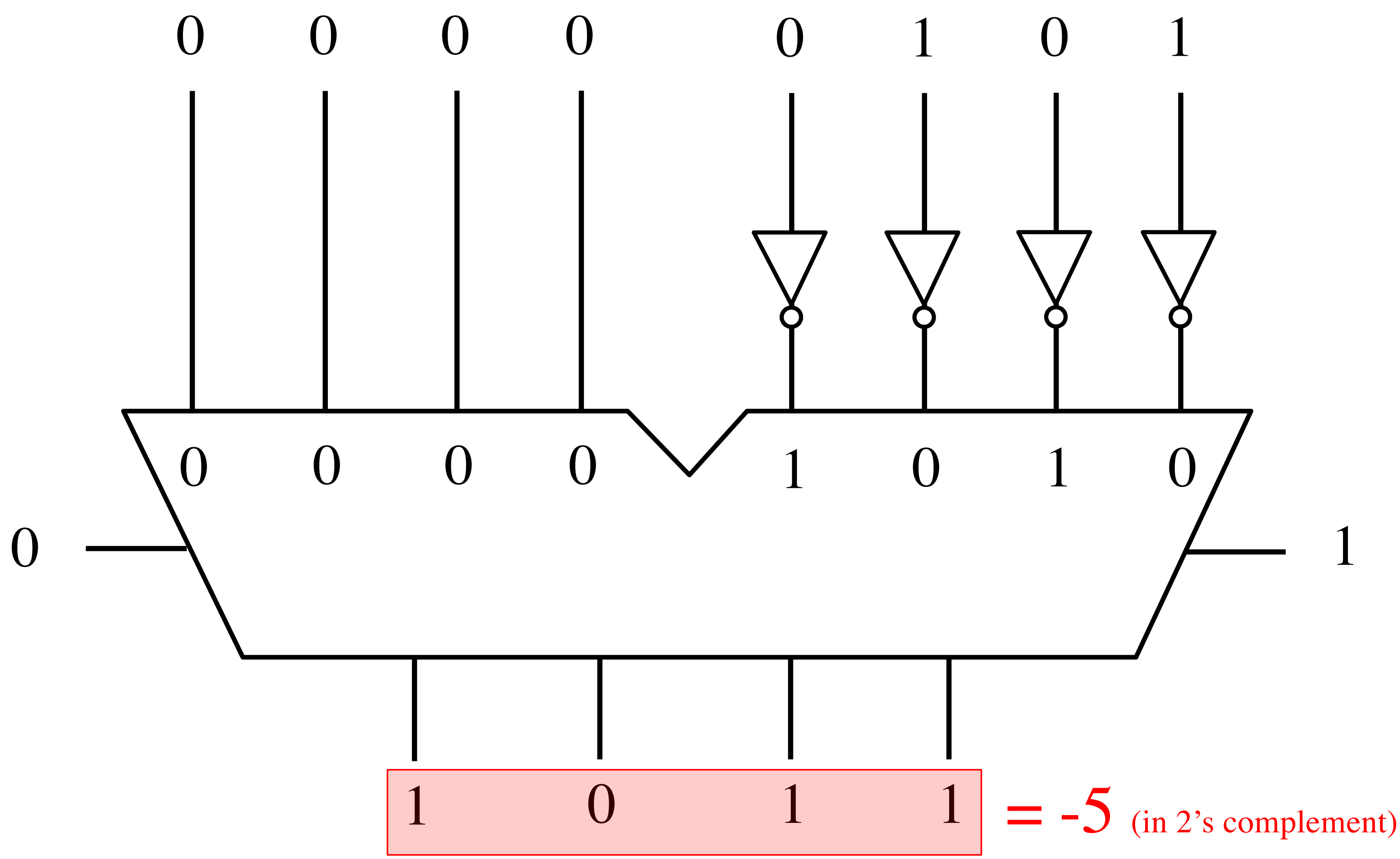
Circuit #2 for negating a number stored in 2's complement representation



Circuit #2 for negating a number stored in 2's complement representation

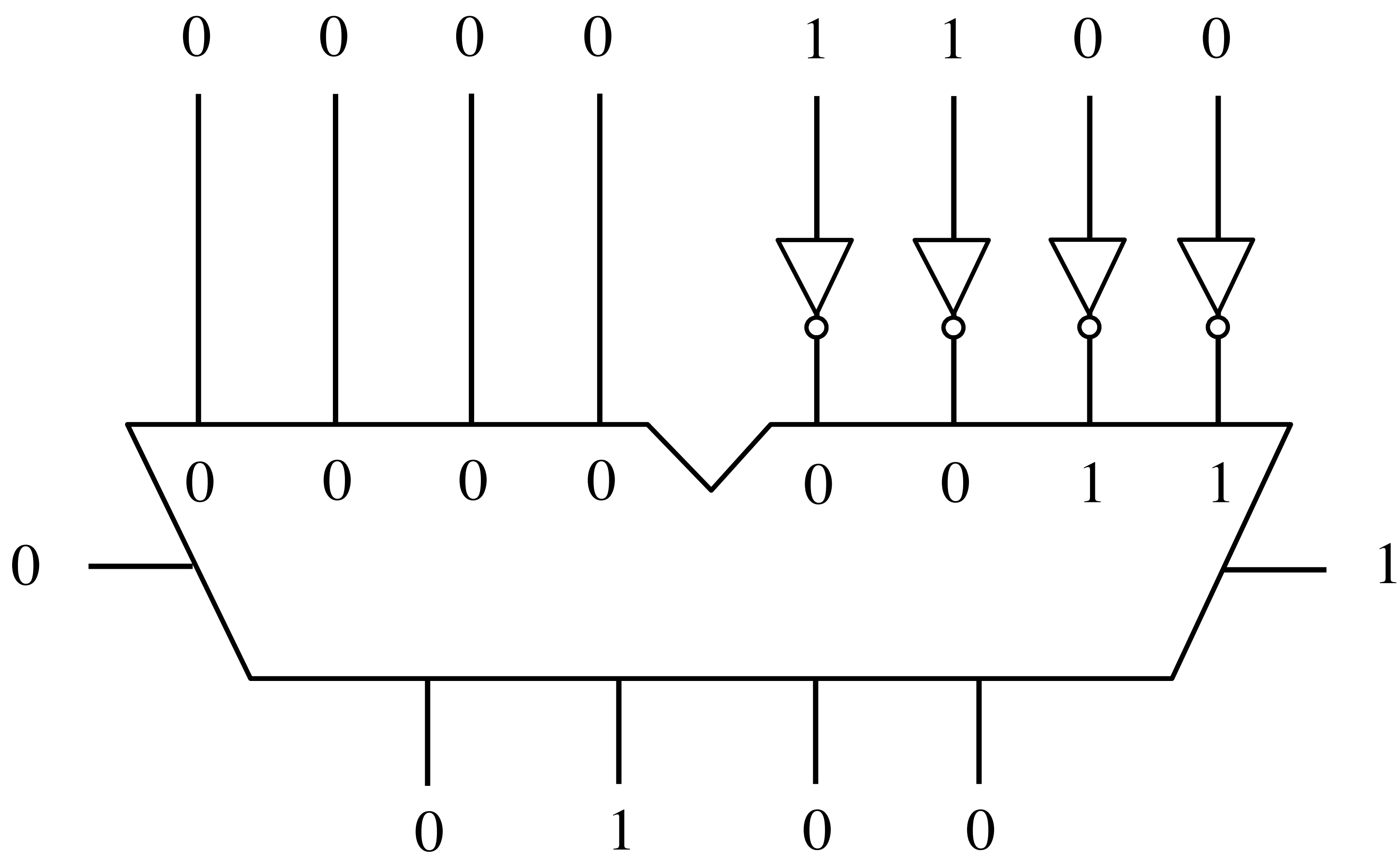


Circuit #2 for negating a number stored in 2's complement representation

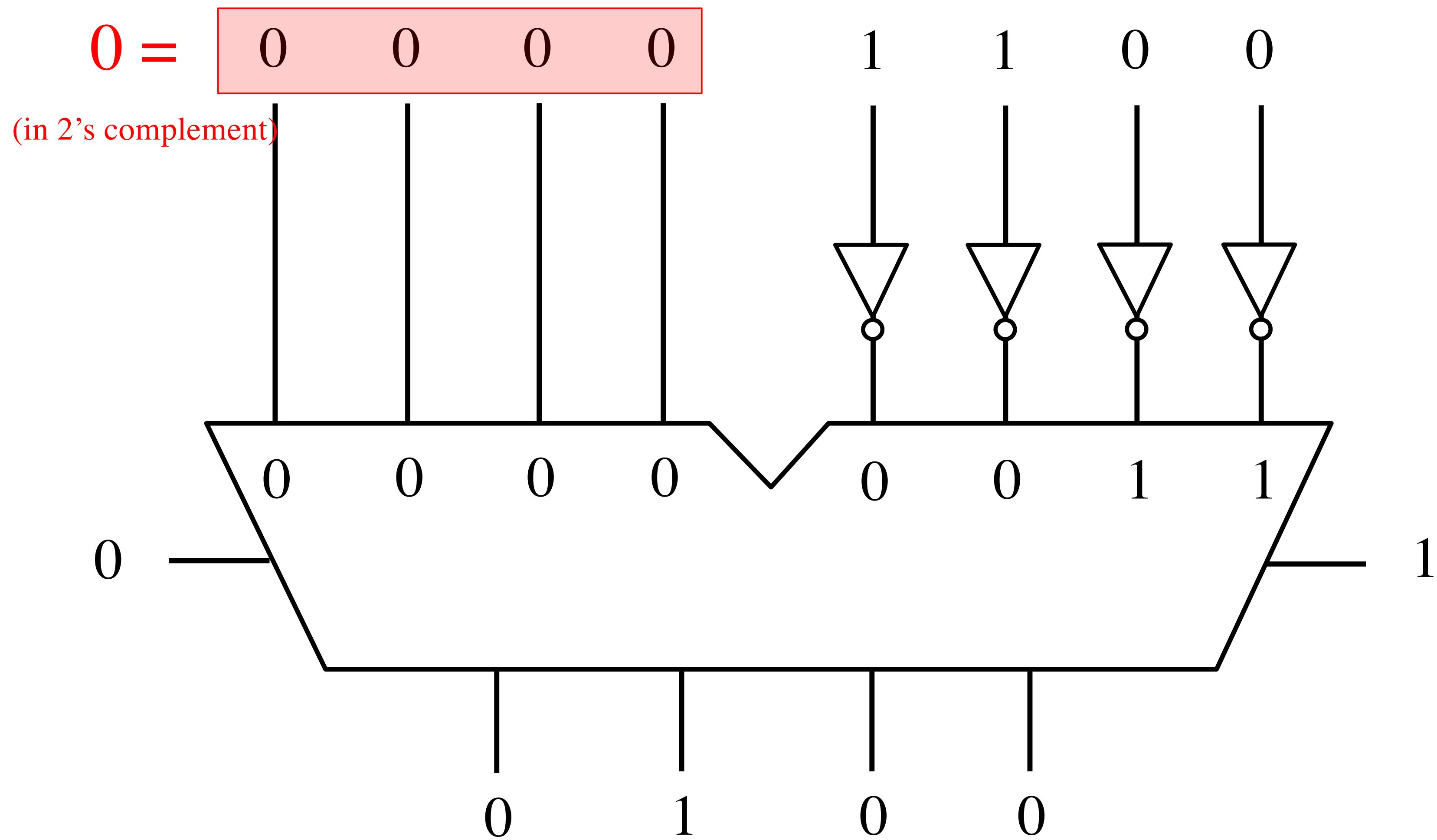


This also works for negating a negative number, thus making it positive

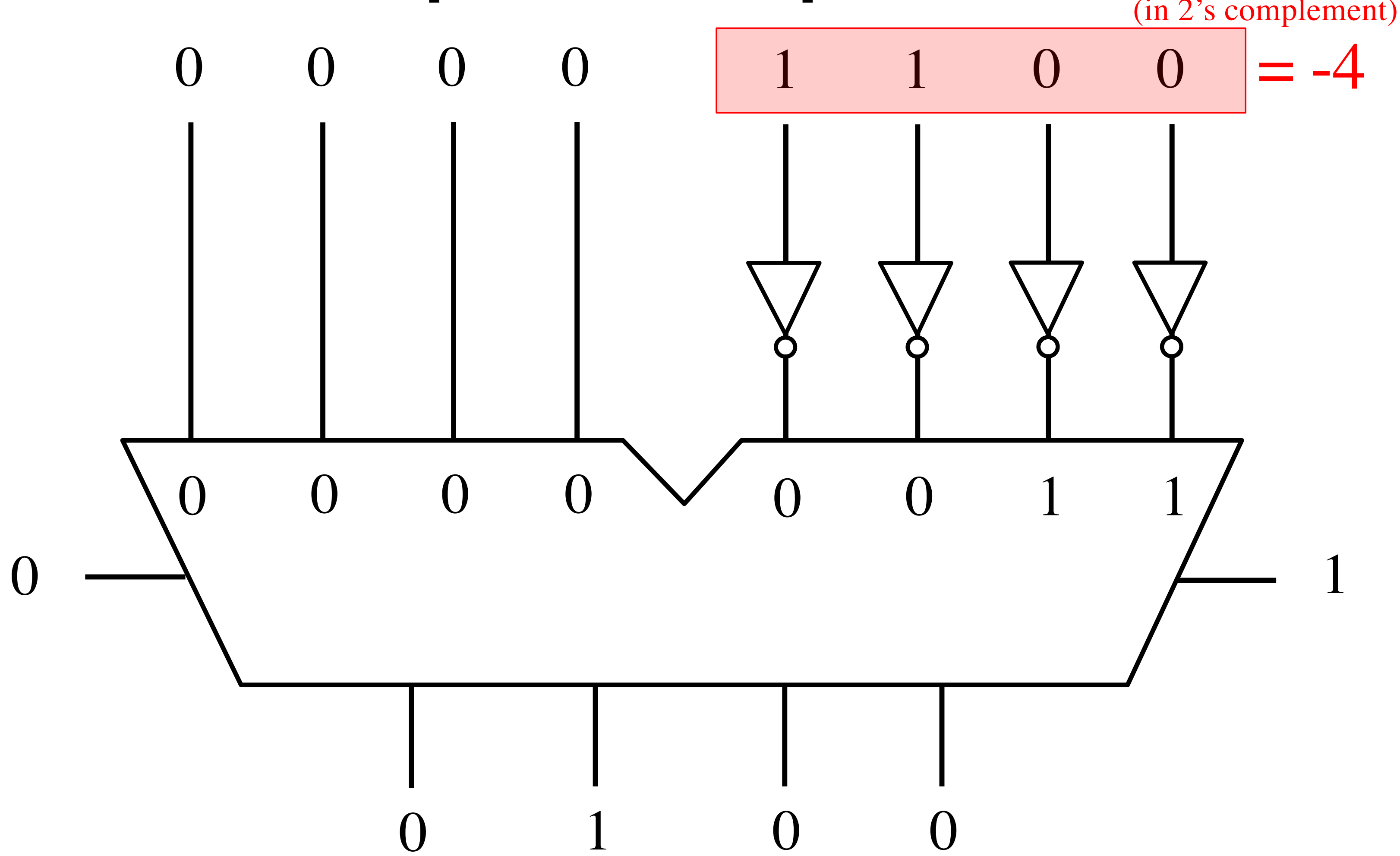
Circuit #2 for negating a number stored in 2's complement representation



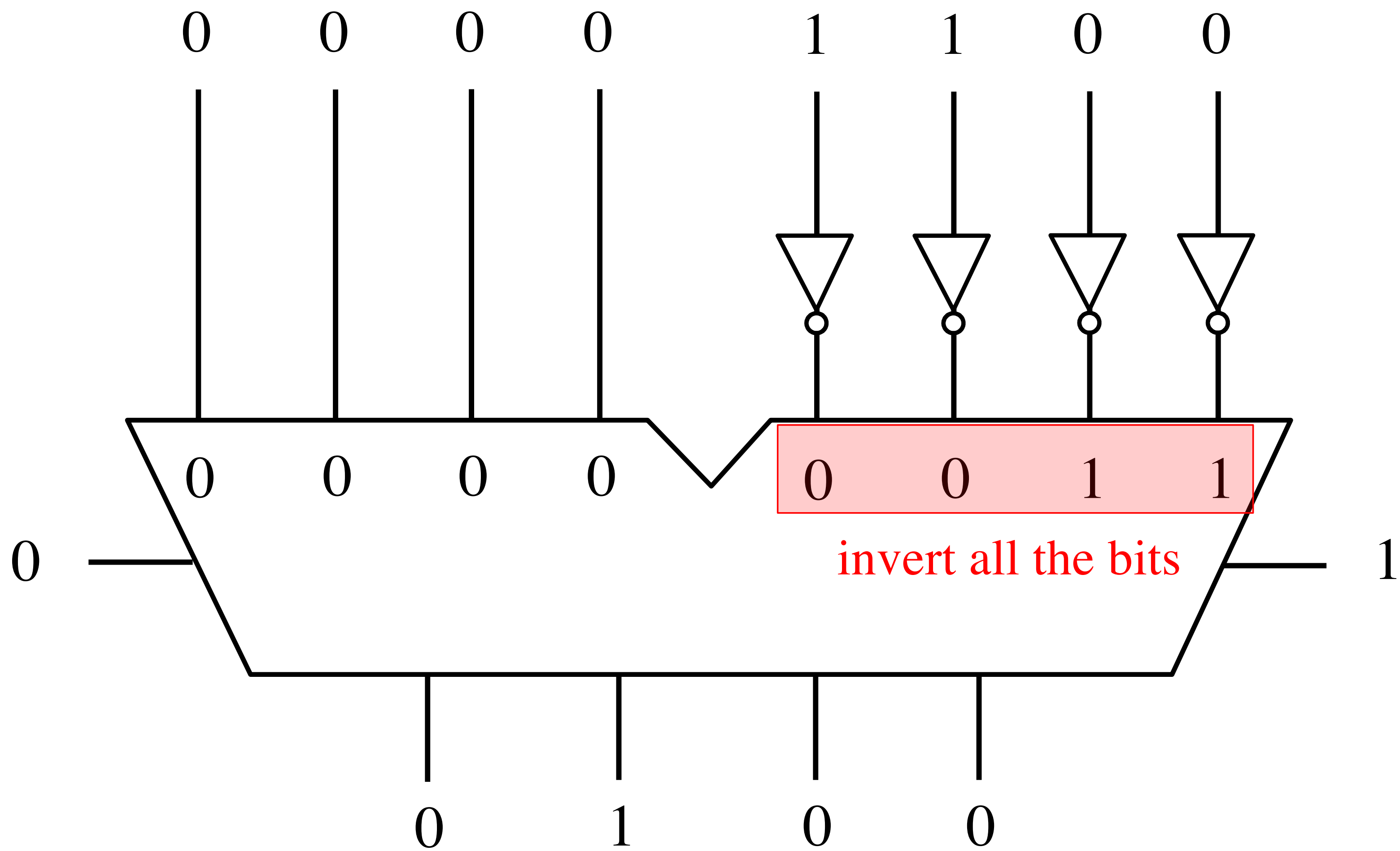
Circuit #2 for negating a number stored in 2's complement representation



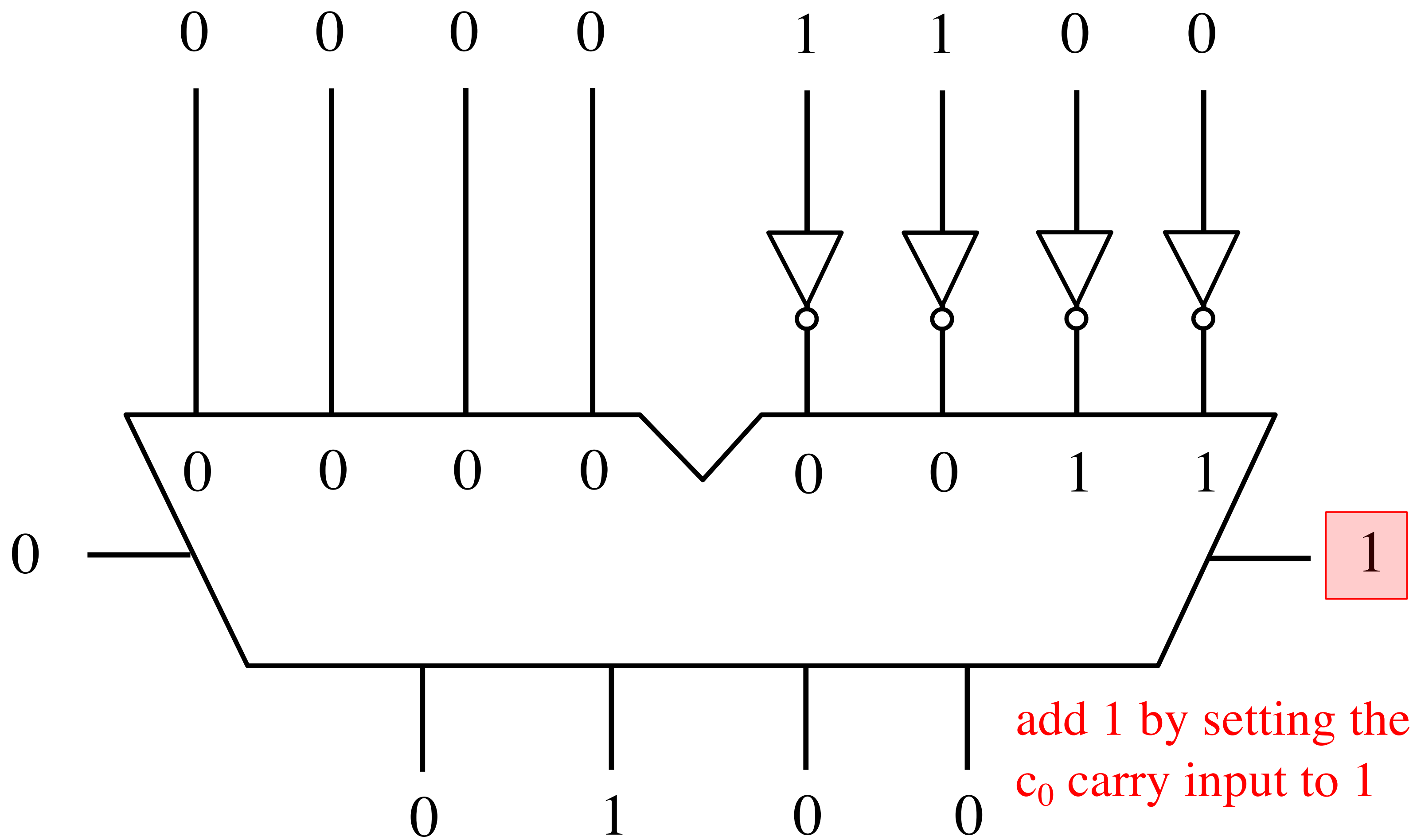
Circuit #2 for negating a number stored in 2's complement representation



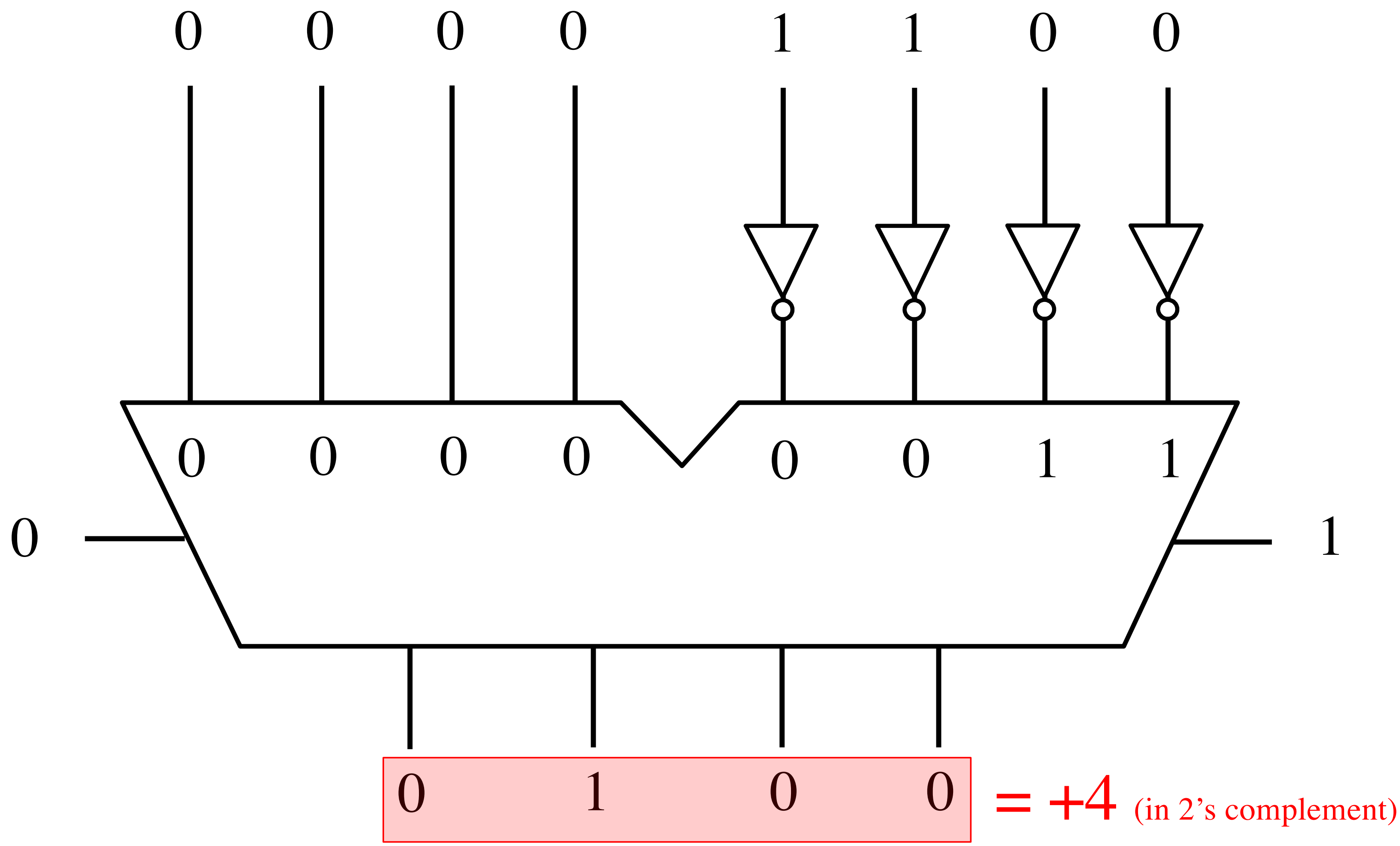
Circuit #2 for negating a number stored in 2's complement representation



Circuit #2 for negating a number stored in 2's complement representation



Circuit #2 for negating a number stored in 2's complement representation



Quick way (for a human) to negate a number stored in 2's complement

- Scan the binary number from right to left
- Copy all bits that are 0 from right to left
- Stop at the first 1
- Copy that 1 as well
- Invert all remaining bits

Negate these numbers stored in 2's complement representation

0 1 0 1

1 1 1 0

1 1 0 0

0 1 1 1

Negate these numbers stored in 2's complement representation

0 1 0 1	1 1 1 0
. . . .	. . . 0
1 1 0 0	0 1 1 1
. . 0 0	. . . .

Copy all bits that are 0 from right to left.

Negate these numbers stored in 2's complement representation

0 1 0 1	1 1 1 0
. . . 1	. . 1 0
1 1 0 0	0 1 1 1
. 1 0 0	. . . 1

Stop at the first 1. Copy that 1 as well.

Negate these numbers stored in 2's complement representation

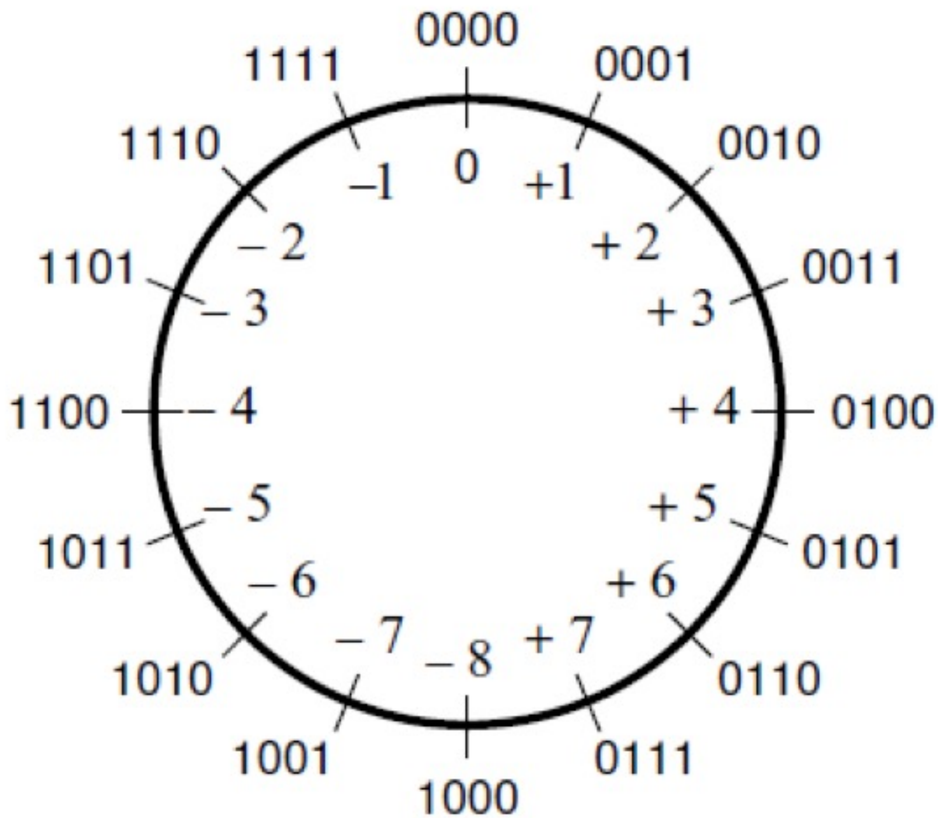
0 1 0 1	1 1 1 0
1 0 1 1	0 0 1 0
1 1 0 0	0 1 1 1
0 1 0 0	1 0 0 1

Invert all remaining bits.

Negate these numbers stored in 2's complement representation

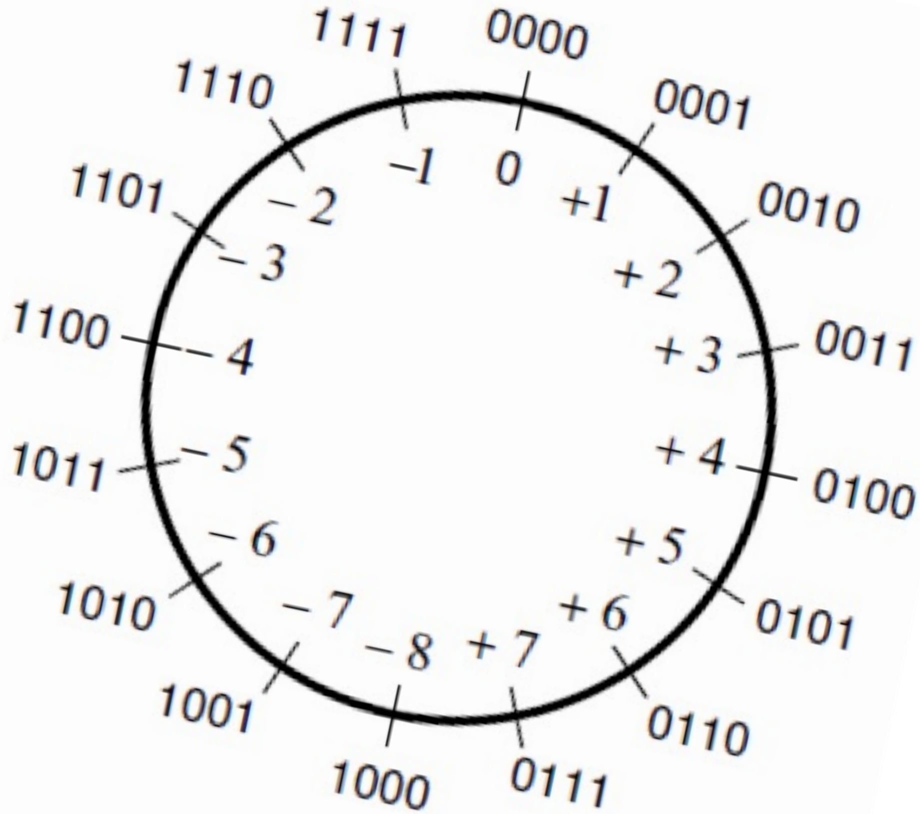
0 1 0 1 = +5	1 1 1 0 = -2
1 0 1 1 = -5	0 0 1 0 = +2
1 1 0 0 = -4	0 1 1 1 = +7
0 1 0 0 = +4	1 0 0 1 = -7

The number circle for 2's complement

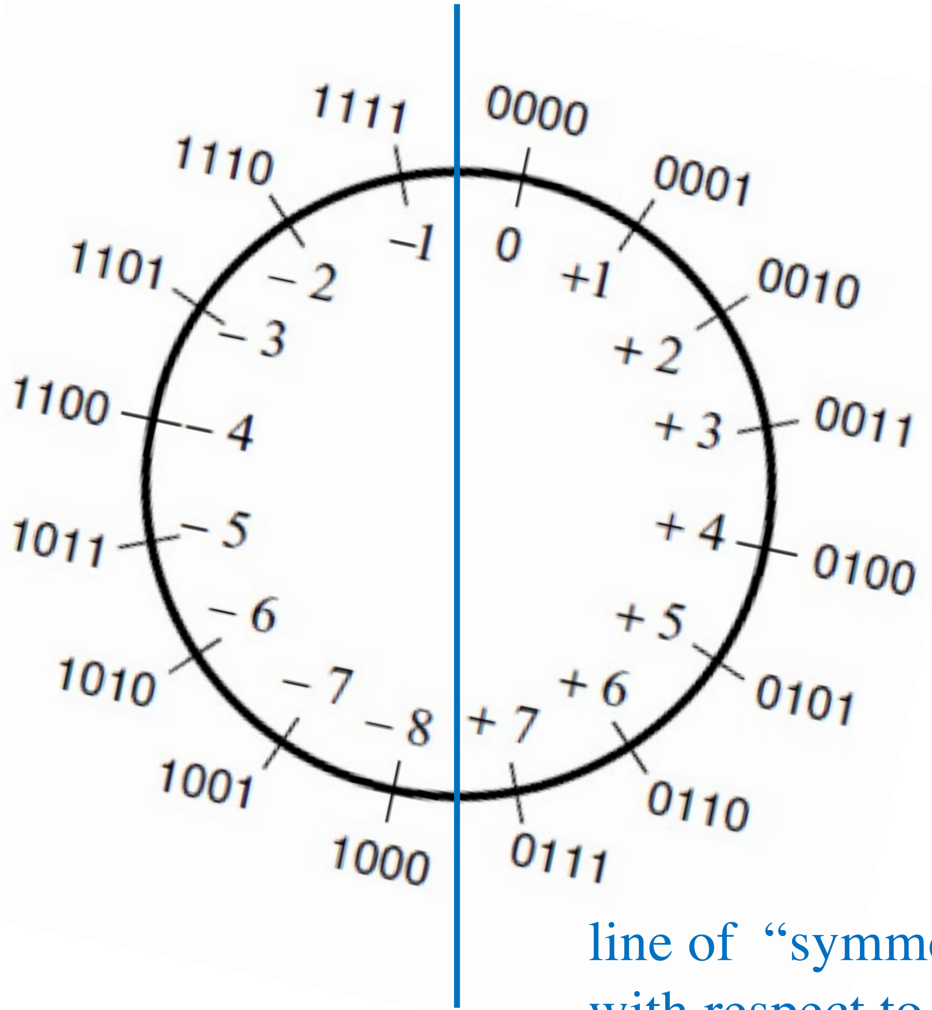


[Figure 3.11a from the textbook]

The number circle for 2's complement

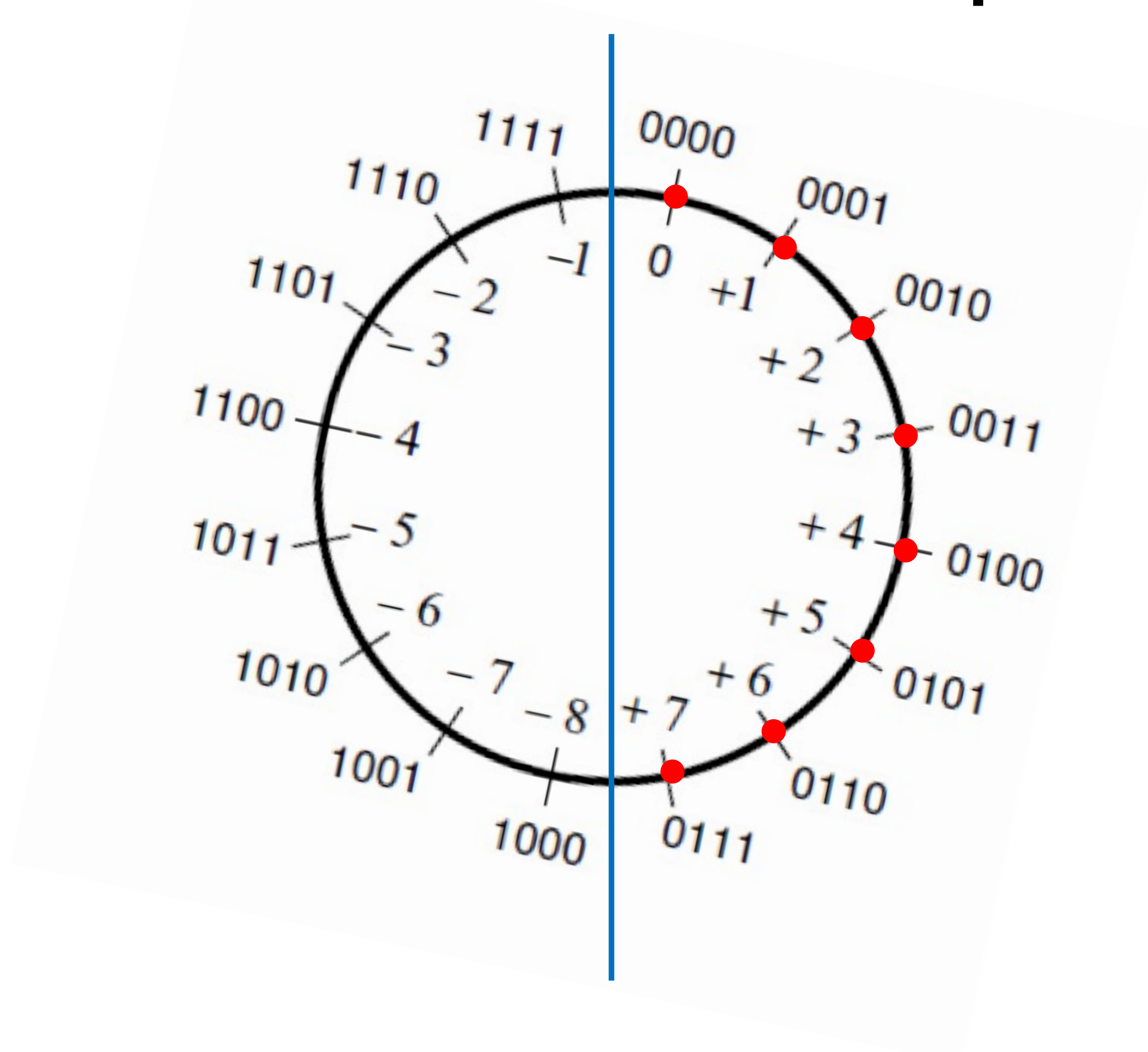


The number circle for 2's complement

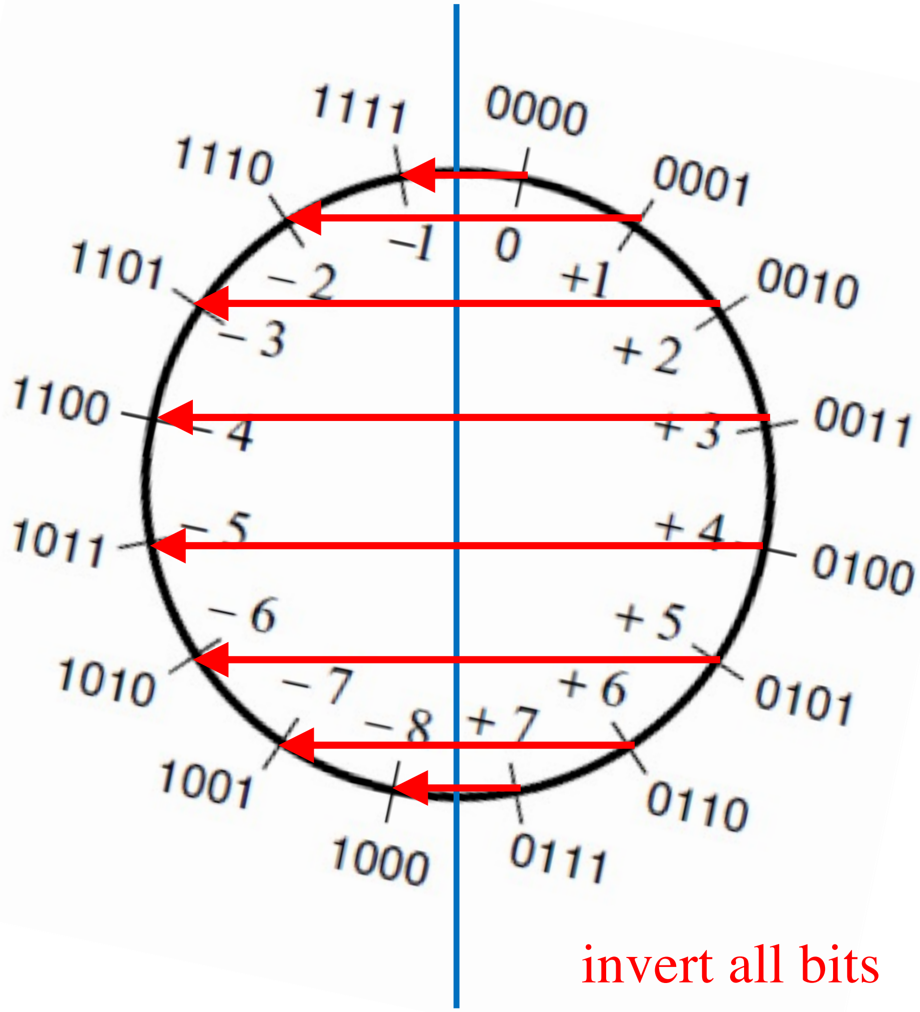


line of "symmetry"
with respect to bit inversion

The number circle for 2's complement

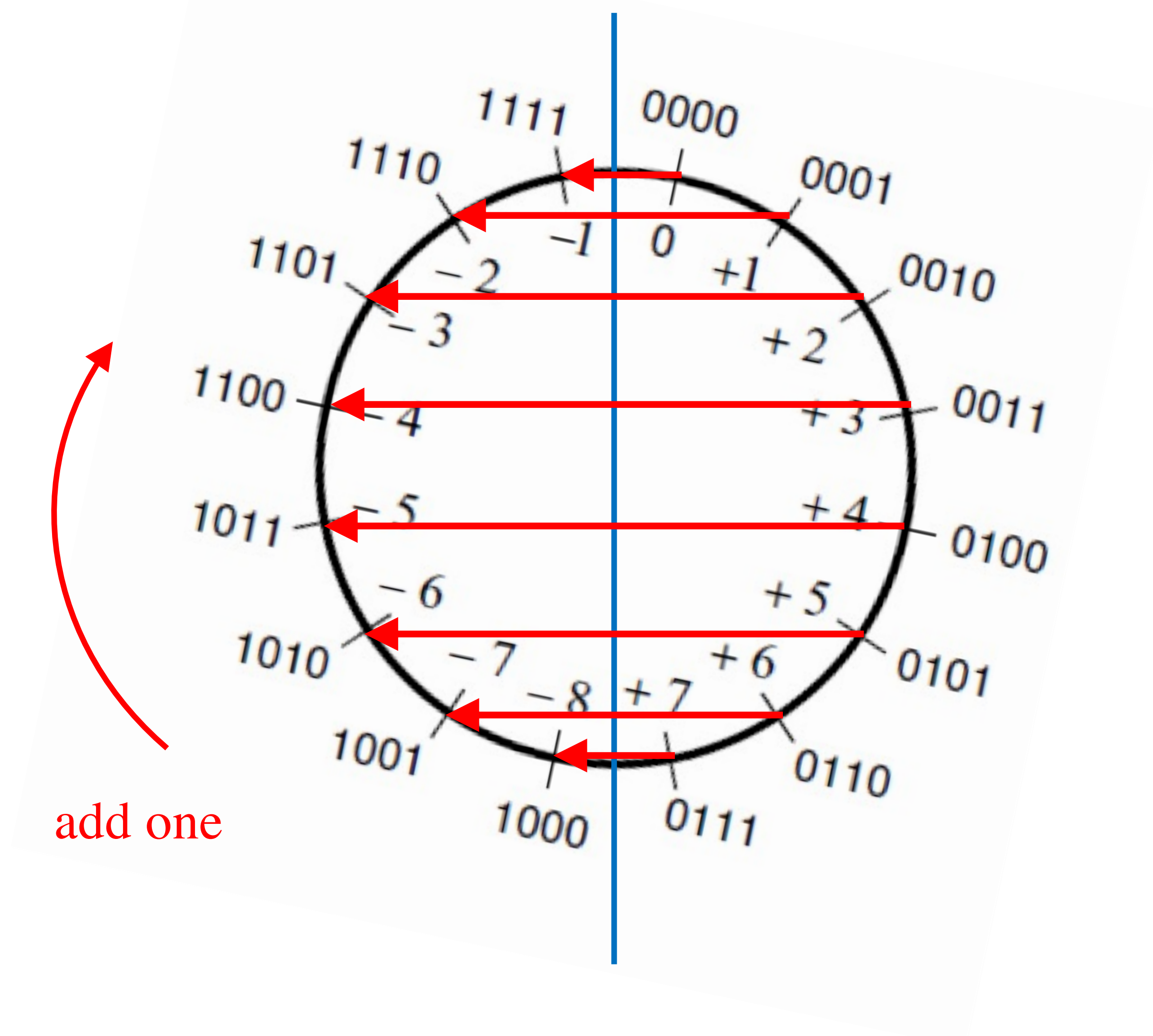


The number circle for 2's complement

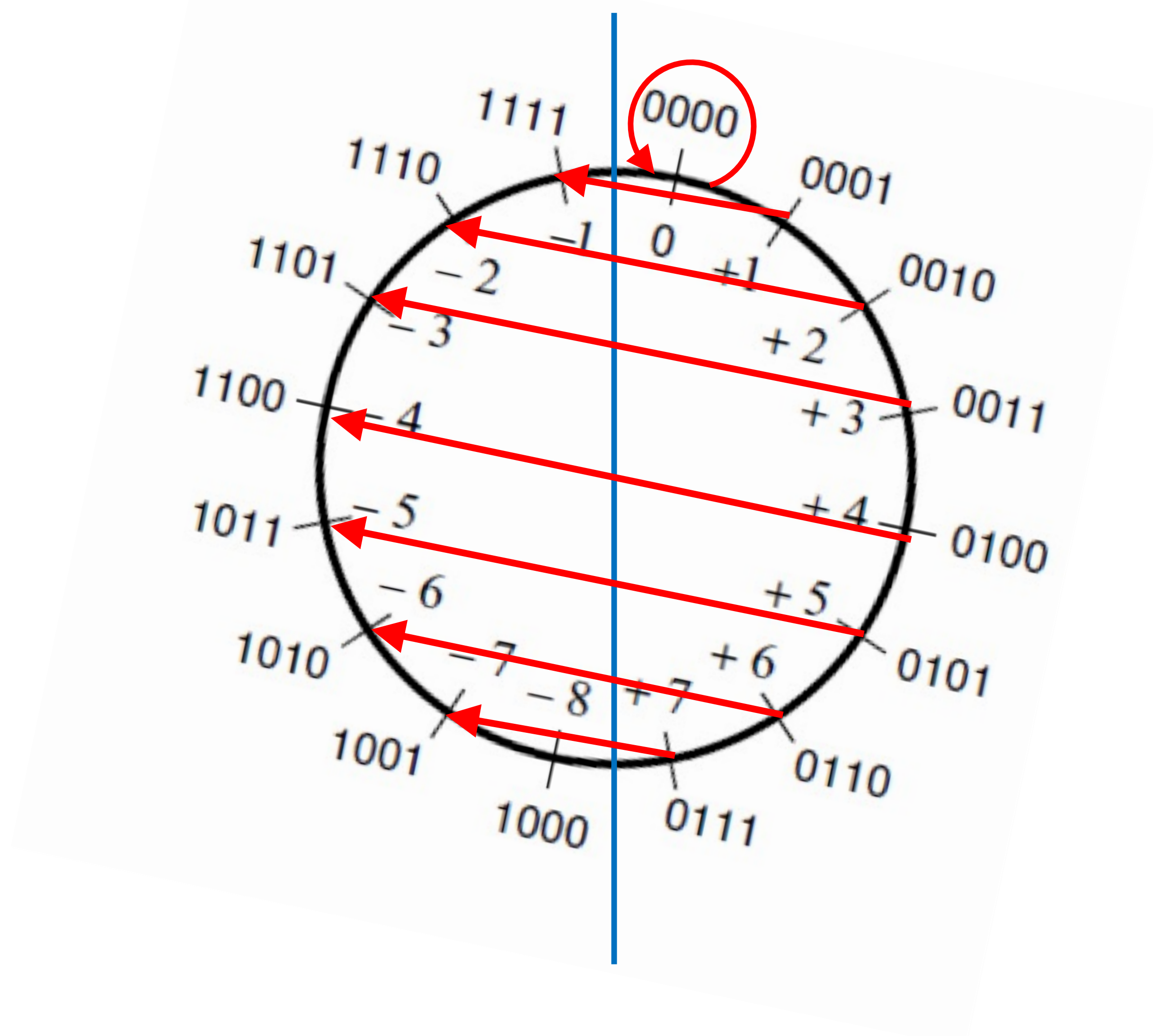


invert all bits

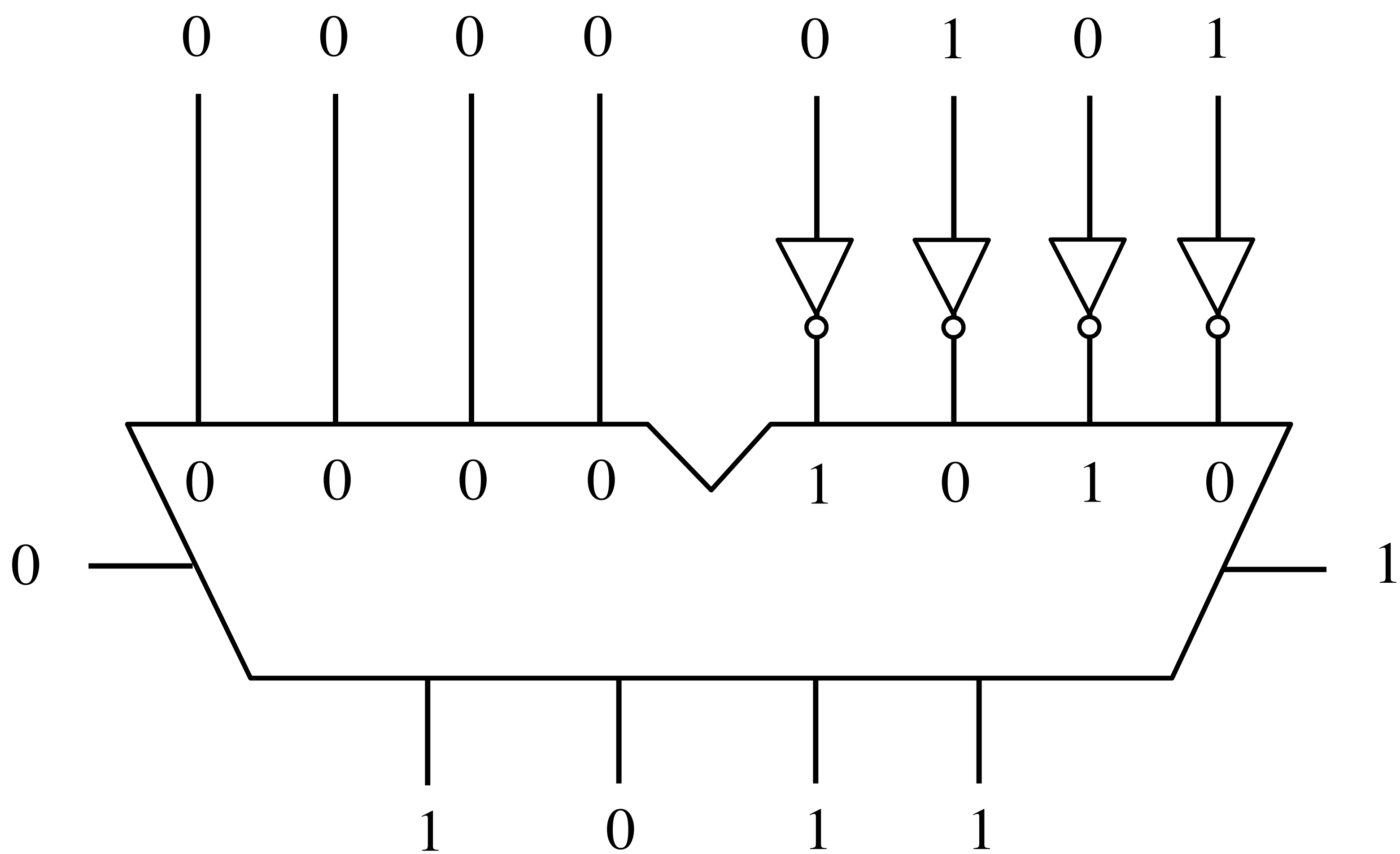
The number circle for 2's complement



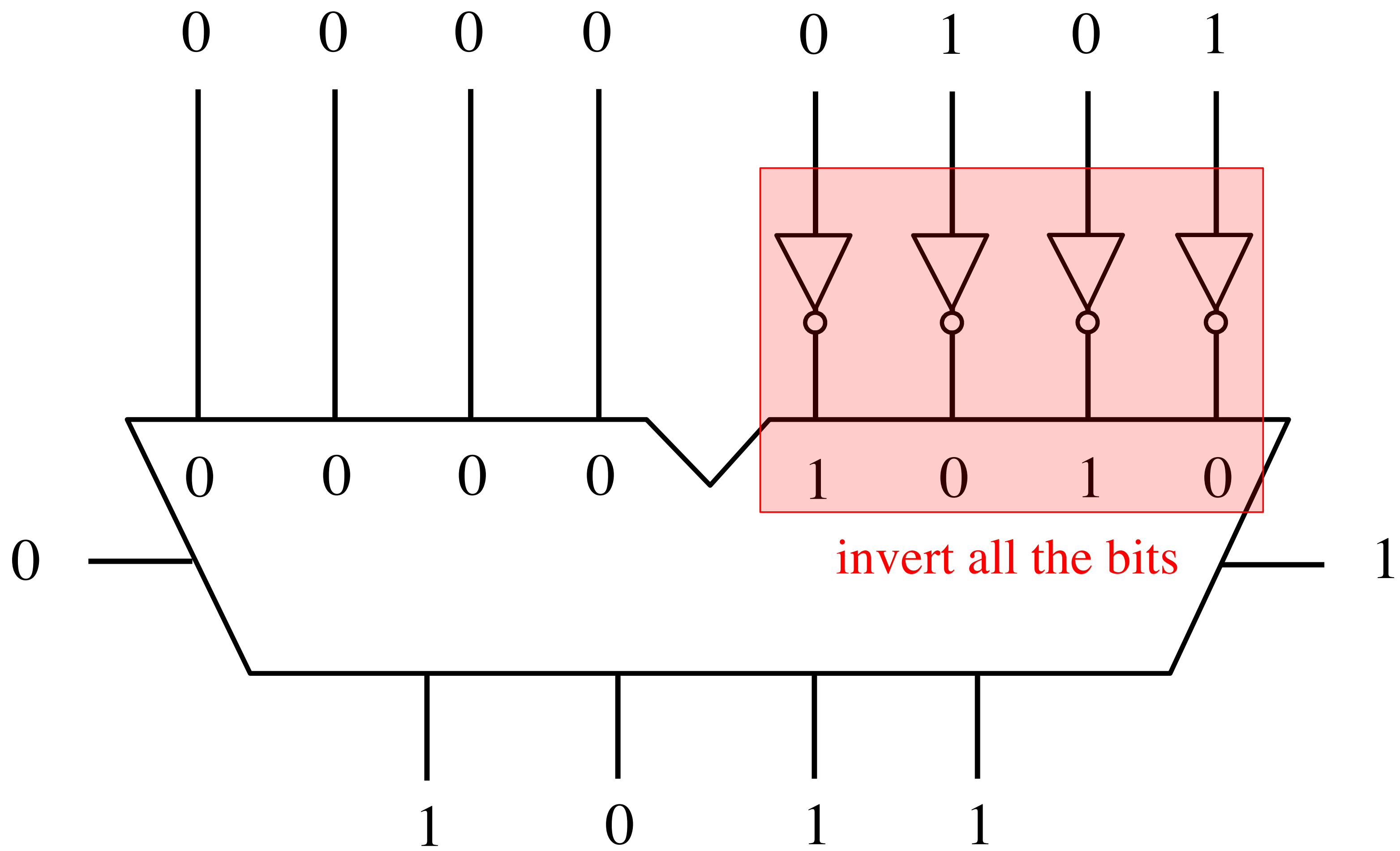
The number circle for 2's complement



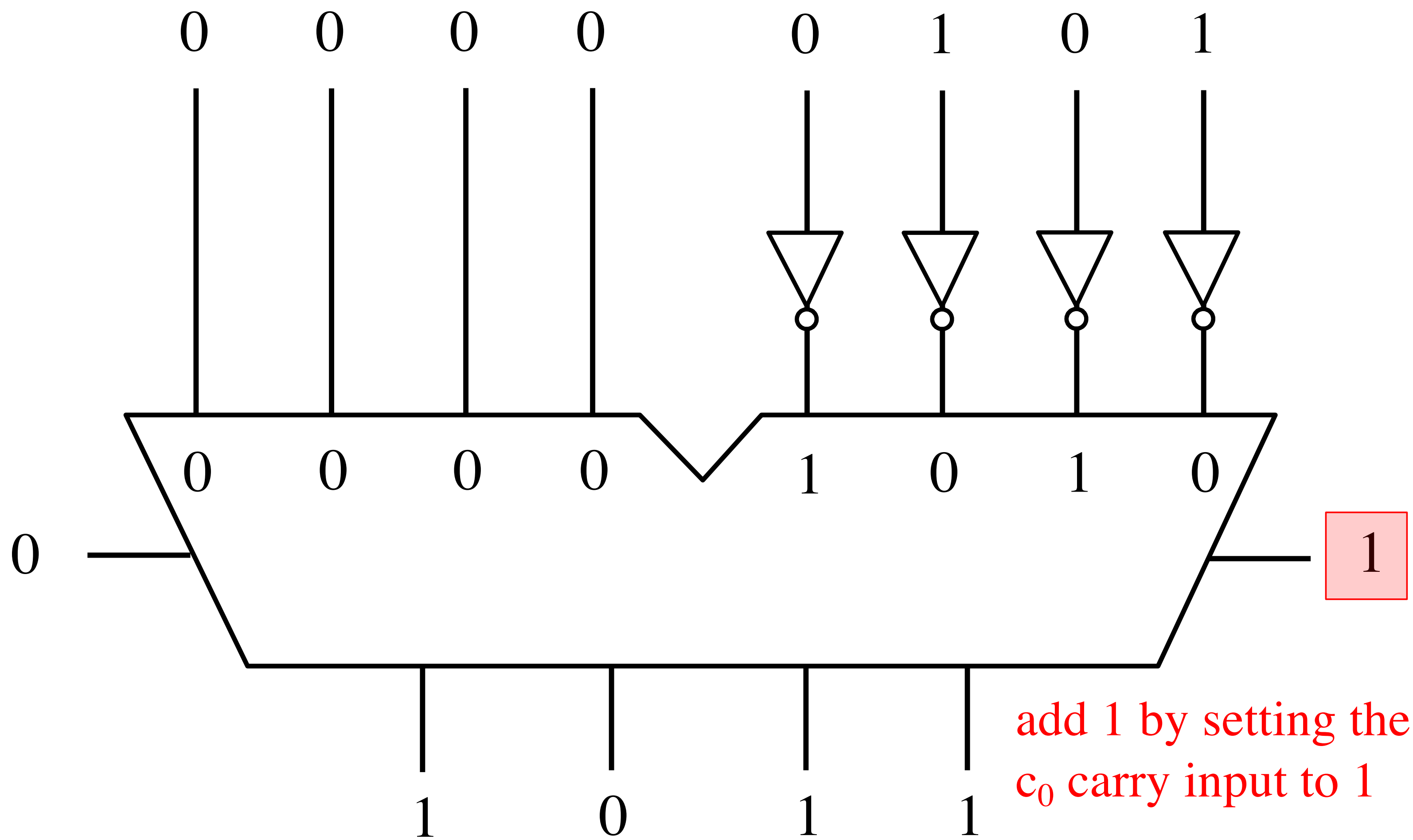
Circuit #2 for negating a number stored in 2's complement representation



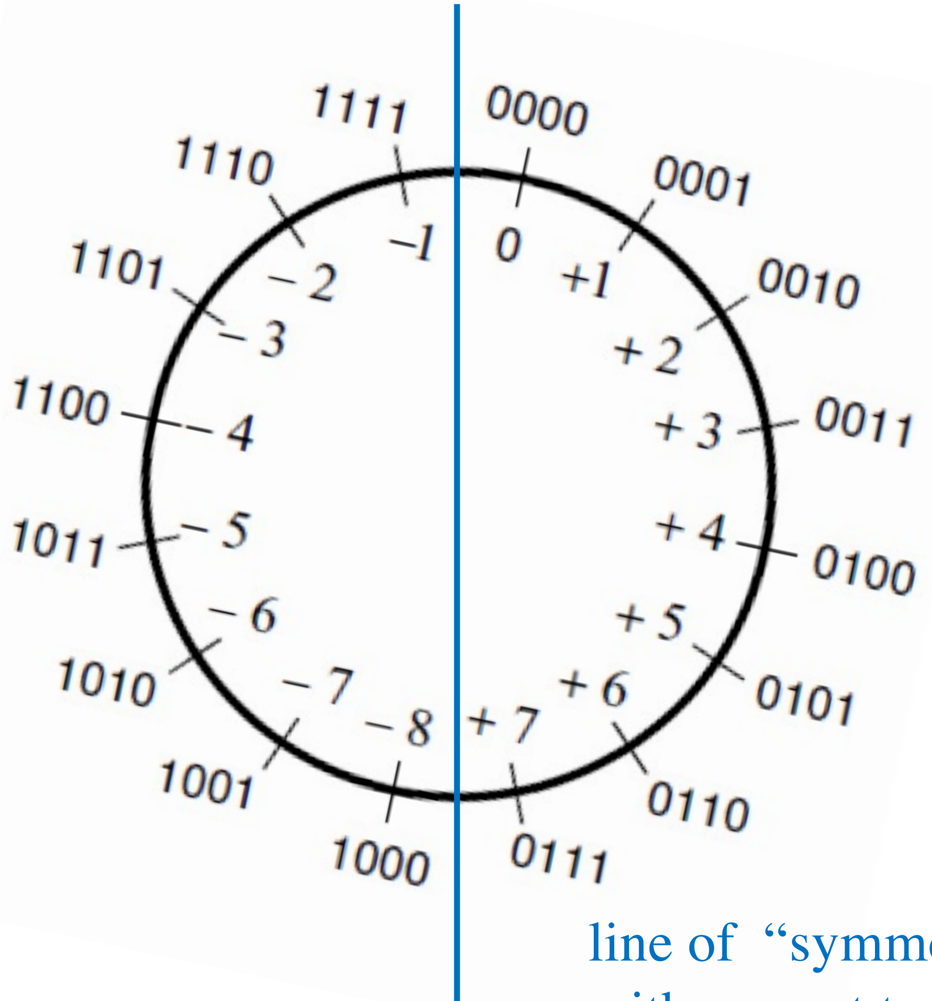
Circuit #2 for negating a number stored in 2's complement representation



Circuit #2 for negating a number stored in 2's complement representation

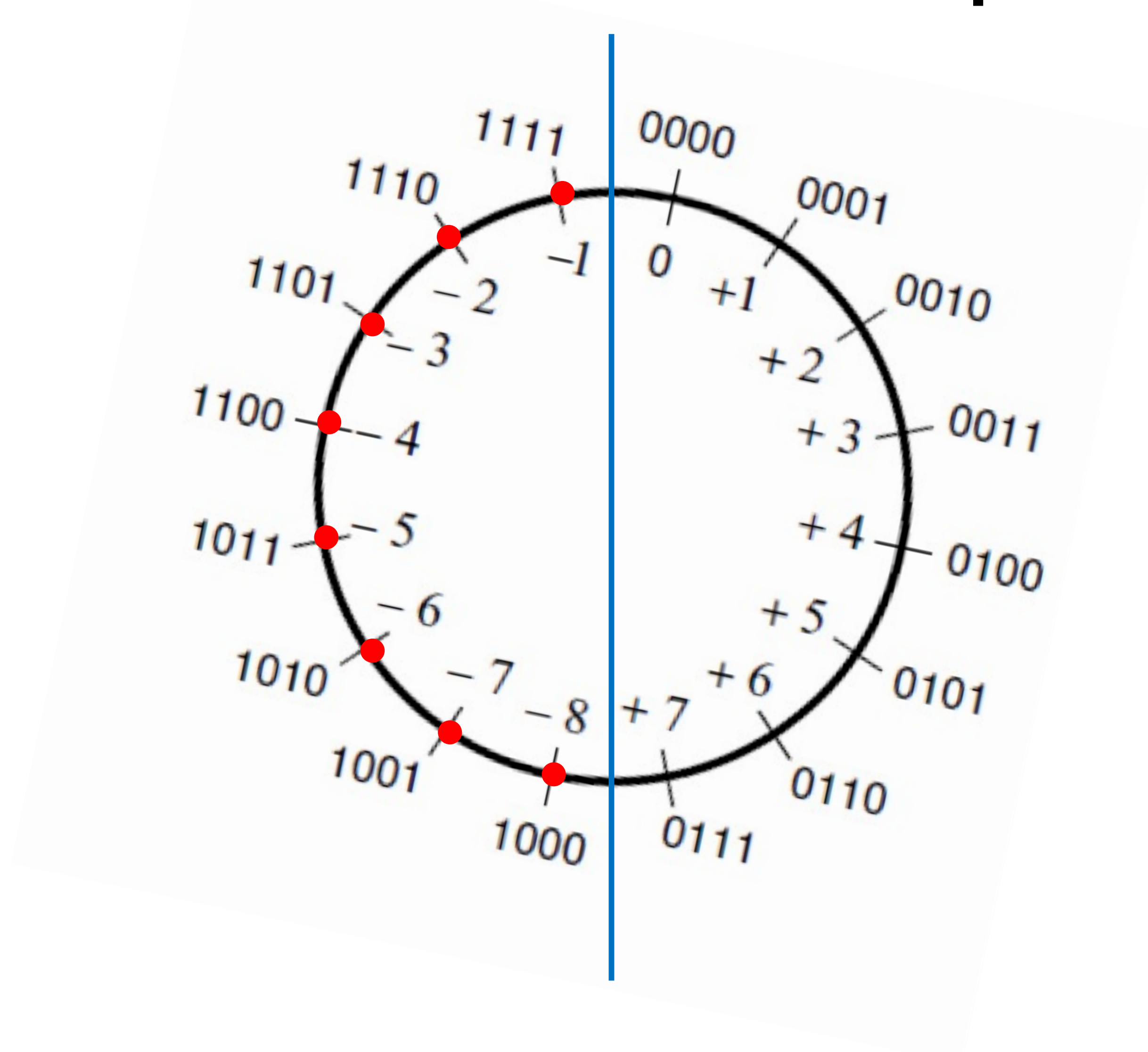


The number circle for 2's complement

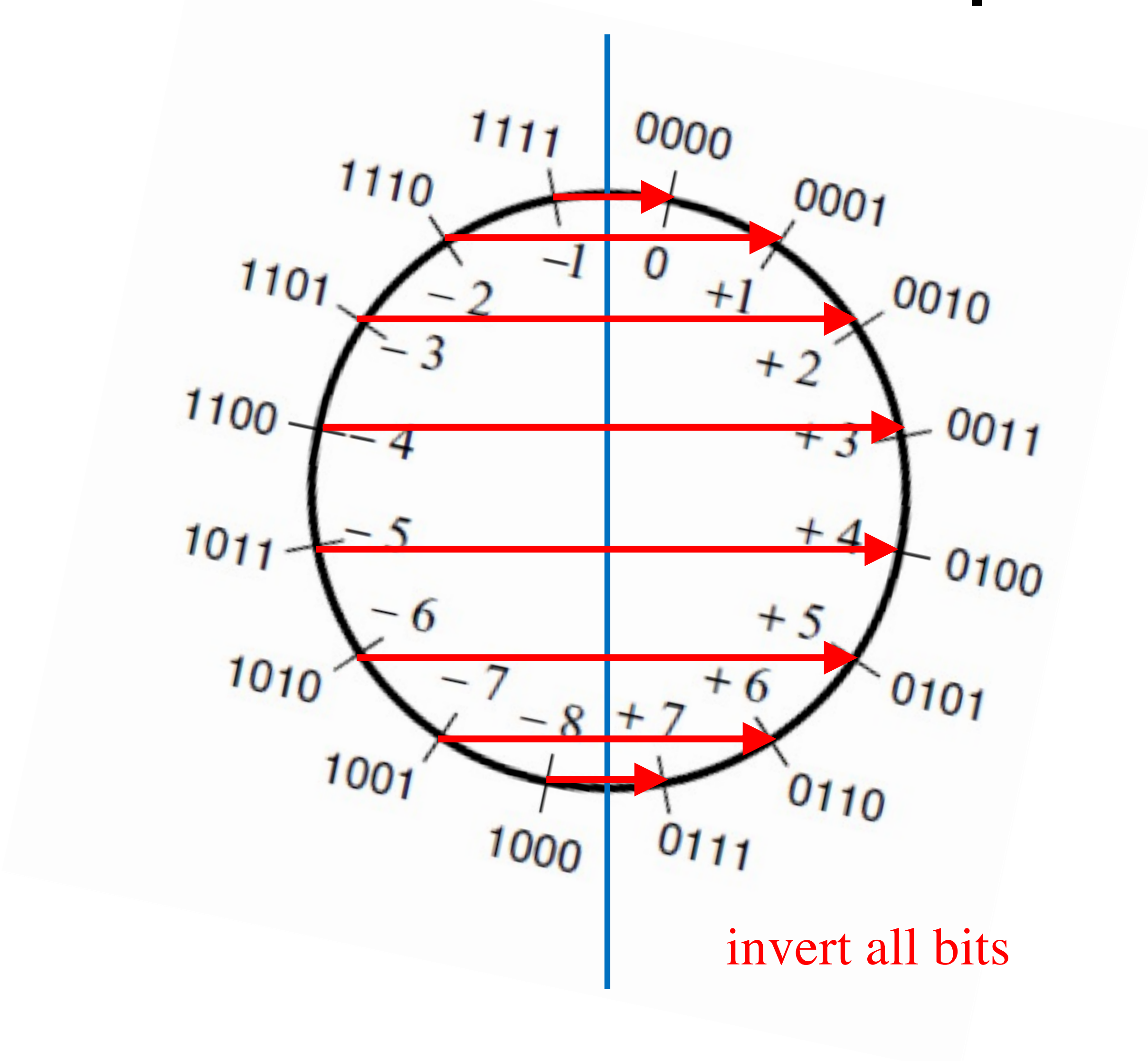


line of "symmetry"
with respect to bit inversion

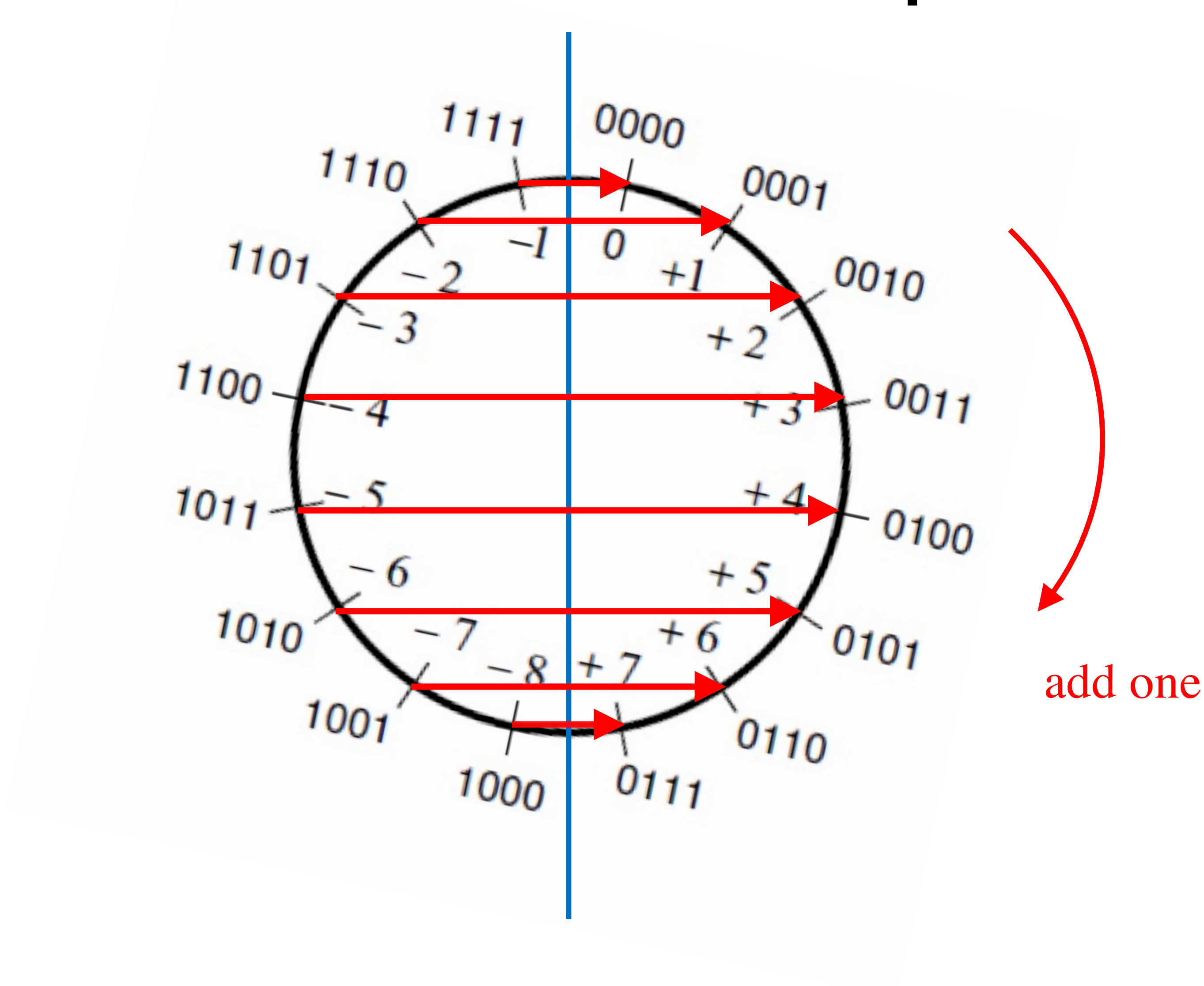
The number circle for 2's complement



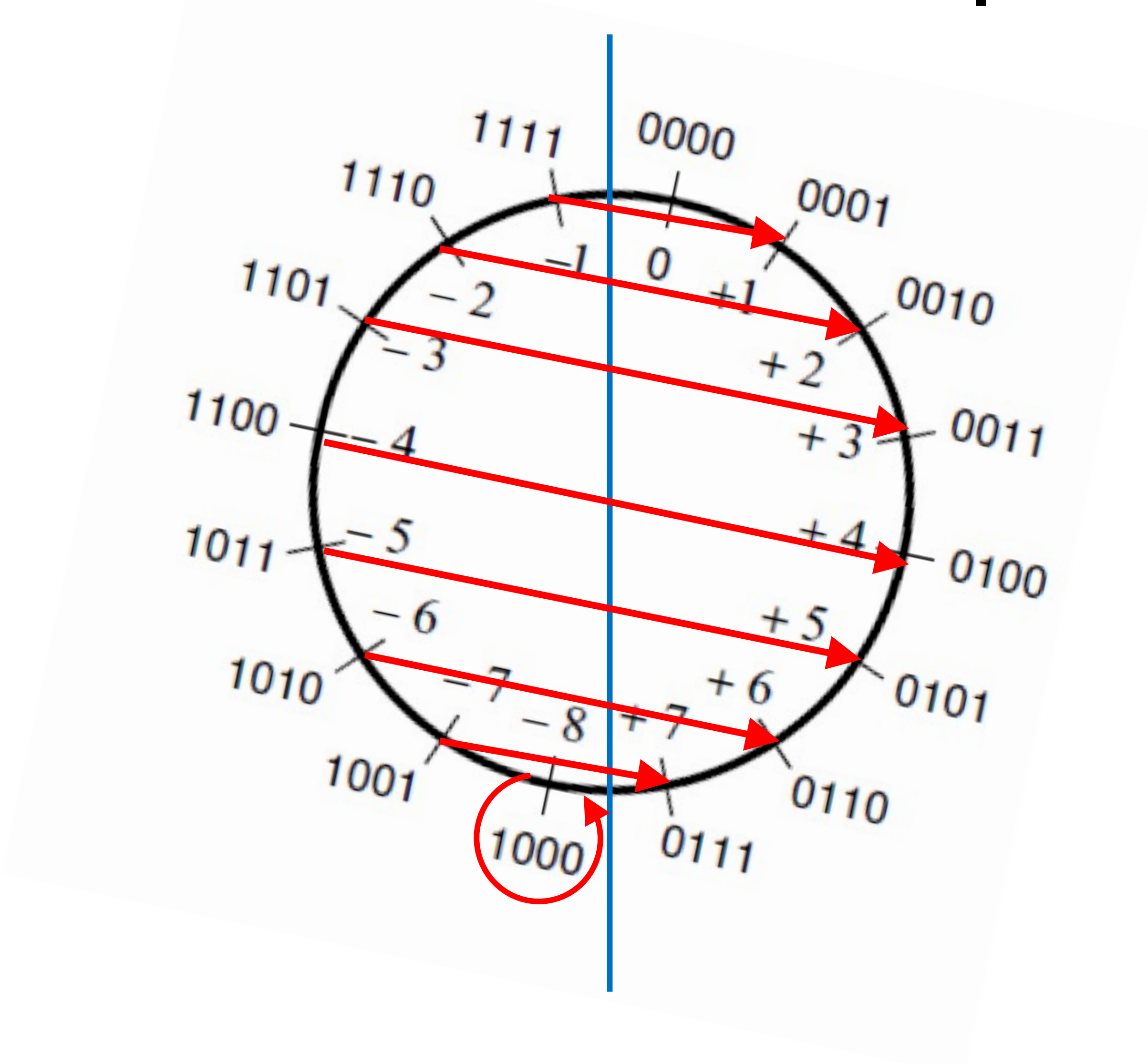
The number circle for 2's complement



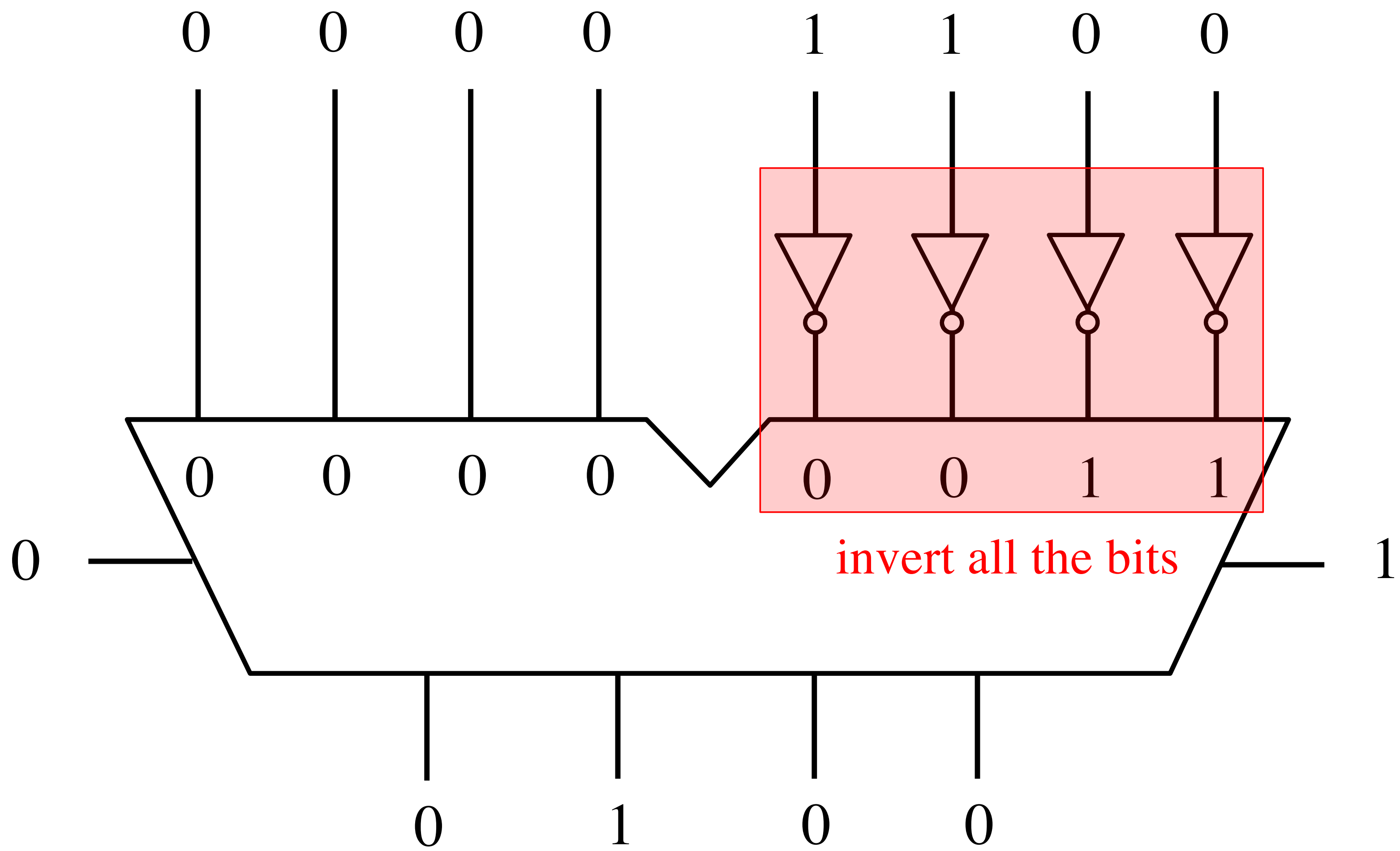
The number circle for 2's complement



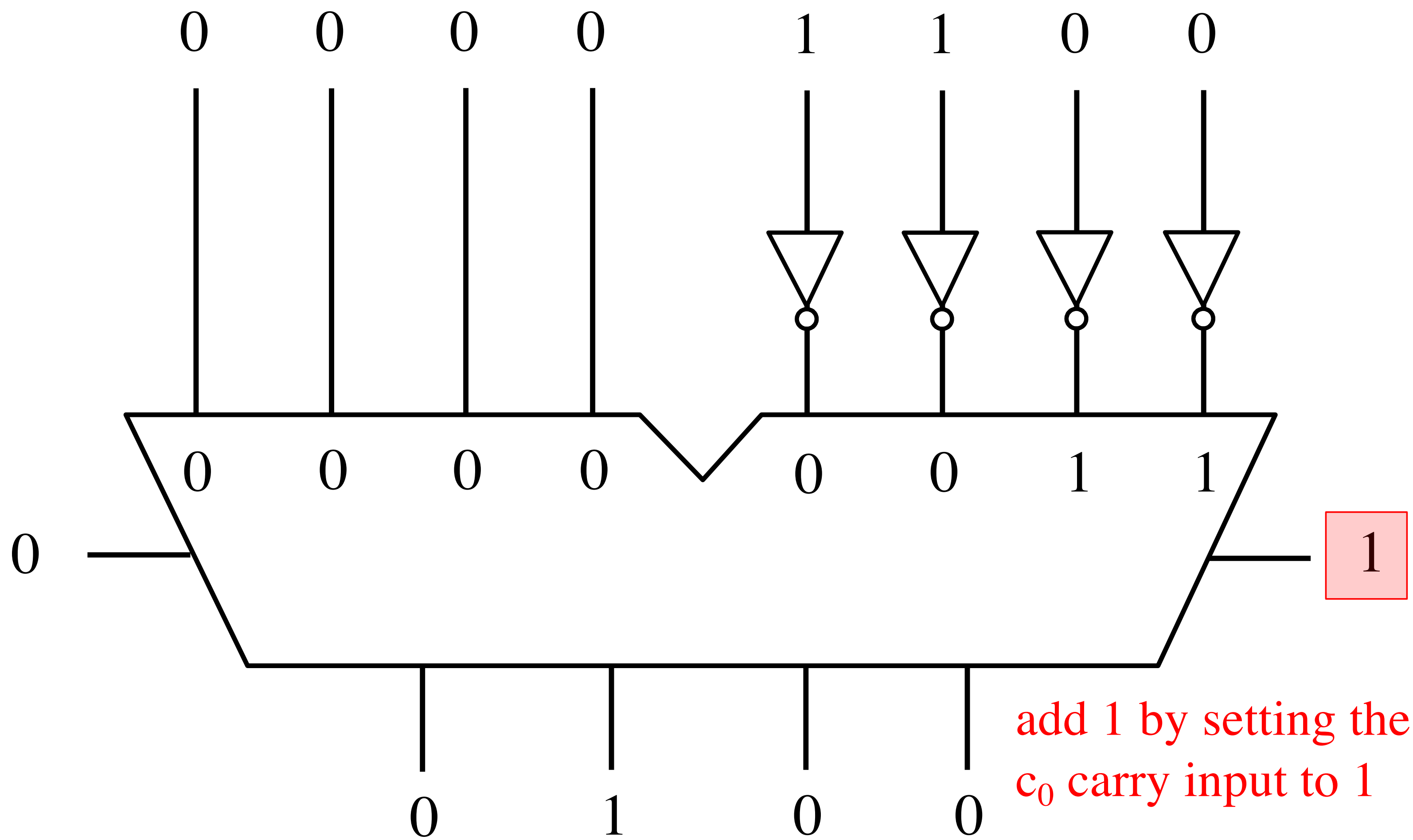
The number circle for 2's complement



Circuit #2 for negating a number stored in 2's complement representation



Circuit #2 for negating a number stored in 2's complement representation



Addition of two numbers stored in 2’s complement representation

There are four cases to consider

- $(+5) \quad + \quad (+2)$
- $(-5) \quad + \quad (+2)$
- $(+5) \quad + \quad (-2)$
- $(-5) \quad + \quad (-2)$

There are four cases to consider

- `(+5)  +  (+2)      positive plus positive`
- `(−5)  +  (+2)      negative plus positive`
- `(+5)  +  (−2)      positive plus negative`
- `(−5)  +  (−2)      negative plus negative`

A) Example of 2's complement addition

$$\begin{array}{rr} (+5) \\ +\ (+2) \\ \hline (+7) \end{array} \qquad \begin{array}{rr} & 0\,1\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

B) Example of 2's complement addition

$$\begin{array}{r} (-5) \\ +\ (+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,1\,0\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

C) Example of 2's complement addition

$$\begin{array}{rr} & (+\,5) \\ + & (-2) \\ \hline & (+\,3) \end{array} \qquad \begin{array}{rr} & 0\,1\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline 1 & 0\,0\,1\,1 \end{array}$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

D) Example of 2's complement addition

$$
\begin{array}{rr}
(-5) & 1\,0\,1\,1 \\
+\ (-2) & +\ 1\,1\,1\,0 \\
\hline
(-7) & 1\ 1\,0\,0\,1
\end{array}
$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Naming Ambiguity: 2's Complement

2's complement has two different meanings:

- representation for signed integer numbers

- algorithm for computing the 2's complement
  (regardless of the representation of the number)

Naming Ambiguity: 2's Complement

2's complement has two different meanings:

- **representation for signed integer numbers**
  in 2's complement

- **algorithm for computing the 2's complement**
  (regardless of the representation of the number)
  take the 2's complement (or negate)

Subtraction of two numbers stored in 2's complement representation

There are four cases to consider

- $(+5) - (+2)$
- $(-5) - (+2)$
- $(+5) - (-2)$
- $(-5) - (-2)$

There are four cases to consider

- `(+5) - (+2)` positive minus positive
- `(−5) - (+2)` negative minus positive
- `(+5) - (−2)` positive minus negative
- `(−5) - (−2)` negative minus negative

There are four cases to consider

- `(+5)   -   (+2)`
- `(-5)   -   (+2)`
- `(+5)   -   (-2)`
- `(-5)   -   (-2)`

There are four cases to consider

- $(+5) - (+2) = (+5) + (-2)$
- $(-5) - (+2) = (-5) + (-2)$
- $(+5) - (-2) = (+5) + (+2)$
- $(-5) - (-2) = (-5) + (+2)$

There are four cases to consider

- $(+5) - (+2) = (+5) + (-2)$
- $(-5) - (+2) = (-5) + (-2)$
- $(+5) - (-2) = (+5) + (+2)$
- $(-5) - (-2) = (-5) + (+2)$

We can change subtraction into addition ...

There are four cases to consider

- $(+5) \quad - \quad (+2) \quad = \quad (+5) \quad + \quad (-2)$
- $(-5) \quad - \quad (+2) \quad = \quad (-5) \quad + \quad (-2)$
- $(+5) \quad - \quad (-2) \quad = \quad (+5) \quad + \quad (+2)$
- $(-5) \quad - \quad (-2) \quad = \quad (-5) \quad + \quad (+2)$

… if we negate the second number.

There are four cases to consider

- $(+5) - (+2) = (+5) + (-2)$
- $(-5) - (+2) = (-5) + (-2)$
- $(+5) - (-2) = (+5) + (+2)$
- $(-5) - (-2) = (-5) + (+2)$

These are the four addition cases
(arranged in a shuffled order)

Example of 2' s complement <u>subtraction</u>

$$
\begin{array}{r}
(+5) \\
-\ (+2) \\
\hline
(+3)
\end{array}
\qquad
\begin{array}{r}
0\,1\,0\,1 \\
-\ 0\,0\,1\,0 \\
\hline
\\
\end{array}
\quad \Longrightarrow \quad
\begin{array}{r}
0\,1\,0\,1 \\
+\ 1\,1\,1\,0 \\
\hline
1\,0\,0\,1\,1
\end{array}
$$

ignore

$\Longrightarrow$ means take the 2's complement (or negate)

[Figure 3.10 from the textbook]

Example of 2' s complement subtraction

$$
\begin{array}{r}
(+5) \\
-\ (+2) \\
\hline
(+3)
\end{array}
\qquad
\begin{array}{r}
0\,1\,0\,1 \\
-\ 0\,0\,1\,0 \\
\hline
\\
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{r}
0\,1\,0\,1 \\
+\ 1\,1\,1\,0 \\
\hline
1\,0\,0\,1\,1
\end{array}
$$

ignore

Notice that the minus changes to a plus.

$\Longrightarrow$ means take the 2's complement (or negate)

[Figure 3.10 from the textbook]

Example of 2' s complement <u>subtraction</u>

$$
\begin{array}{r}
(+5) \\
-\ (+2) \\
\hline
(+3)
\end{array}
\qquad
\begin{array}{r}
0\,1\,0\,1 \\
-\ 0\,0\,1\,0 \\
\hline
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{r}
0\,1\,0\,1 \\
+\ 1\,1\,1\,0 \\
\hline
1\,0\,0\,1\,1
\end{array}
$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2' s complement <u>subtraction</u>

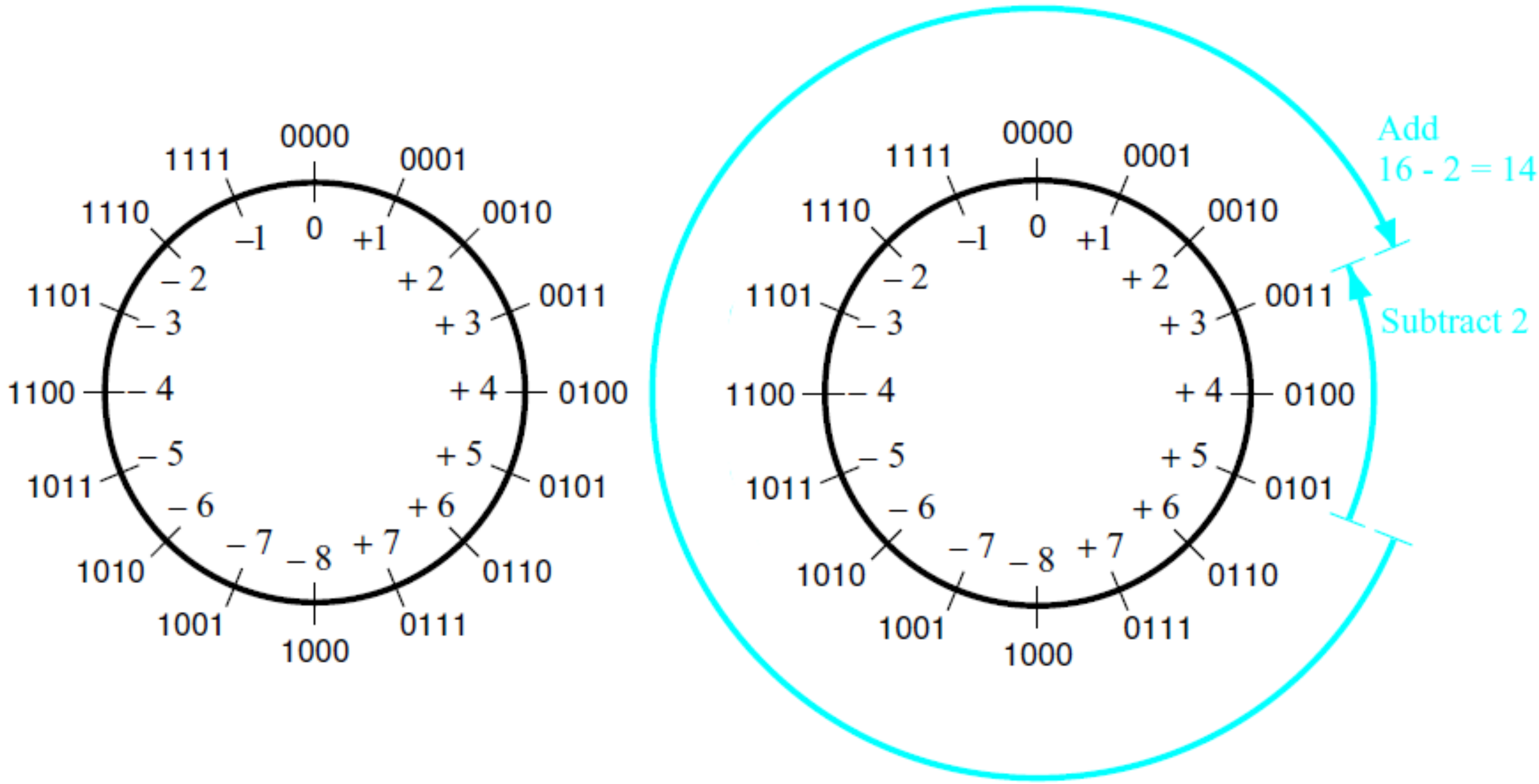
$$\begin{array}{r} (+5) \\ -\,(+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\;0\,0\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 0\,1\,0\,1 \\ +\;1\,1\,1\,0 \\ \hline 1\;0\,0\,1\,1 \end{array}$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Graphical interpretation of four-bit 2's complement numbers



(a) The number circle (b) Subtracting 2 by adding its 2's complement

[Figure 3.11 from the textbook]

Example of 2' s complement subtraction

$$\begin{array}{r} (-5) \\ -\ (+2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 1\,0\,1\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\ 1\,0\,0\,1 \end{array}$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2's complement subtraction

$$\begin{array}{r} (+5) \\ -\ (-2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 1\,1\,1\,0 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 0\,1\,0\,1 \\ +\ 0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2’ s complement subtraction

$$
\begin{array}{r} (-5) \\ -\ (-2) \\ \hline (-3) \end{array}
\qquad
\begin{array}{r} 1\,0\,1\,1 \\ -\ 1\,1\,1\,0 \\ \hline \end{array}
\quad \Longrightarrow \quad
\begin{array}{r} 1\,0\,1\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,1\,0\,1 \end{array}
$$

$b_3b_2b_1b_0$	2’s complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Taking the 2's complement negates the number

decimal	$b_3\,b_2\,b_1\,b_0$	take the 2's complement	$b_3\,b_2\,b_1\,b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

Taking the 2's complement negates the number

decimal	$b_3\, b_2\, b_1\, b_0$	take the 2's complement	$b_3\, b_2\, b_1\, b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

This is an exception

Taking the 2' s complement negates the number

decimal	$b_3\,b_2\,b_1\,b_0$	take the 2's complement	$b_3\,b_2\,b_1\,b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

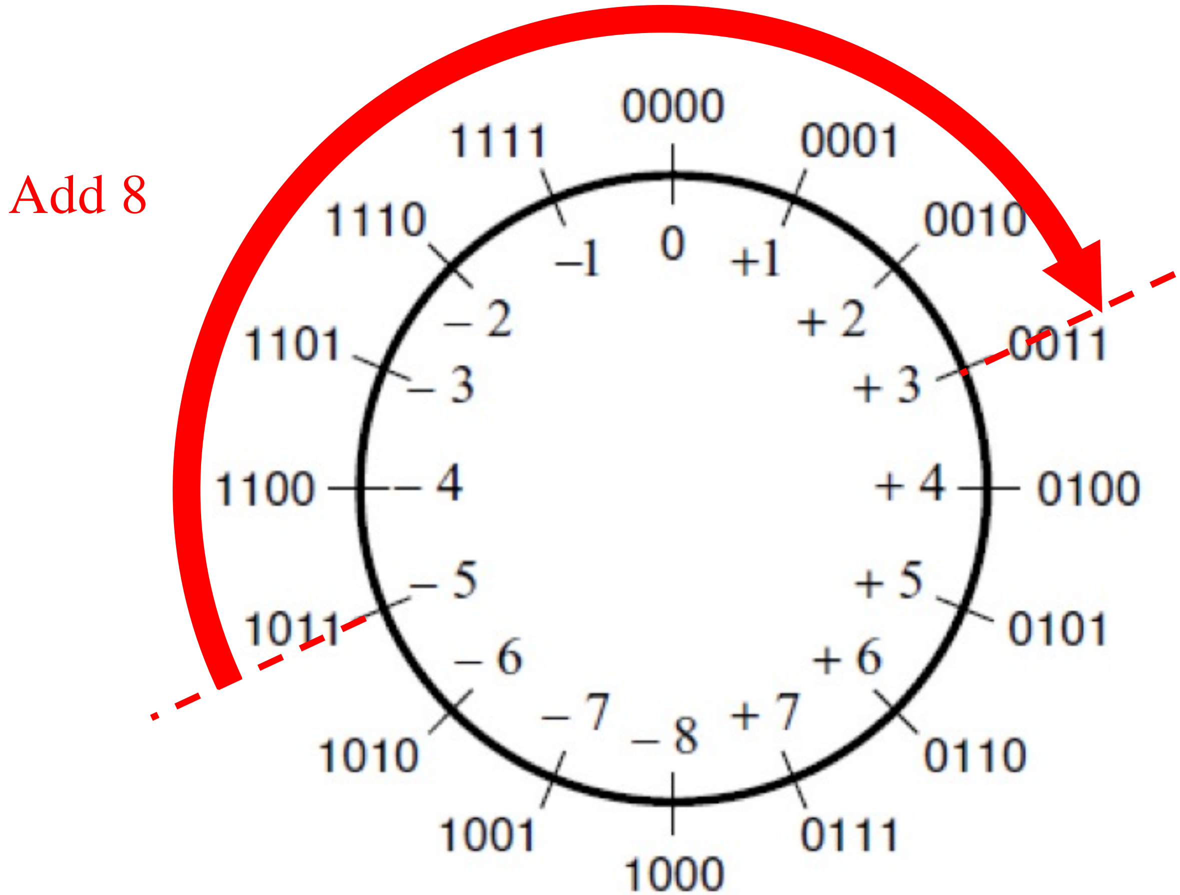
And this one too.

But that exception does not matter

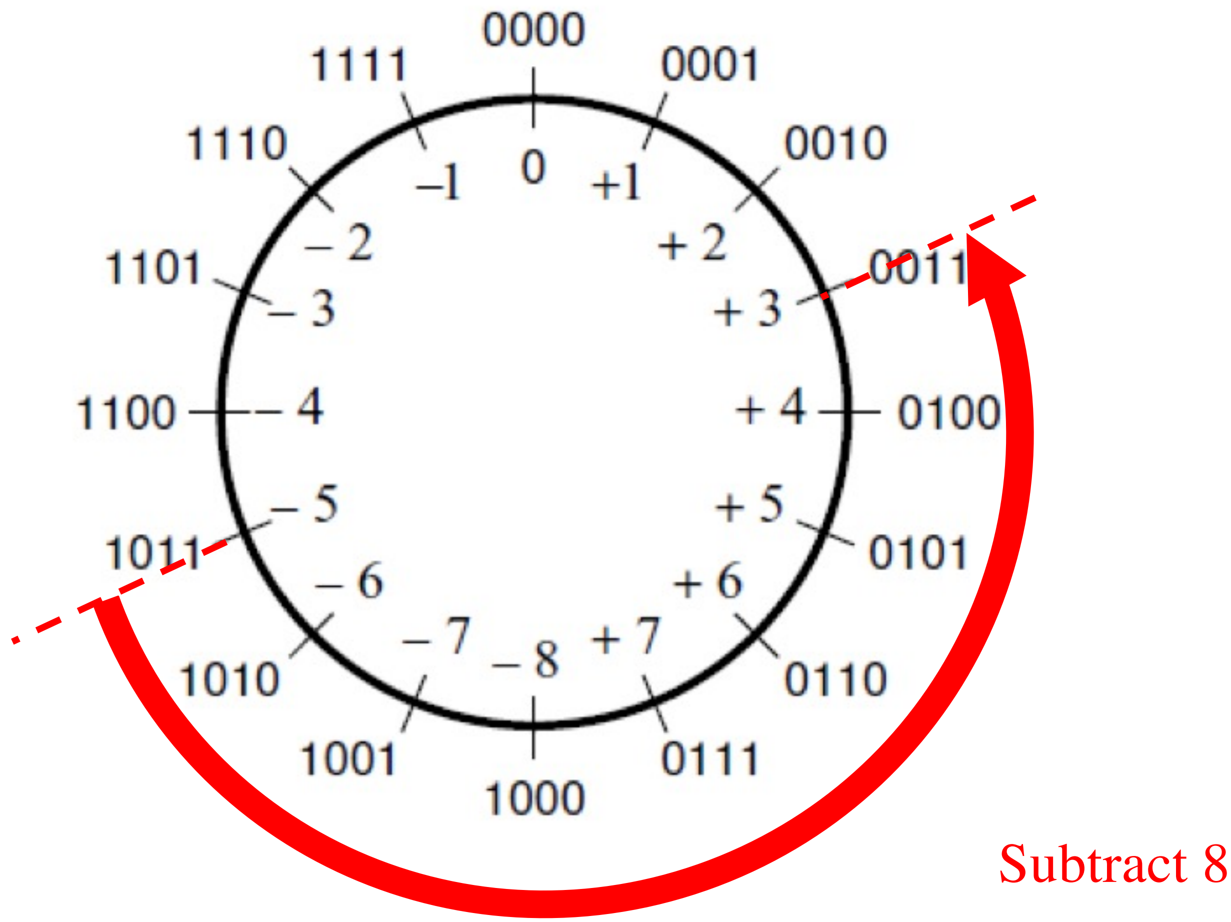
$$\begin{array}{r} (-5) \\ -\ (-8) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 1\,0\,0\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 1\,0\,1\,1 \\ +\ 1\,0\,0\,0 \\ \hline 1\,0\,0\,1\,1 \end{array}$$

ignore

But that exception does not matter



But that exception does not matter



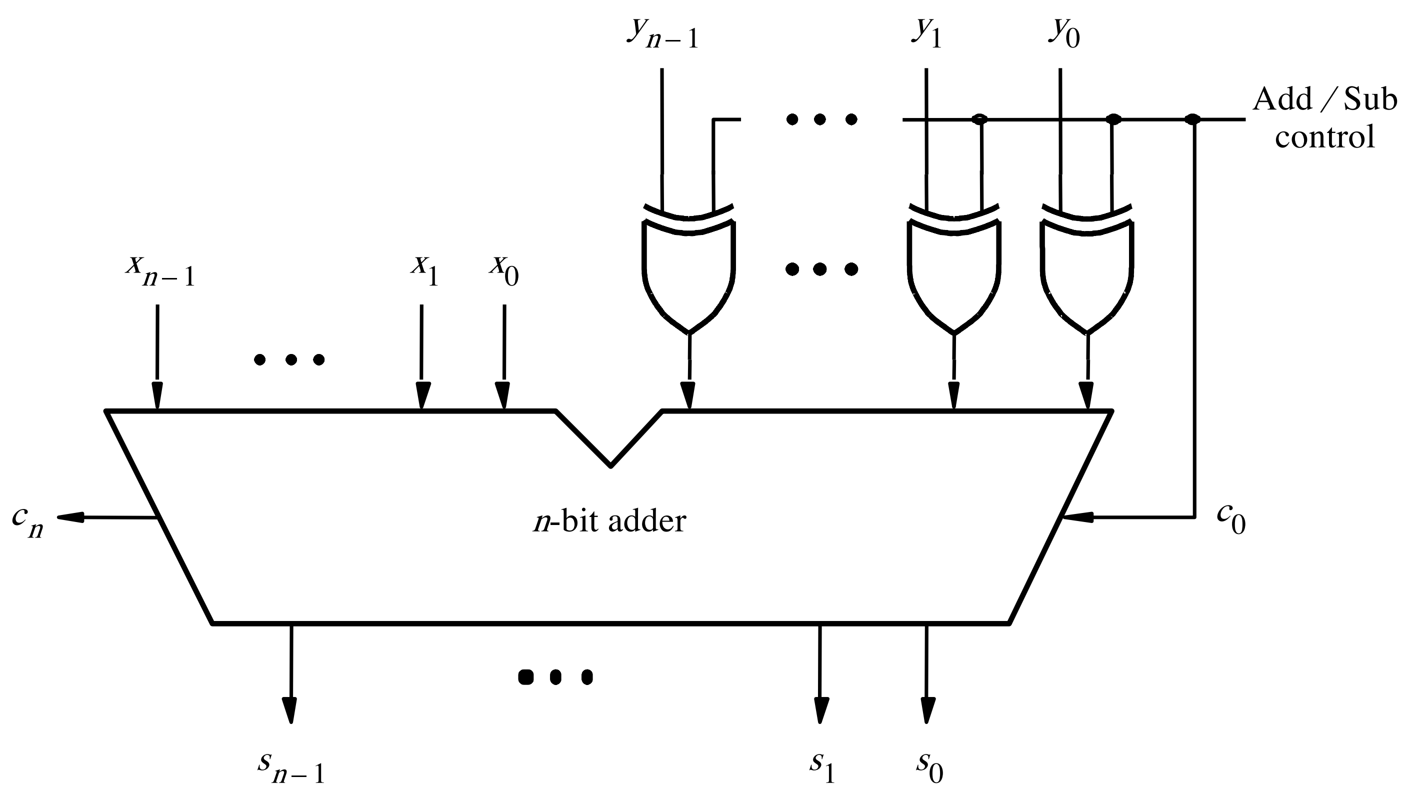
Take-Home Message

Take-Home Message

- **Subtraction can be performed by simply negating the second number and adding it to the first, regardless of the signs of the two numbers.**

- **Thus, the same adder circuit can be used to perform <u>both addition and subtraction</u> !!!**

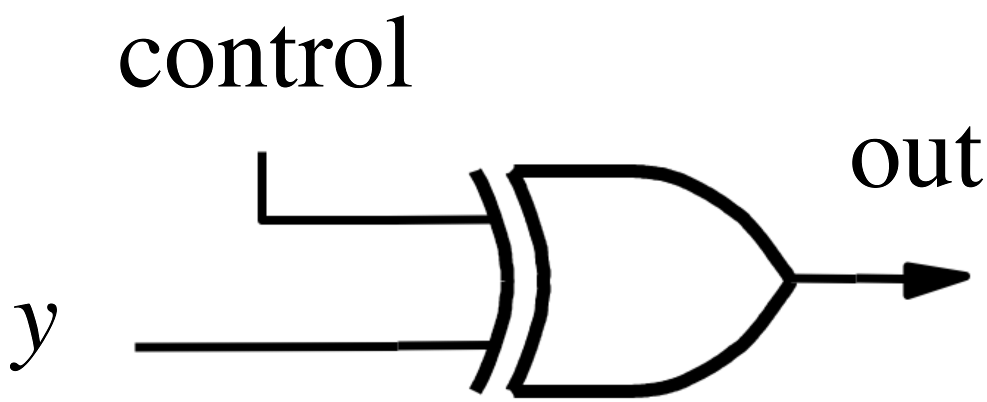
Adder/subtractor unit



[Figure 3.12 from the textbook]

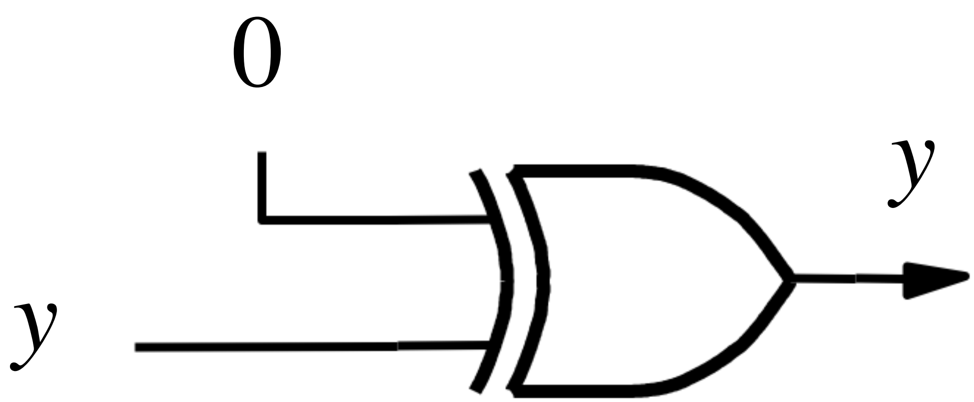
XOR Tricks

control	y	out
0	0	0
0	1	1
1	0	1
1	1	0



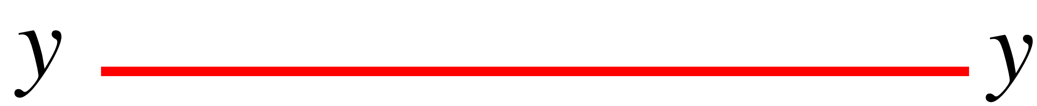
XOR as a repeater

control	y	out
0	0	0
0	1	1



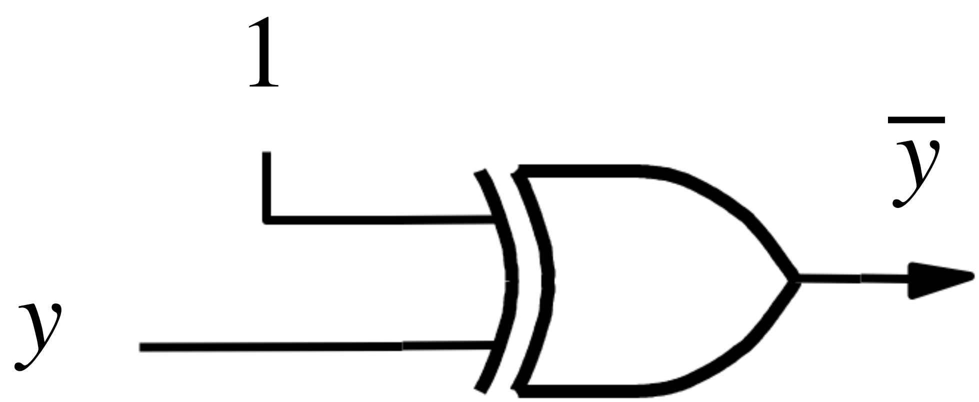
XOR as a repeater

control	y	out
0	0	0
0	1	1



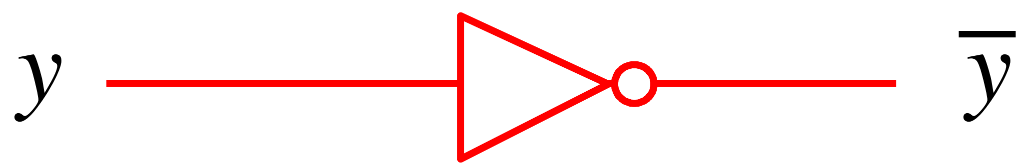
XOR as an inverter

control	y	out
1	0	1
1	1	0

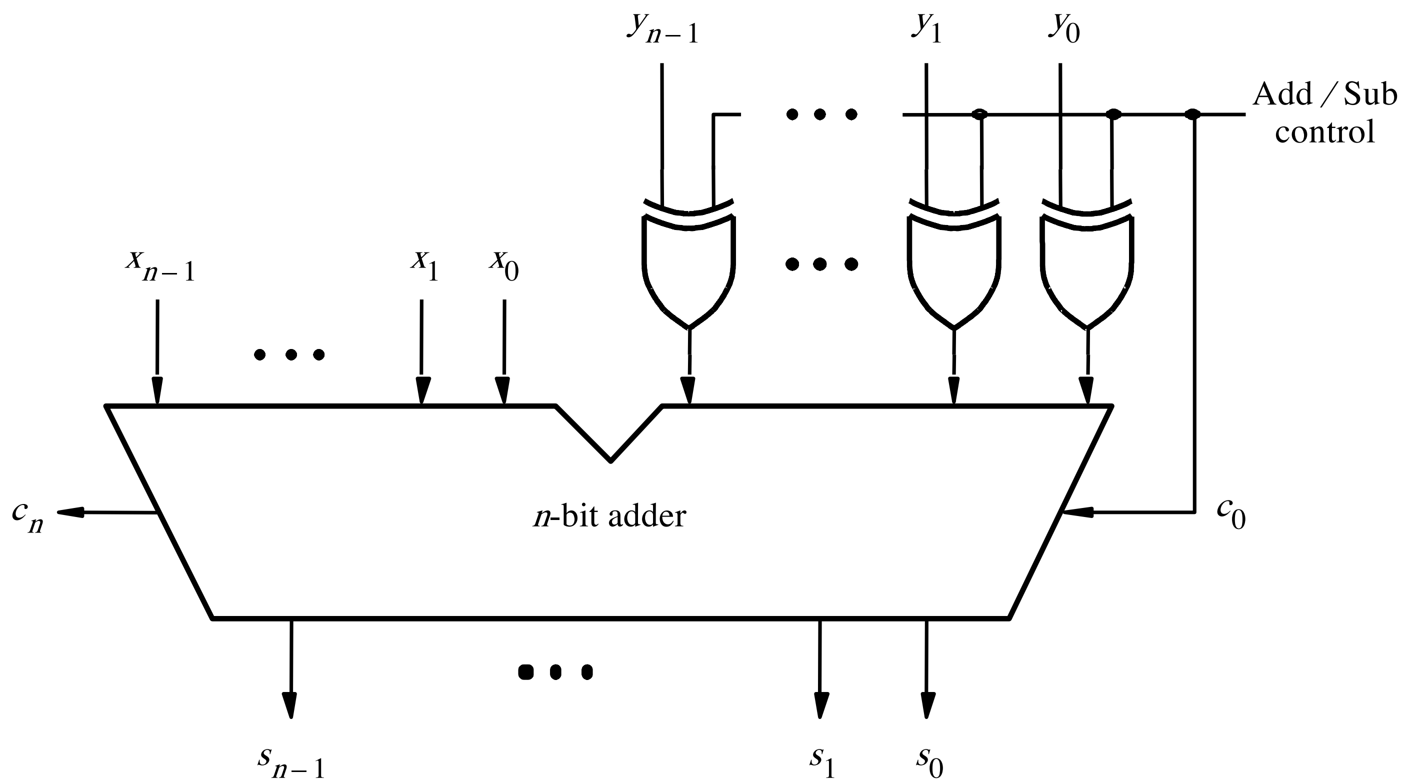


XOR as an inverter

control	y	out
1	0	1
1	1	0

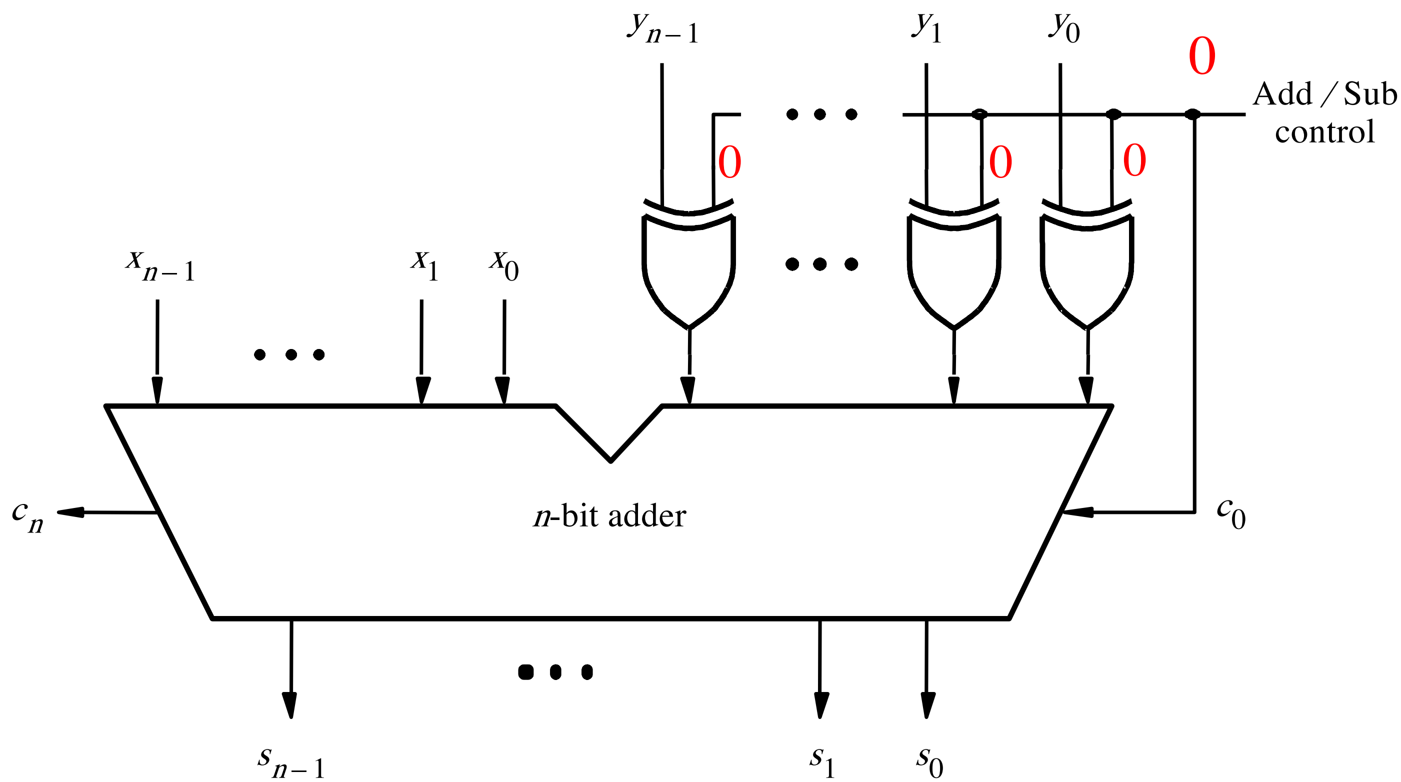


Addition: when control = 0



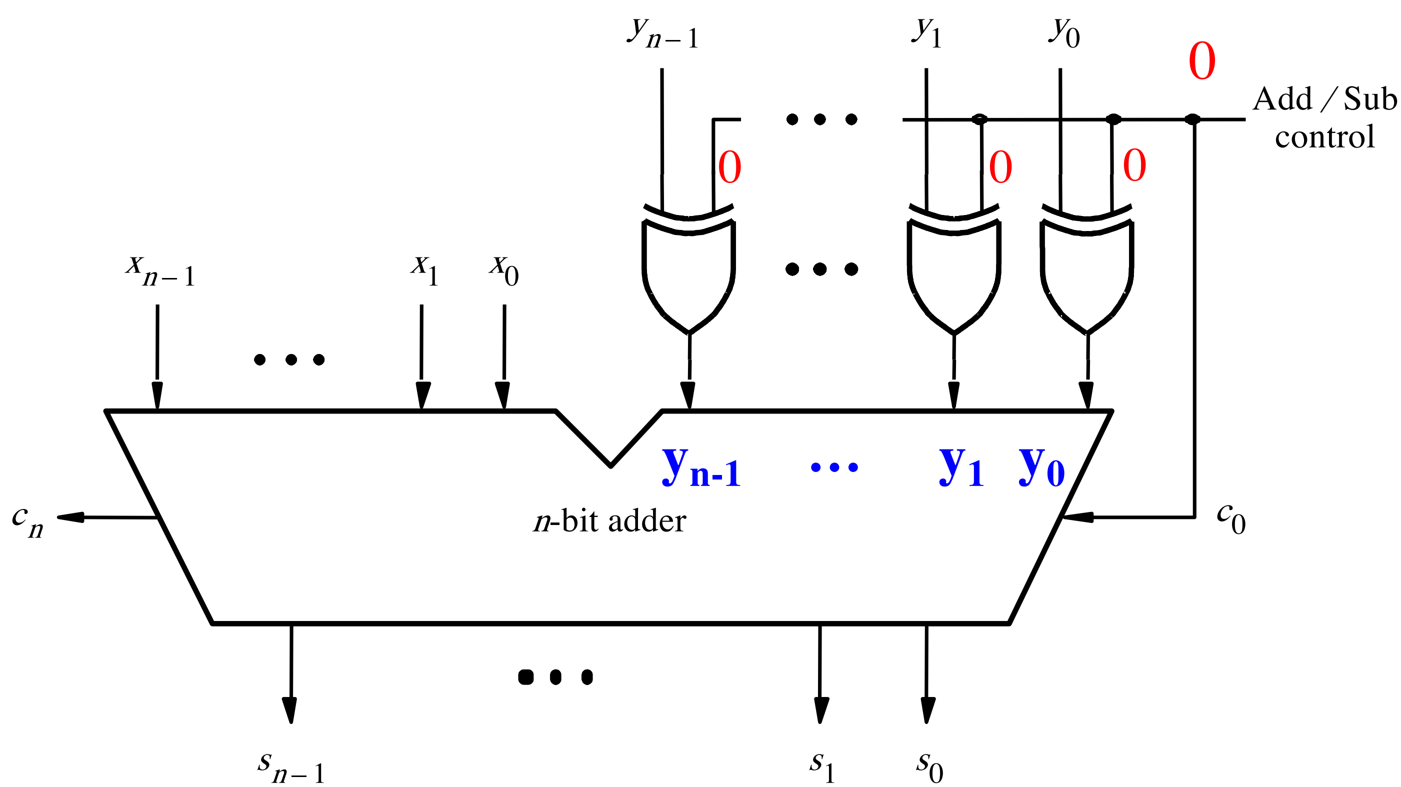
[Figure 3.12 from the textbook]

Addition: when control = 0



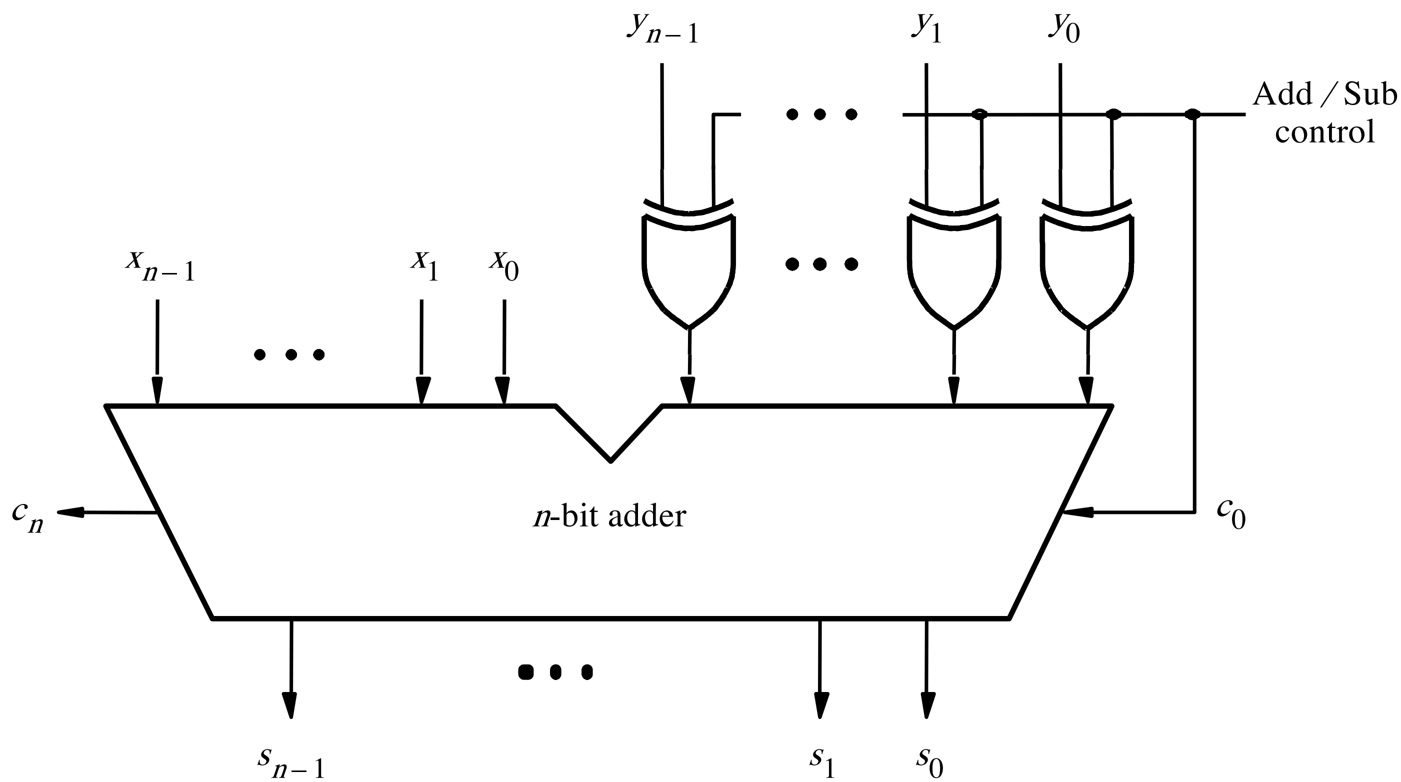
[Figure 3.12 from the textbook]

Addition: when control = 0



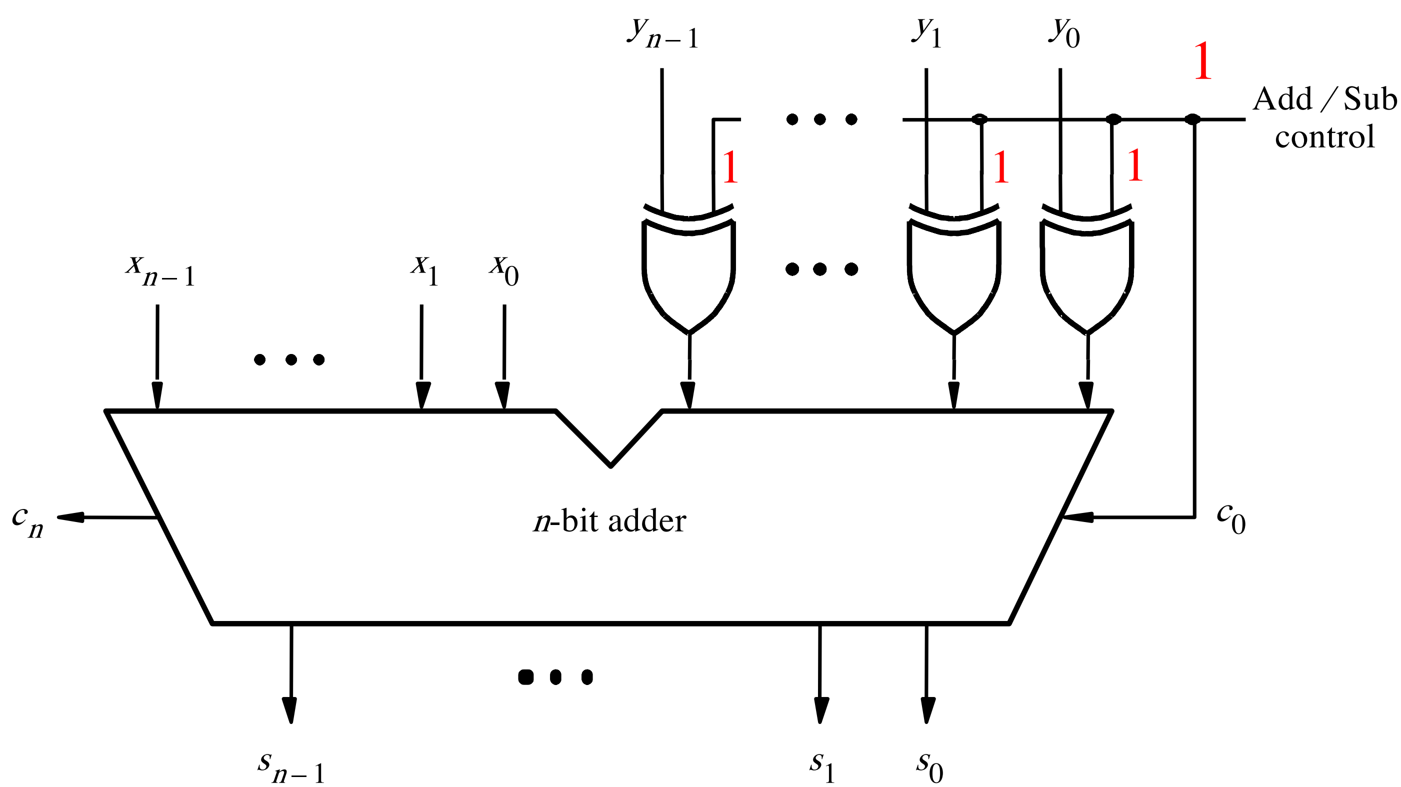
[Figure 3.12 from the textbook]

Subtraction: when control = 1



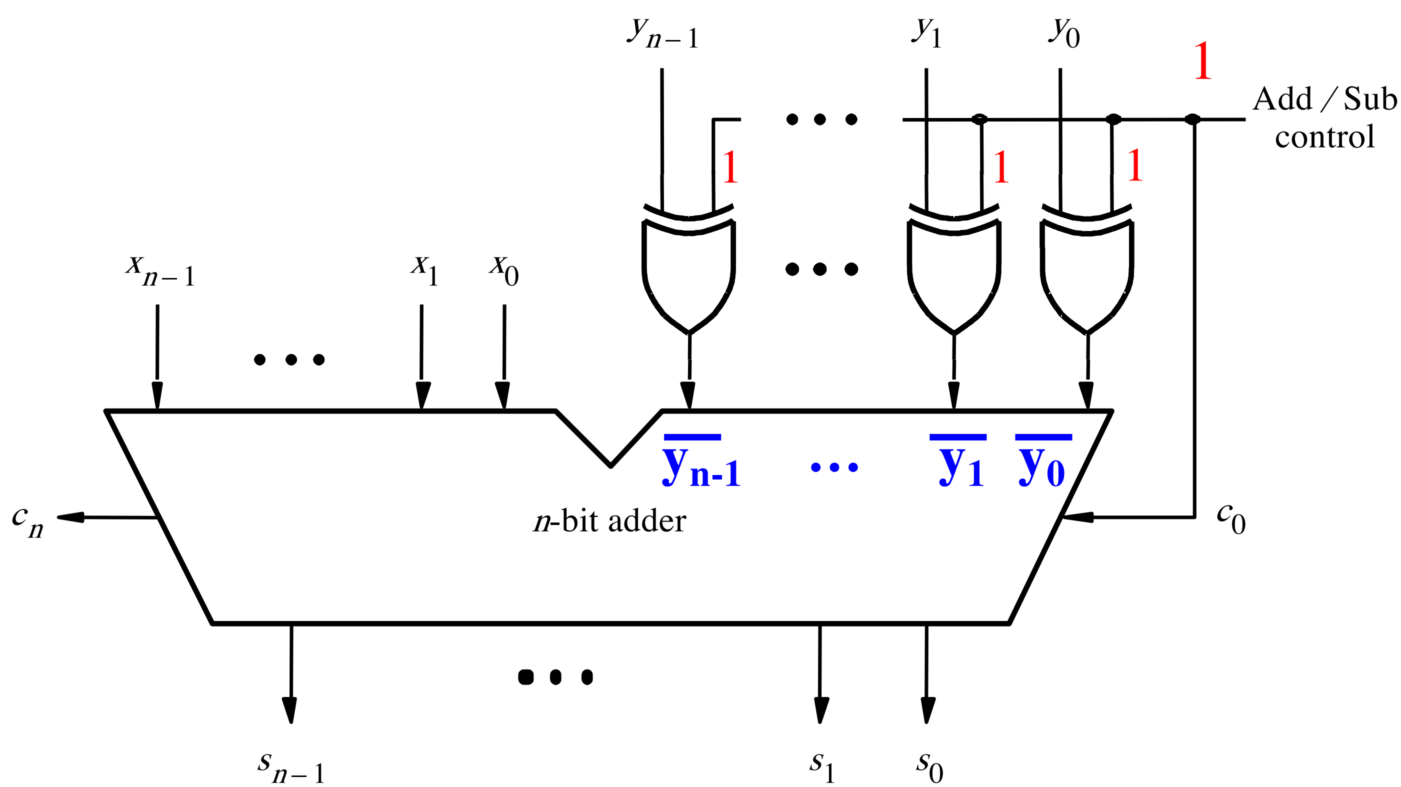
[Figure 3.12 from the textbook]

Subtraction: when control = 1



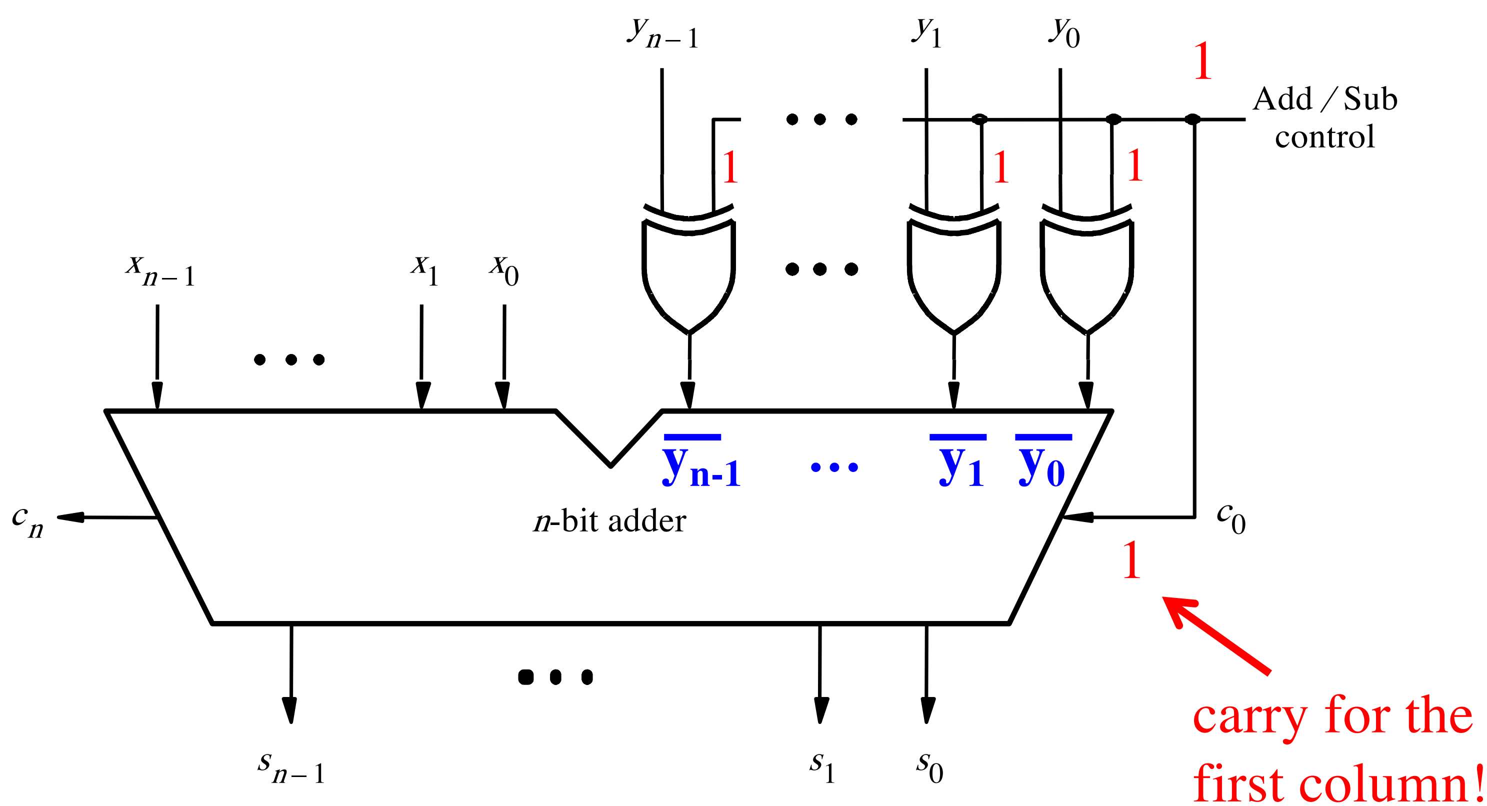
[Figure 3.12 from the textbook]

Subtraction: when control = 1



[Figure 3.12 from the textbook]

Subtraction: when control = 1

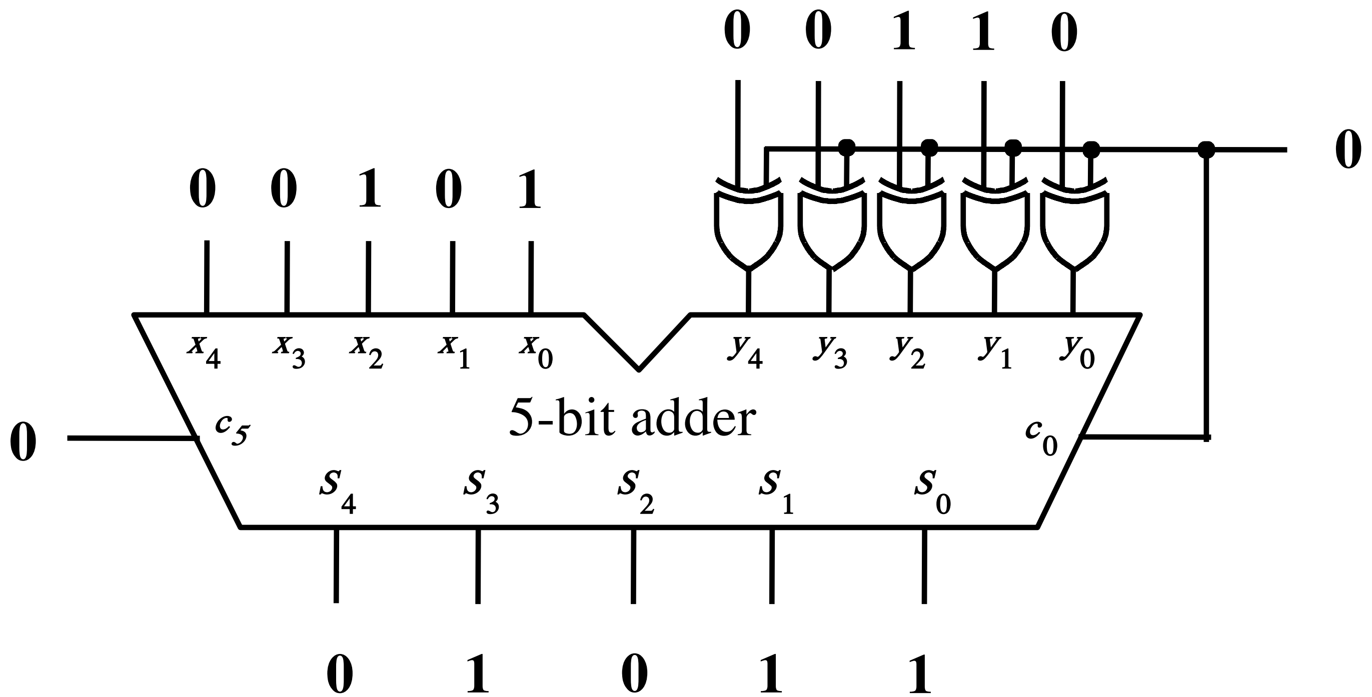


[Figure 3.12 from the textbook]

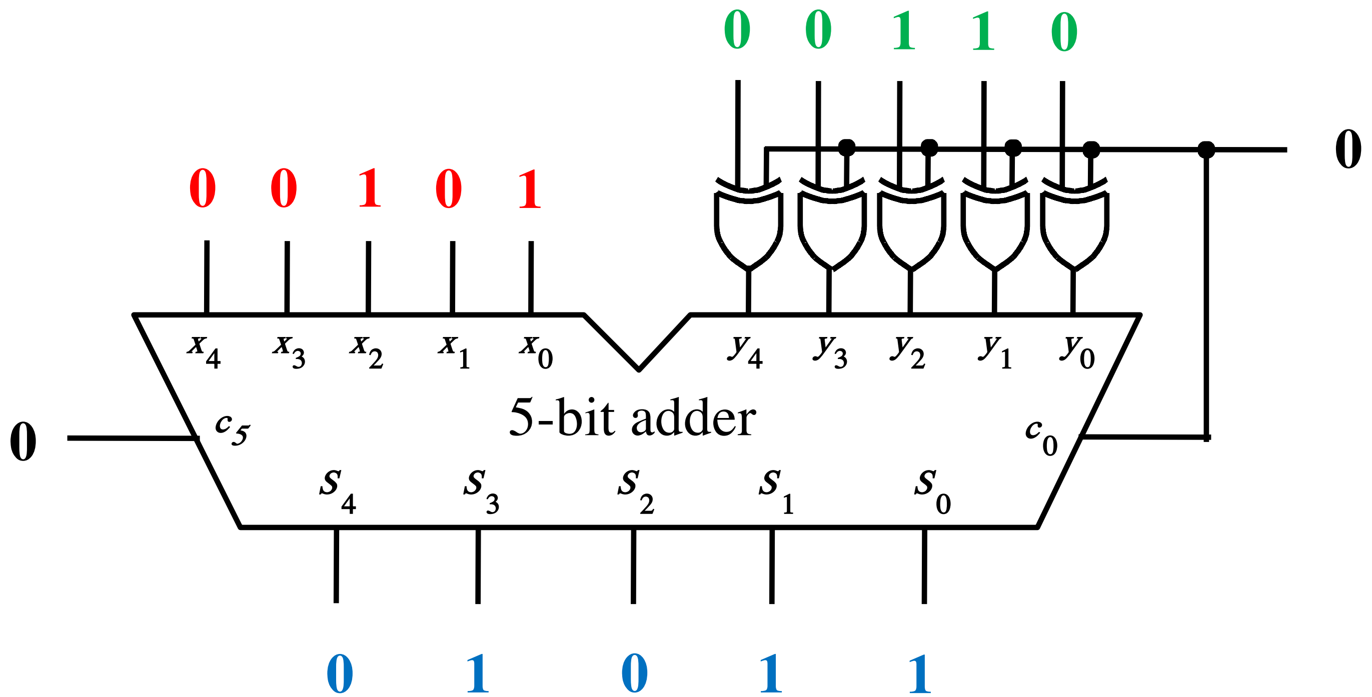
Addition Examples:

all inputs and outputs are given in 2's complement representation

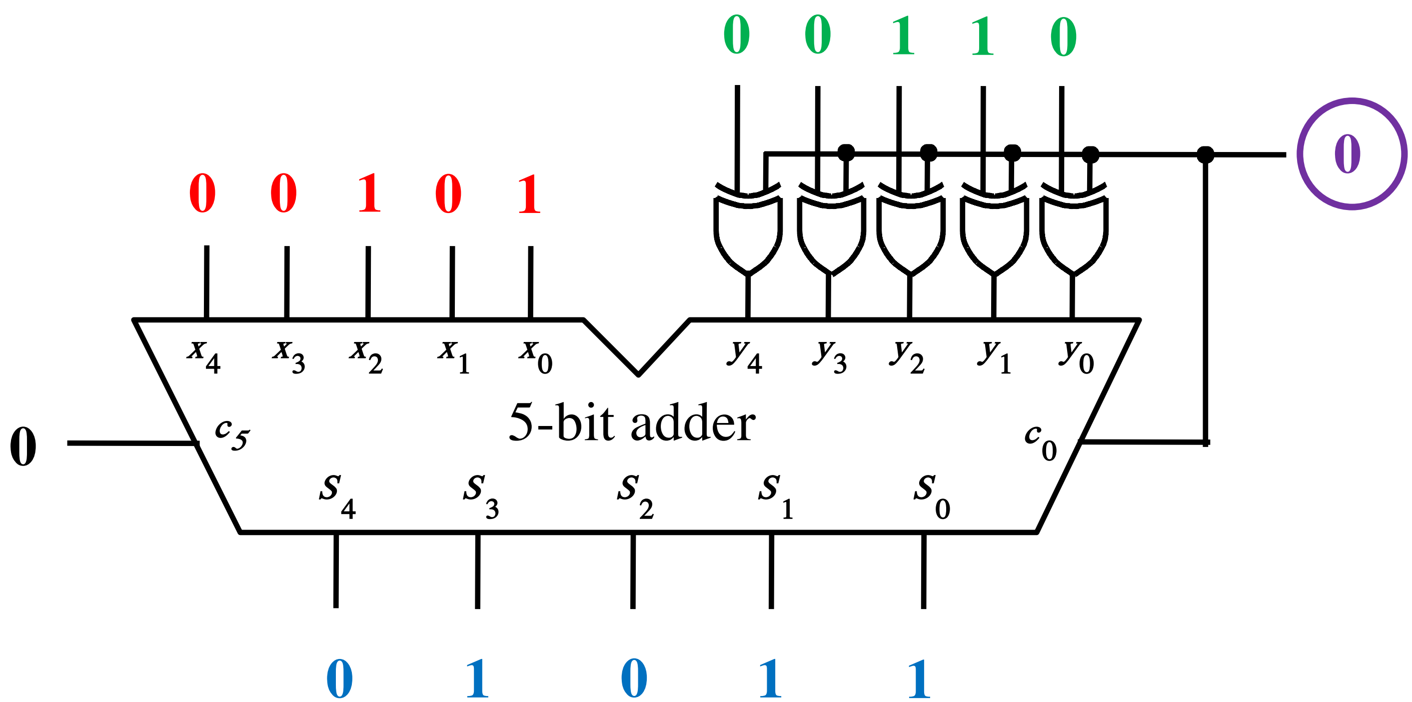
Addition: 5 + 6 = 11



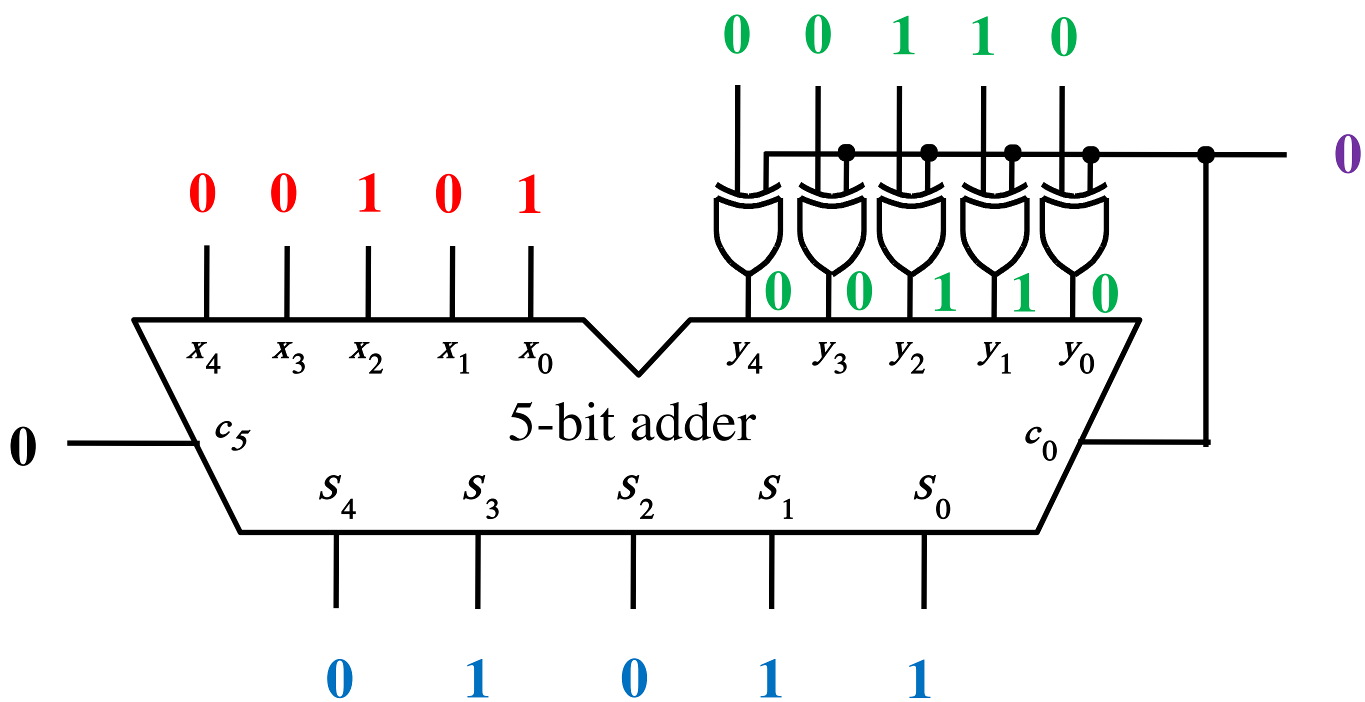
Addition: 5 + 6 = 11



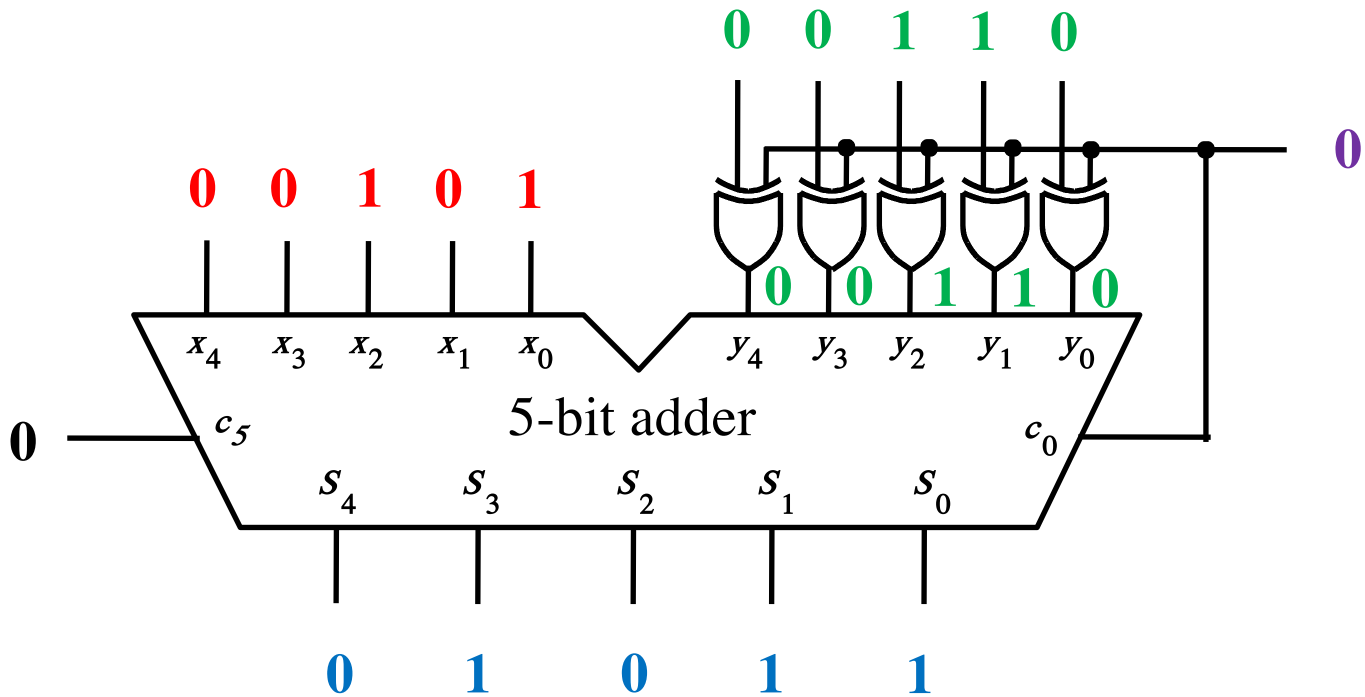
Addition: 5 + 6 = 11



Addition: 5 + 6 = 11

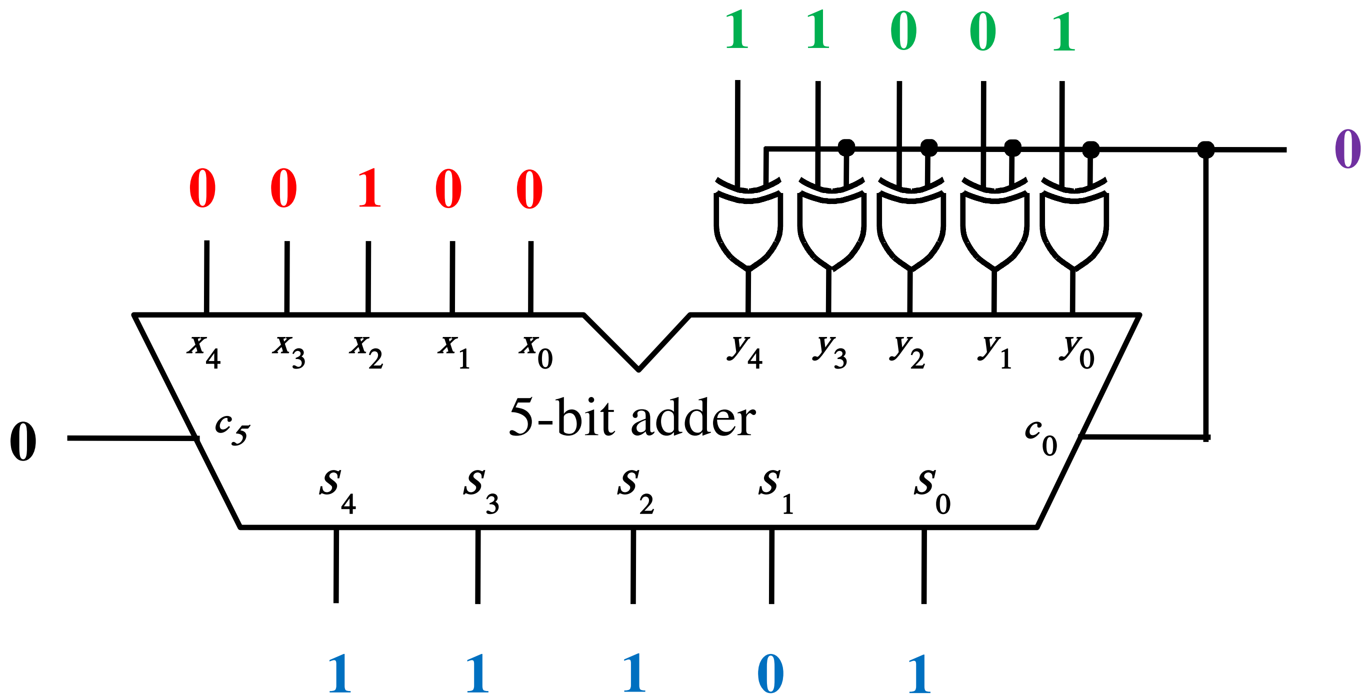


Addition: 5 + 6 = 11

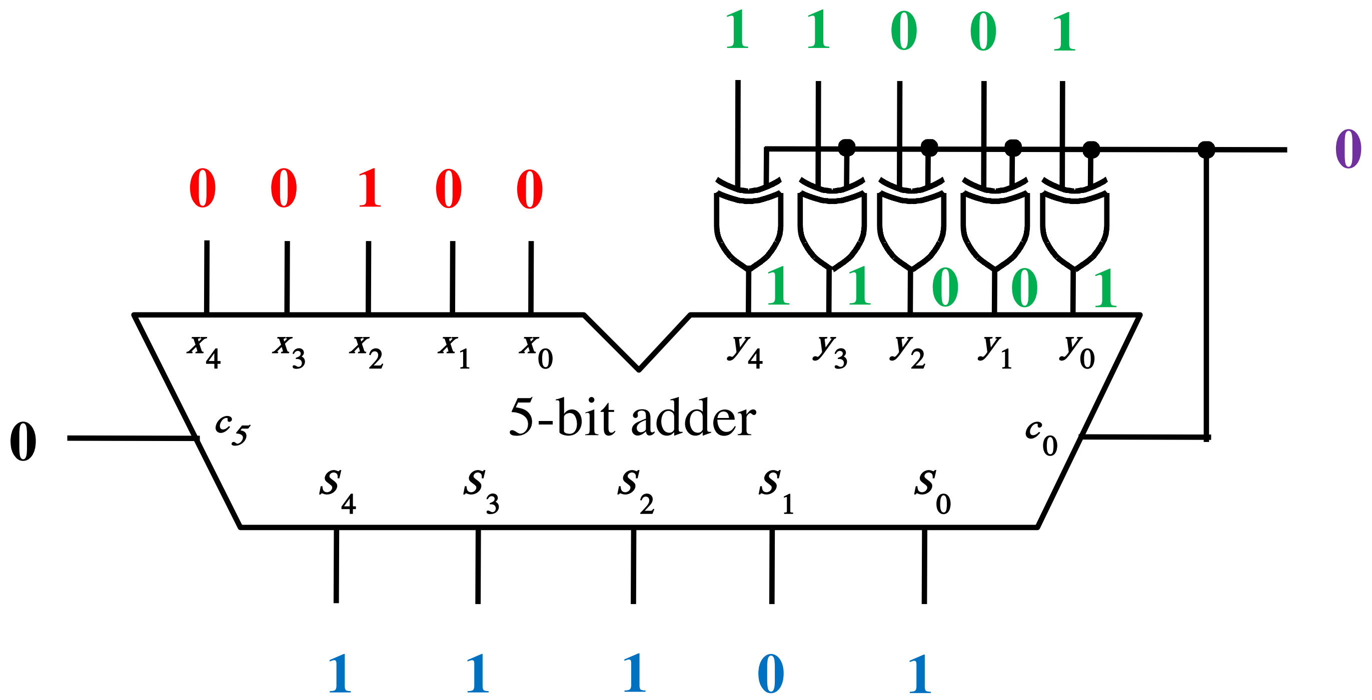


$$
\begin{array}{rccccc}
 & & & & & 0 \\
+ & 0 & 0 & 1 & 0 & 1 \\
 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 1 & 0 & 1 & 1
\end{array}
$$

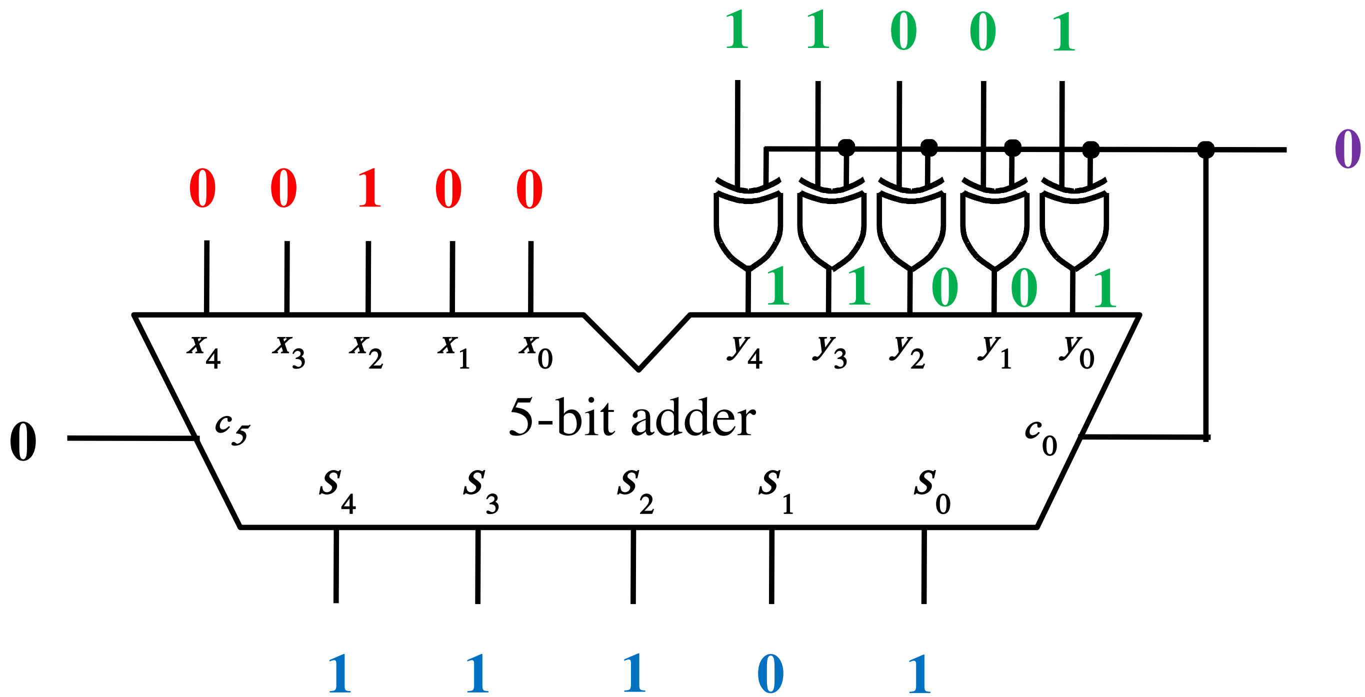
Addition: 4 + (-7) = -3



Addition: 4 + (-7) = -3



Addition: 4 + (-7) = -3

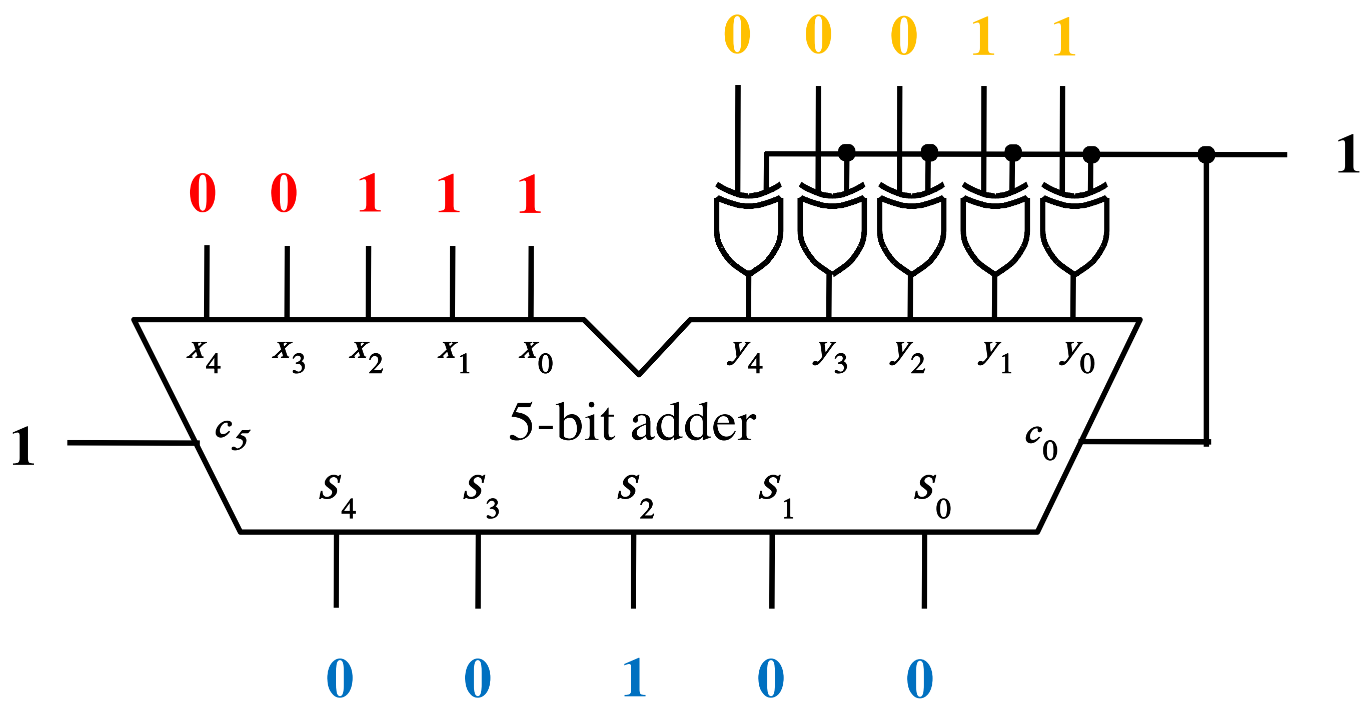


$$
\begin{array}{r}
0 \\
+\; 0\,0\,1\,0\,0 \\
1\,1\,0\,0\,1 \\
\hline
0\;1\,1\,1\,0\,1
\end{array}
$$

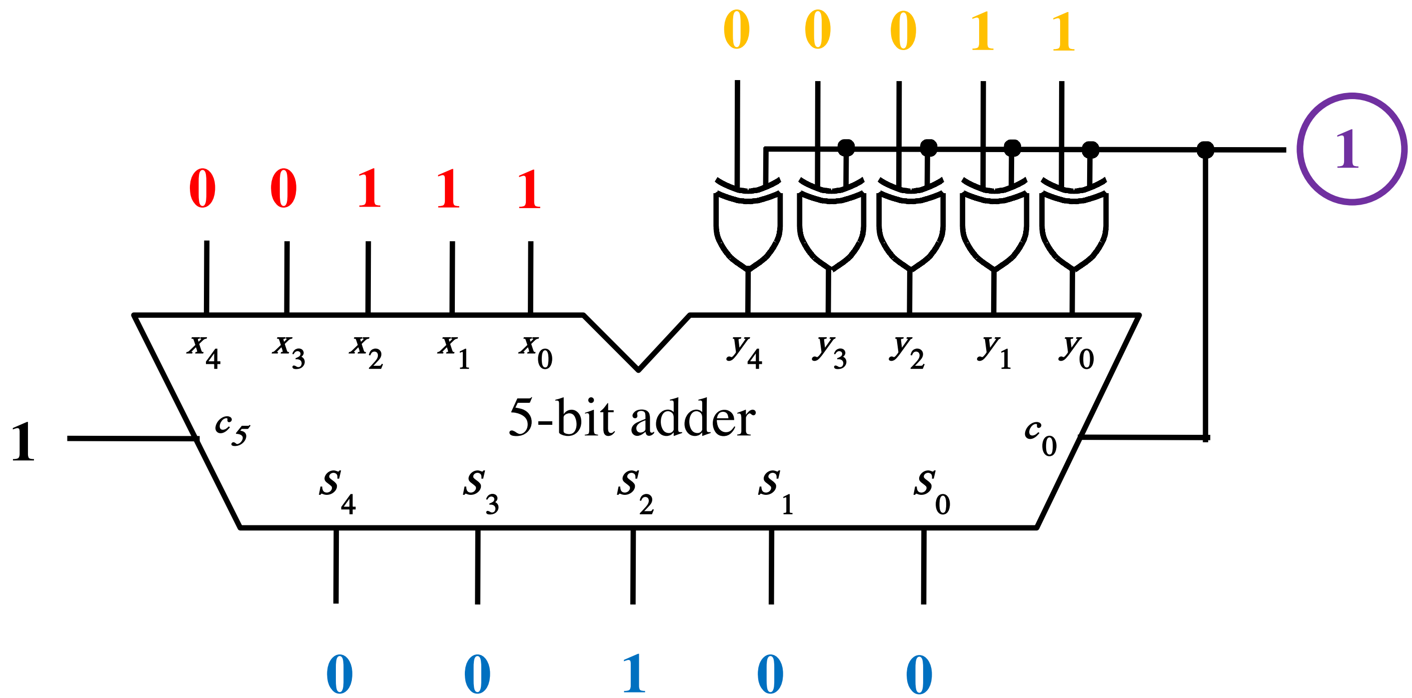
Subtraction Examples:

all inputs and outputs are given in 2's complement representation

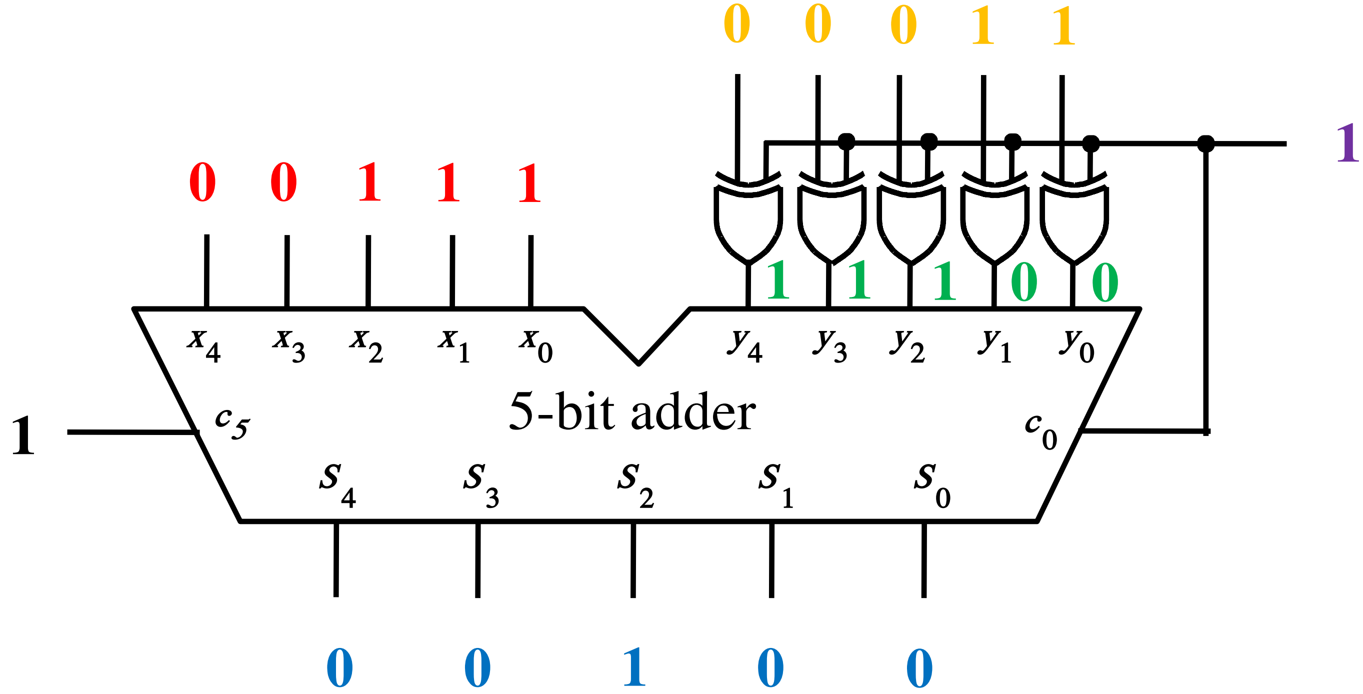
Subtraction: 7 - 3 = 4



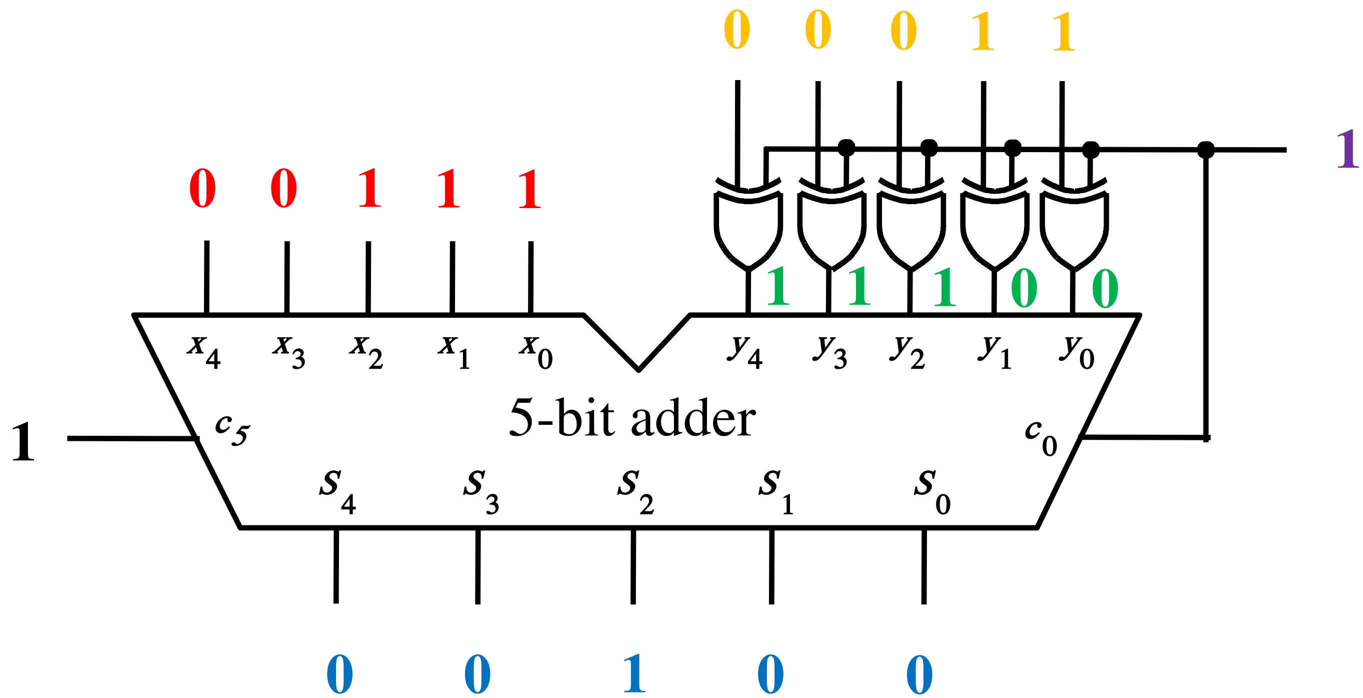
Subtraction: 7 - 3 = 4



Subtraction: 7 - 3 = 4

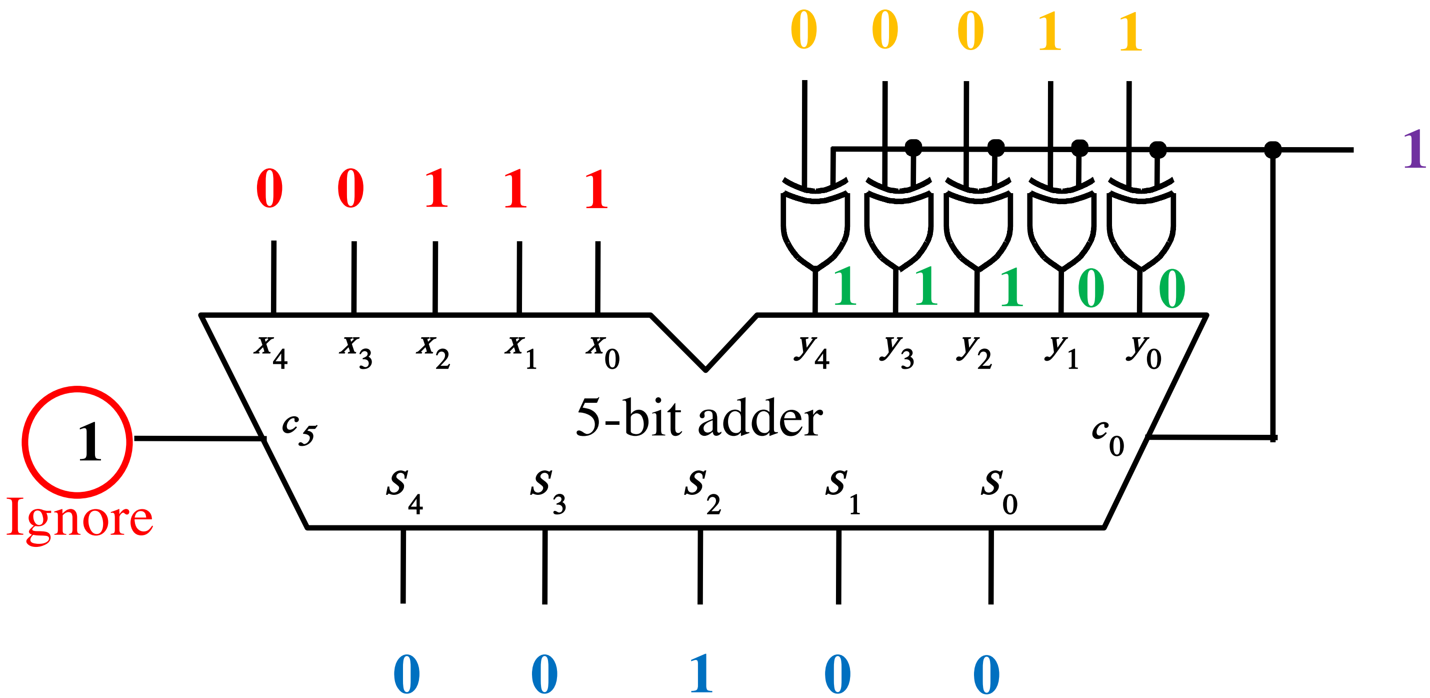


Subtraction: 7 - 3 = 4



$$\begin{array}{r} 1 \\ +\;0\,0\,1\,1\,1 \\ 1\,1\,1\,0\,0 \\ \hline 1\,0\,0\,1\,0\,0 \end{array}$$

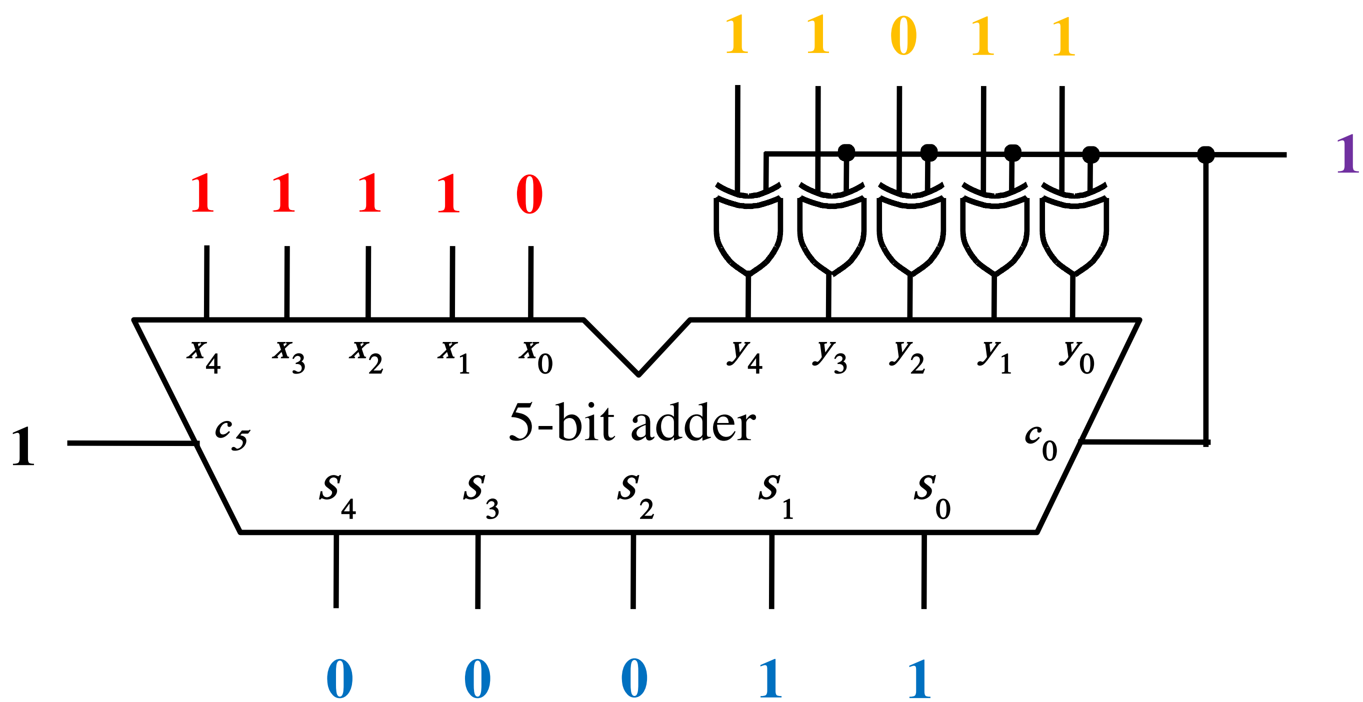
Subtraction: 7 - 3 = 4



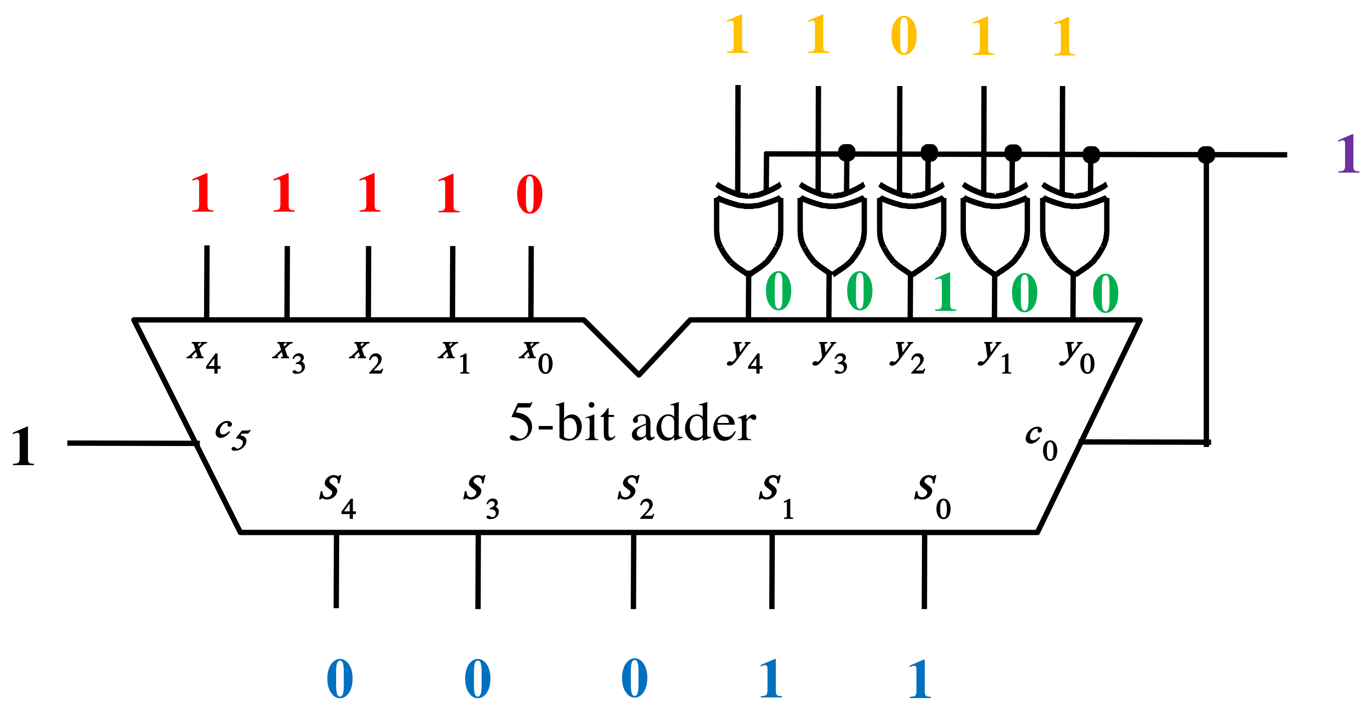

```
       1
+  0 0 1 1 1
   1 1 1 0 0
 ___________
 1 0 0 1 0 0
```

Ignore

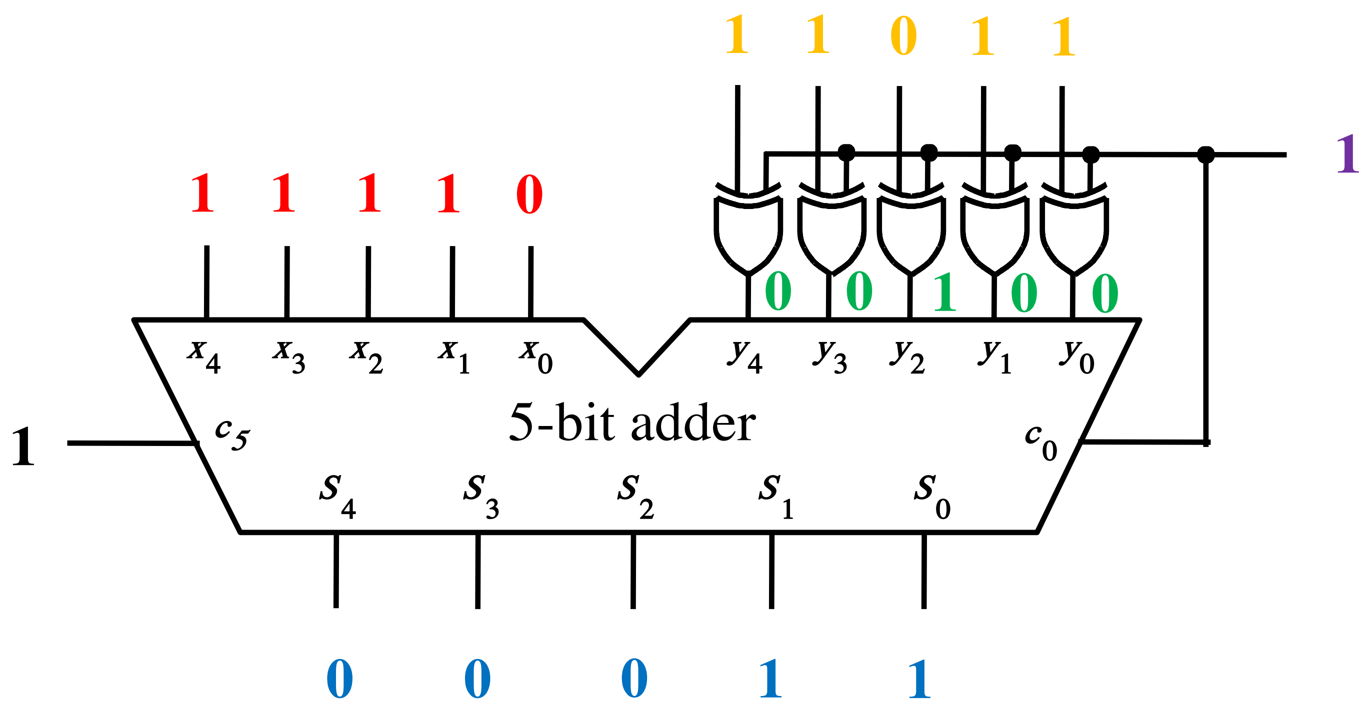
Subtraction: (–2) – (–5) = 3



Subtraction: (–2) – (–5) = 3

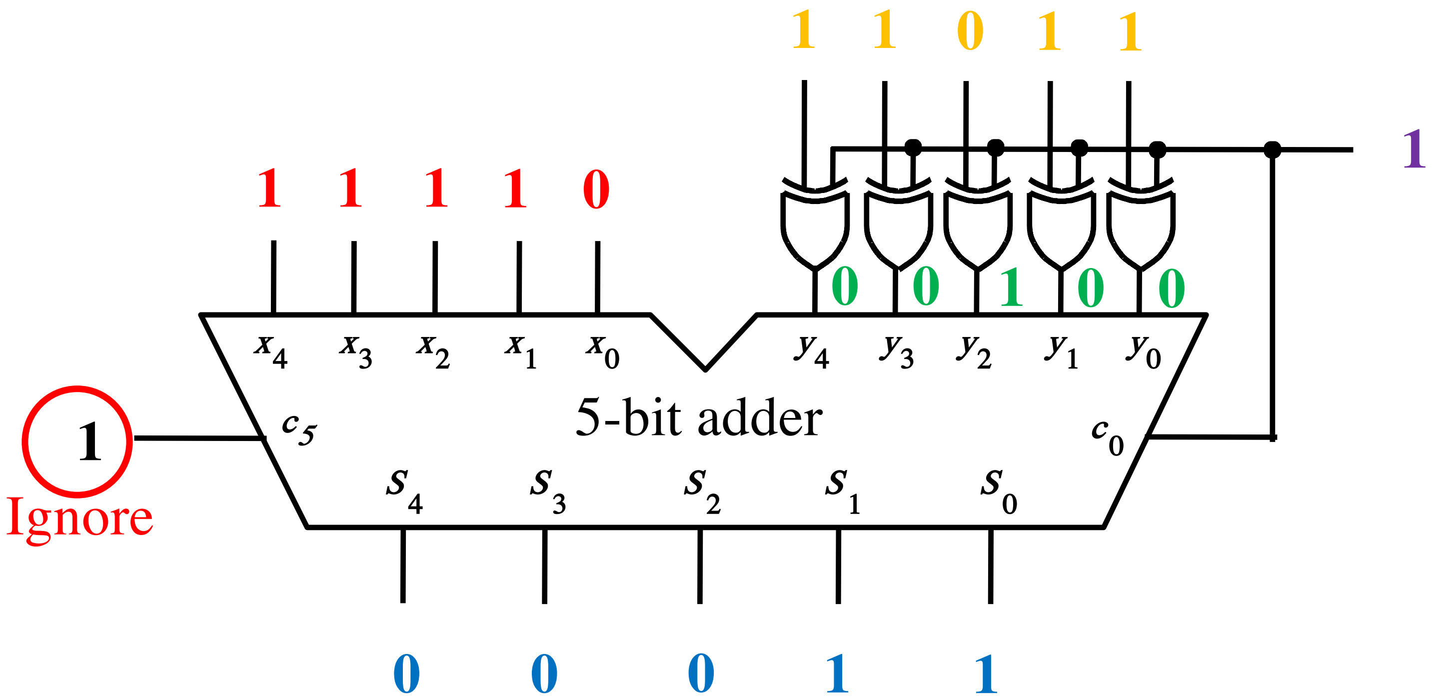


Subtraction: (–2) – (–5) = 3



$$
\begin{array}{r}
1 \\
+\;\; 1\,1\,1\,1\,0 \\
0\,0\,1\,0\,0 \\
\hline
1\,0\,0\,0\,1\,1
\end{array}
$$

Subtraction: (–2) – (–5) = 3



$$
\begin{array}{r}
1 \\
+\ 11110 \\
00100 \\
\hline
1\ 00011
\end{array}
$$

Ignore

Detecting Overflow

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0111 \\ +\ 0010 \\ \hline 1001 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1001 \\ +\ 0010 \\ \hline 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0111 \\ +\ 1110 \\ \hline 10101 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1001 \\ +\ 1110 \\ \hline 10111 \end{array}$$

[Figure 3.13 from the textbook]

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} \color{green}{0\,1\,1\,0\,0} \\ 0\,1\,1\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} \color{green}{0\,0\,0\,0\,0} \\ 1\,0\,0\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} \color{green}{1\,1\,1\,0\,0} \\ 0\,1\,1\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} \color{green}{1\,0\,0\,0\,0} \\ 1\,0\,0\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

Include the carry bits: $c_4\ c_3\ c_2\ c_1\ c_0$

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} \boxed{0\ 1}\ 1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 0\ 1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} \boxed{0\ 0}\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 1\ 1 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} \boxed{1\ 1}\ 1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 0\ 1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} \boxed{1\ 0}\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 1\ 1 \end{array}$$

Include the carry bits: $\boxed{c_4\ c_3}\ c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 1}\ 1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 0}\ 0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 1}\ 1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 0}\ 0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

Include the carry bits: $\boxed{c_4\ c_3}\ c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$, $c_3 = 1$	$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array}$	$\begin{array}{r} \boxed{0\ 1}\,1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 0\ 1 \end{array}$	
	$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array}$	$\begin{array}{r} \boxed{0\ 0}\,0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 1\ 1 \end{array}$	$c_4 = 0$, $c_3 = 0$
$c_4 = 1$, $c_3 = 1$	$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array}$	$\begin{array}{r} \boxed{1\ 1}\,1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 0\ 1 \end{array}$	
	$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array}$	$\begin{array}{r} \boxed{1\ 0}\,0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 1\ 1 \end{array}$	$c_4 = 1$, $c_3 = 0$

Overflow occurs only in these two cases.

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{cr} & \boxed{0\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{cr} & \boxed{0\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{cr} & \boxed{1\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{cr} & \boxed{1\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\text{Overflow} = c_3\overline{c}_4 + \overline{c}_3 c_4$$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} \boxed{0\ 1}\ 1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} \boxed{0\ 0}\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} \boxed{1\ 1}\ 1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} \boxed{1\ 0}\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\text{Overflow} = \underbrace{c_3\overline{c}_4 + \overline{c}_3 c_4}_{\text{XOR}}$$

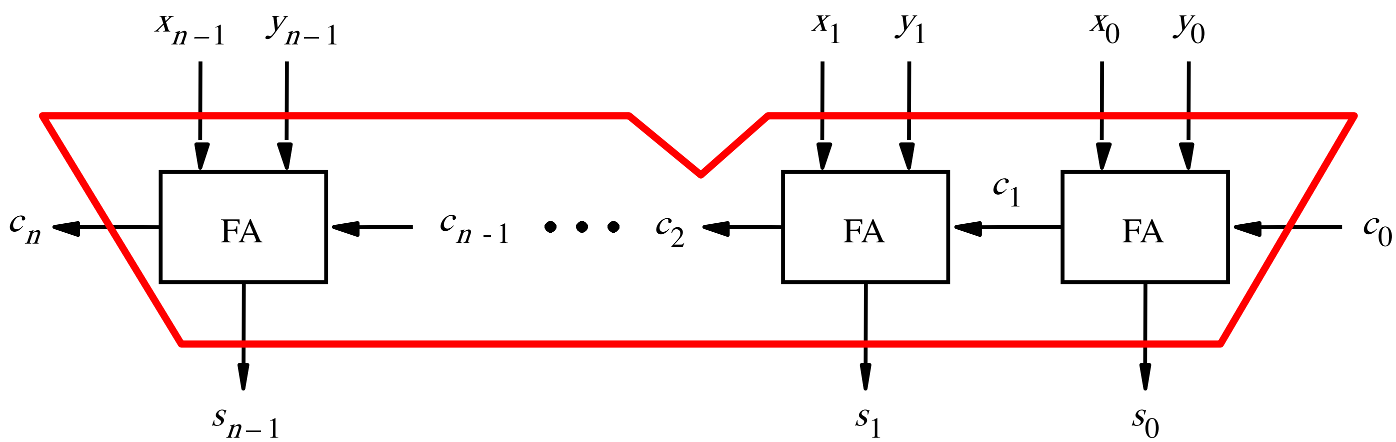
Calculating overflow for 4-bit numbers with only three significant bits

$$\begin{aligned} \text{Overflow} &= c_3\overline{c}_4 + \overline{c}_3 c_4 \\ &= c_3 \oplus c_4 \end{aligned}$$

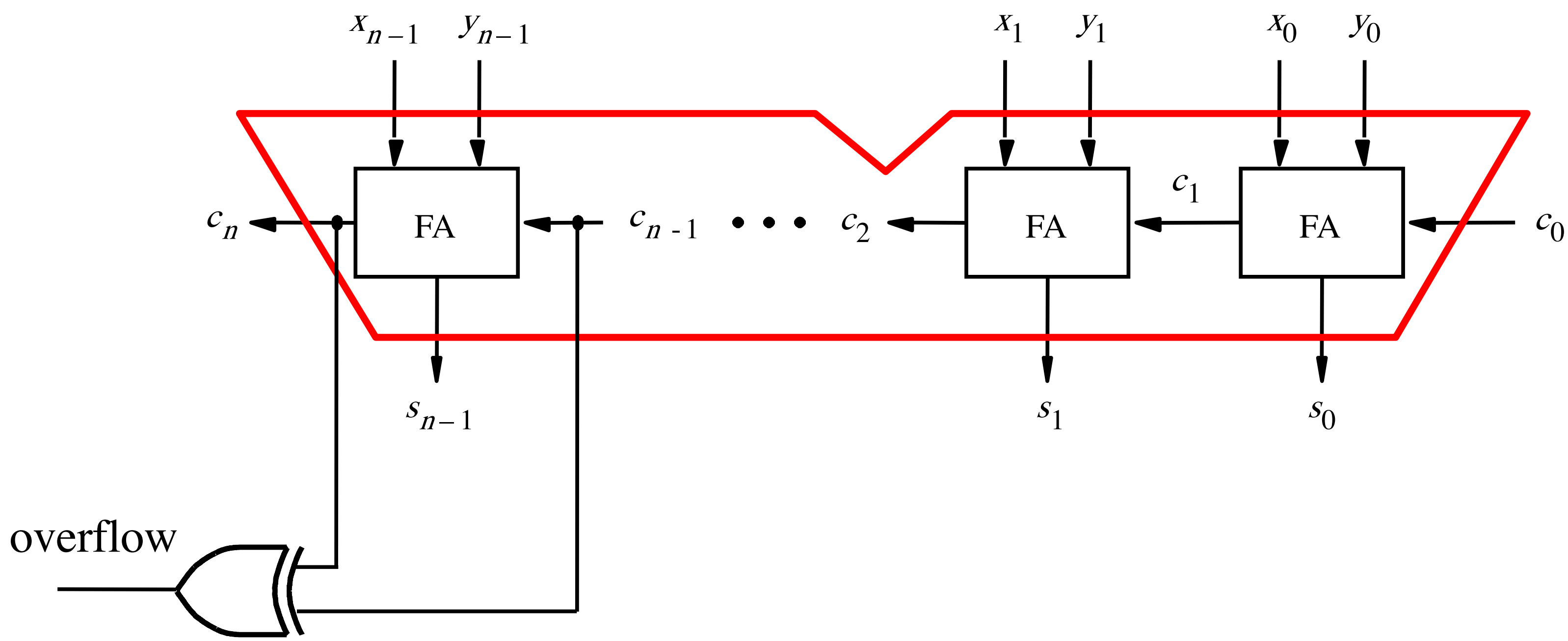
Calculating overflow for n-bit numbers with only n-1 significant bits

$$\text{Overflow} = c_{n-1} \oplus c_n$$

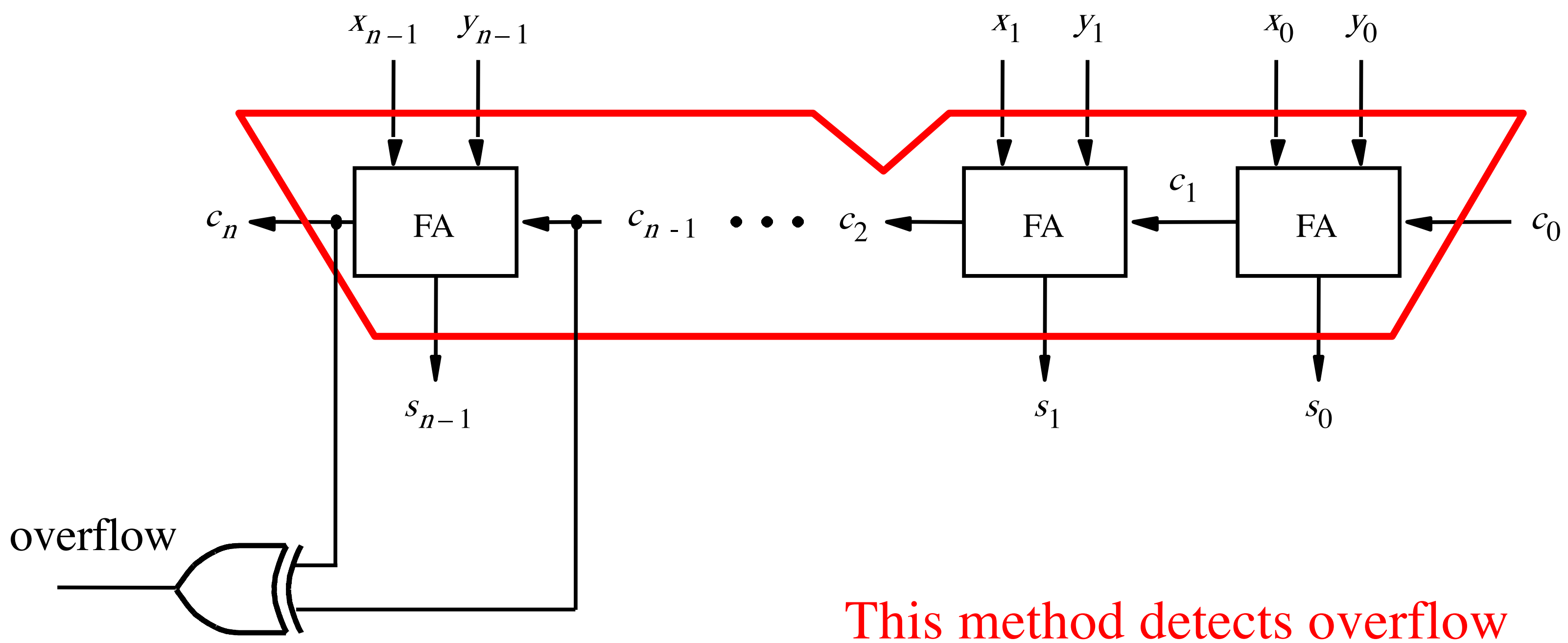
Detecting Overflow



Detecting Overflow
(with one extra XOR)



Detecting Overflow
(with one extra XOR)



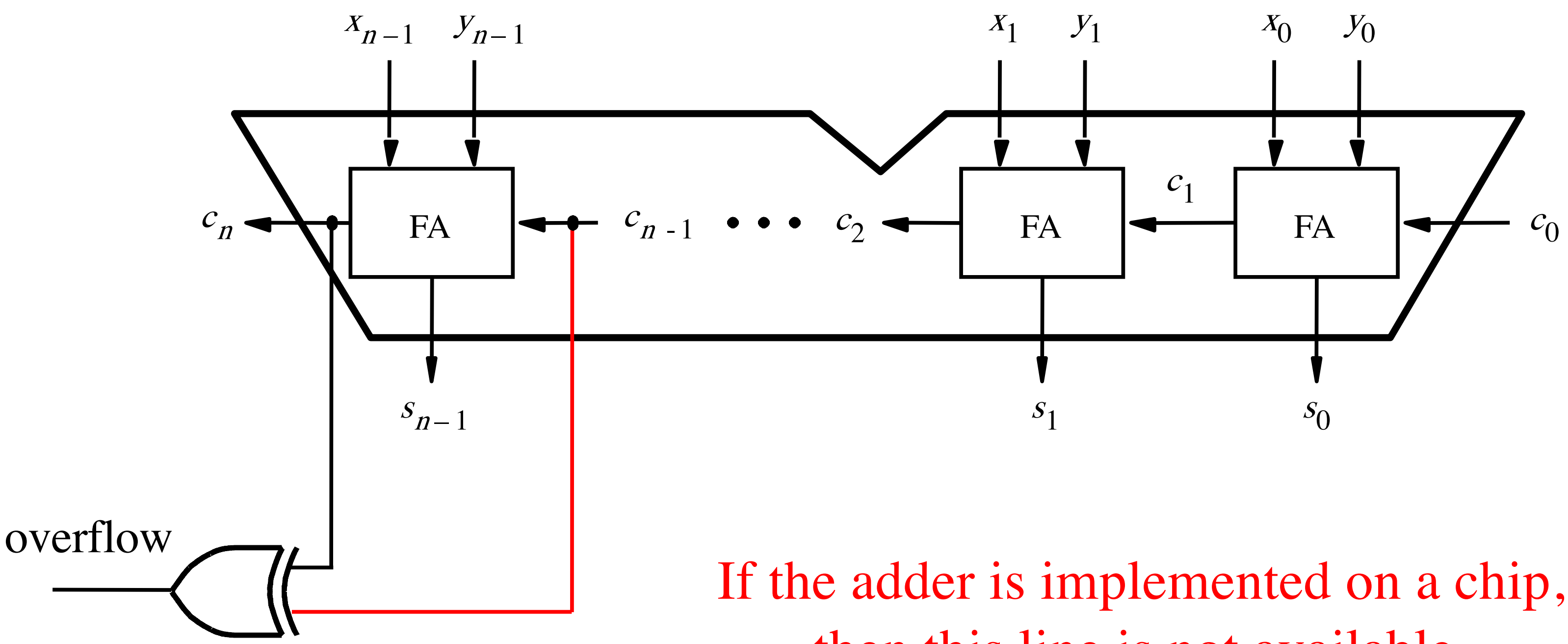
This method detects overflow
for both addition and subtraction.

Detecting Overflow
(alternative method)

Detecting Overflow
(alternative method)

Used if you don't have access to the internal carries of the adder.

Detecting Overflow
(with one extra XOR)



If the adder is implemented on a chip,
then this line is not available.
So, the first method can't be used.

Another way to look at the overflow issue

$$
\begin{array}{rl}
 & X = x_3\ x_2\ x_1\ x_0 \\
+ & Y = y_3\ y_2\ y_1\ y_0 \\
\hline
 & S = s_3\ s_2\ s_1\ s_0
\end{array}
$$

Another way to look at the overflow issue

$$
\begin{array}{rl}
 & X = \boxed{x_3}\ x_2\ x_1\ x_0 \\
+ & Y = \boxed{y_3}\ y_2\ y_1\ y_0 \\
\hline
 & S = \boxed{s_3}\ s_2\ s_1\ s_0
\end{array}
$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

Examples of determination of overflow

$$
\begin{array}{rr}
 & (+7) \\
+ & (+2) \\
\hline
 & (+9)
\end{array}
\qquad
\begin{array}{rr}
 & 0\,1\,1\,1 \\
+ & 0\,0\,1\,0 \\
\hline
 & 1\,0\,0\,1
\end{array}
\qquad\qquad
\begin{array}{rr}
 & (-7) \\
+ & (+2) \\
\hline
 & (-5)
\end{array}
\qquad
\begin{array}{rr}
 & 1\,0\,0\,1 \\
+ & 0\,0\,1\,0 \\
\hline
 & 1\,0\,1\,1
\end{array}
$$

$$
\begin{array}{rr}
 & (+7) \\
+ & (-2) \\
\hline
 & (+5)
\end{array}
\qquad
\begin{array}{rr}
 & 0\,1\,1\,1 \\
+ & 1\,1\,1\,0 \\
\hline
 & 1\,0\,1\,0\,1
\end{array}
\qquad\qquad
\begin{array}{rr}
 & (-7) \\
+ & (-2) \\
\hline
 & (-9)
\end{array}
\qquad
\begin{array}{rr}
 & 1\,0\,0\,1 \\
+ & 1\,1\,1\,0 \\
\hline
 & 1\,0\,1\,1\,1
\end{array}
$$

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ \ 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ \ 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ \ 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ \ 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$
\begin{array}{r}
(+7) \\
+\ (+2) \\
\hline
(+9)
\end{array}
\qquad
\begin{array}{r}
0\,1\,1\,1 \\
+\quad 0\,0\,1\,0 \\
\hline
1\,0\,0\,1
\end{array}
$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$$
\begin{array}{r}
(-7) \\
+\ (+2) \\
\hline
(-5)
\end{array}
\qquad
\begin{array}{r}
1\,0\,0\,1 \\
+\quad 0\,0\,1\,0 \\
\hline
1\,0\,1\,1
\end{array}
$$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$
\begin{array}{r}
(+7) \\
+\ (-2) \\
\hline
(+5)
\end{array}
\qquad
\begin{array}{r}
0\,1\,1\,1 \\
+\quad 1\,1\,1\,0 \\
\hline
1\,0\,1\,0\,1
\end{array}
$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$
\begin{array}{r}
(-7) \\
+\ (-2) \\
\hline
(-9)
\end{array}
\qquad
\begin{array}{r}
1\,0\,0\,1 \\
+\quad 1\,1\,1\,0 \\
\hline
1\,0\,1\,1\,1
\end{array}
$$

In 2's complement, both +9 and -9 are not representable with 4 bits.

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

Overflow occurs only in these two cases.

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$\text{Overflow} = \overline{x}_3\,\overline{y}_3\,s_3 + x_3\,y_3\,\overline{s}_3$$

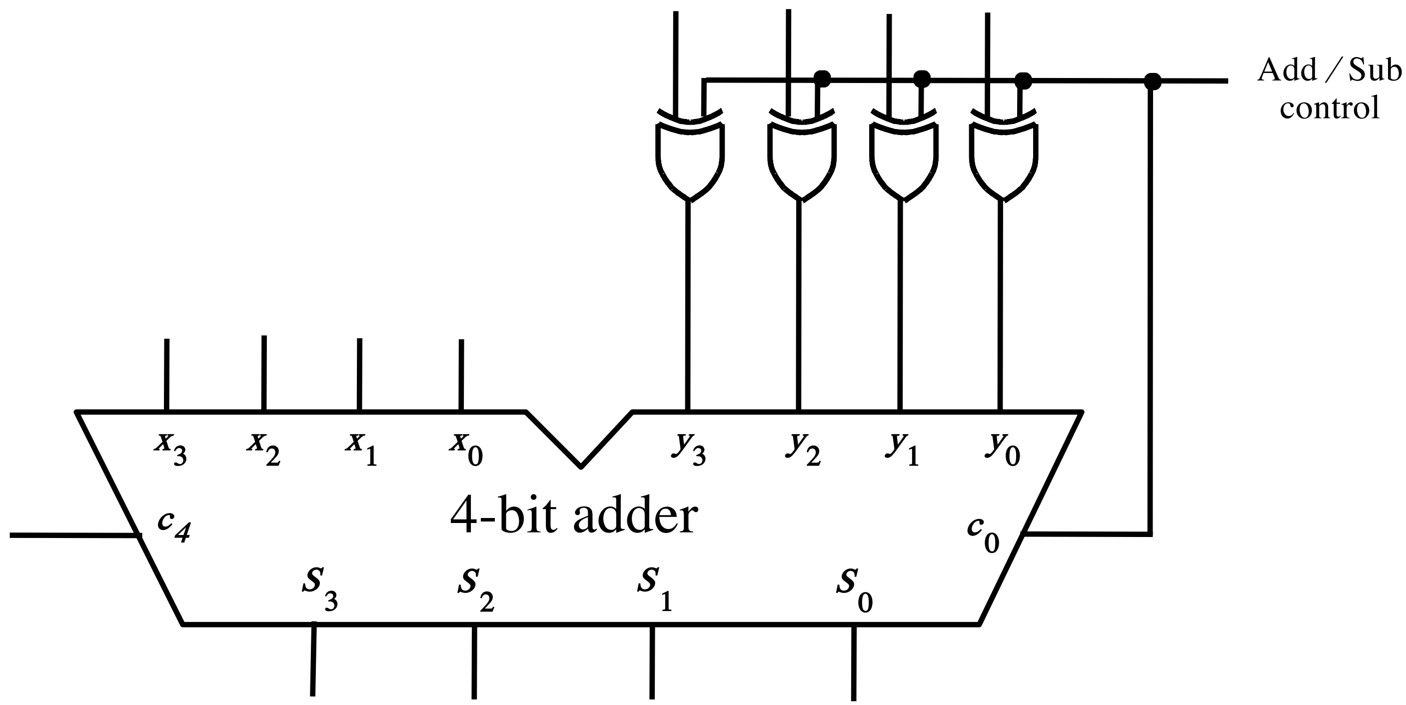
Another way to look at the overflow issue

$$
\begin{array}{cl}
 & X = \boxed{x_3}\ x_2\ x_1\ x_0 \\
+ & Y = \boxed{y_3}\ y_2\ y_1\ y_0 \\
\hline
 & S = \boxed{s_3}\ s_2\ s_1\ s_0
\end{array}
$$

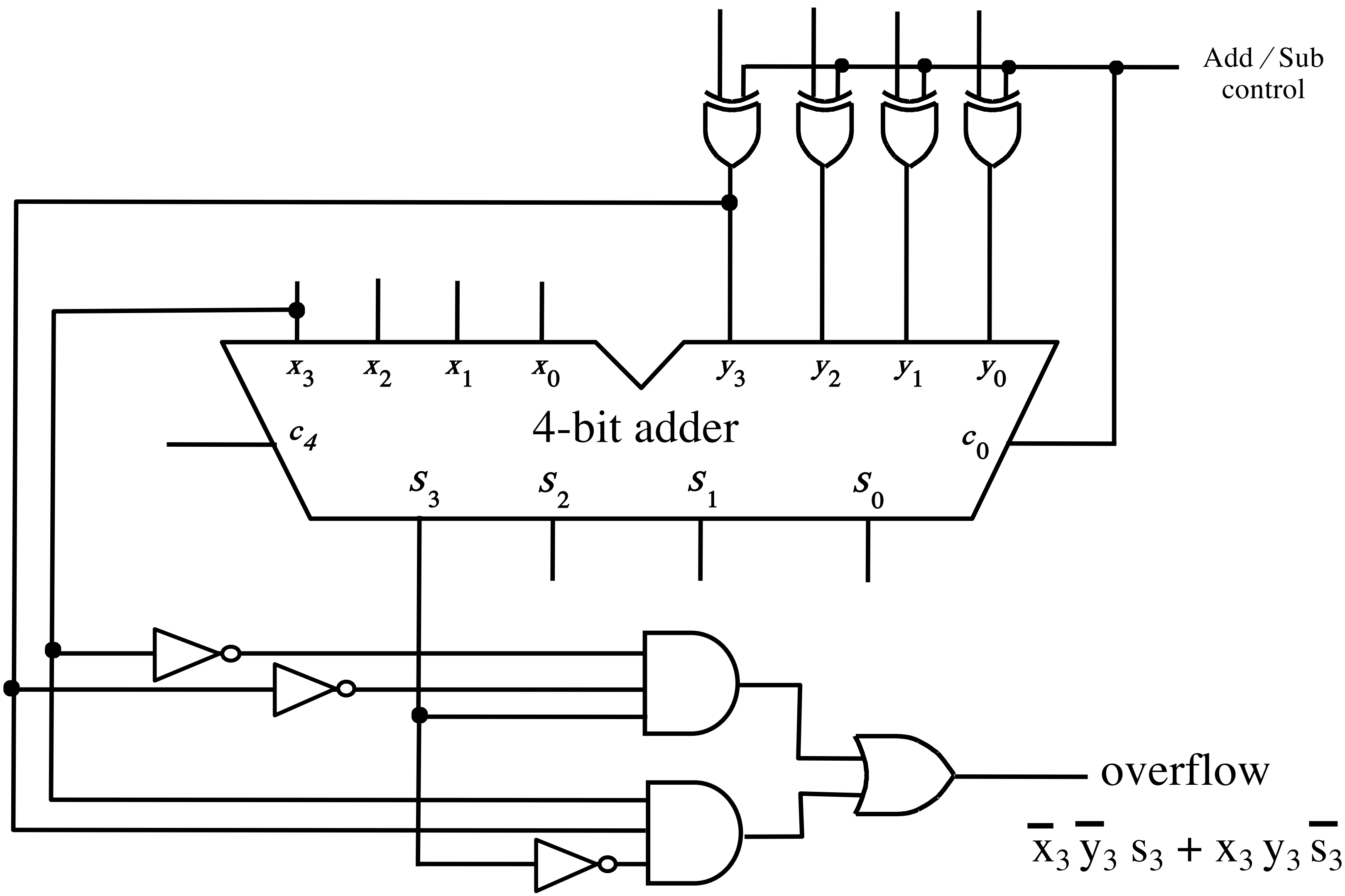
If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

$$\text{Overflow} = \overline{x}_3\, \overline{y}_3\, s_3 + x_3\, y_3\, \overline{s}_3$$

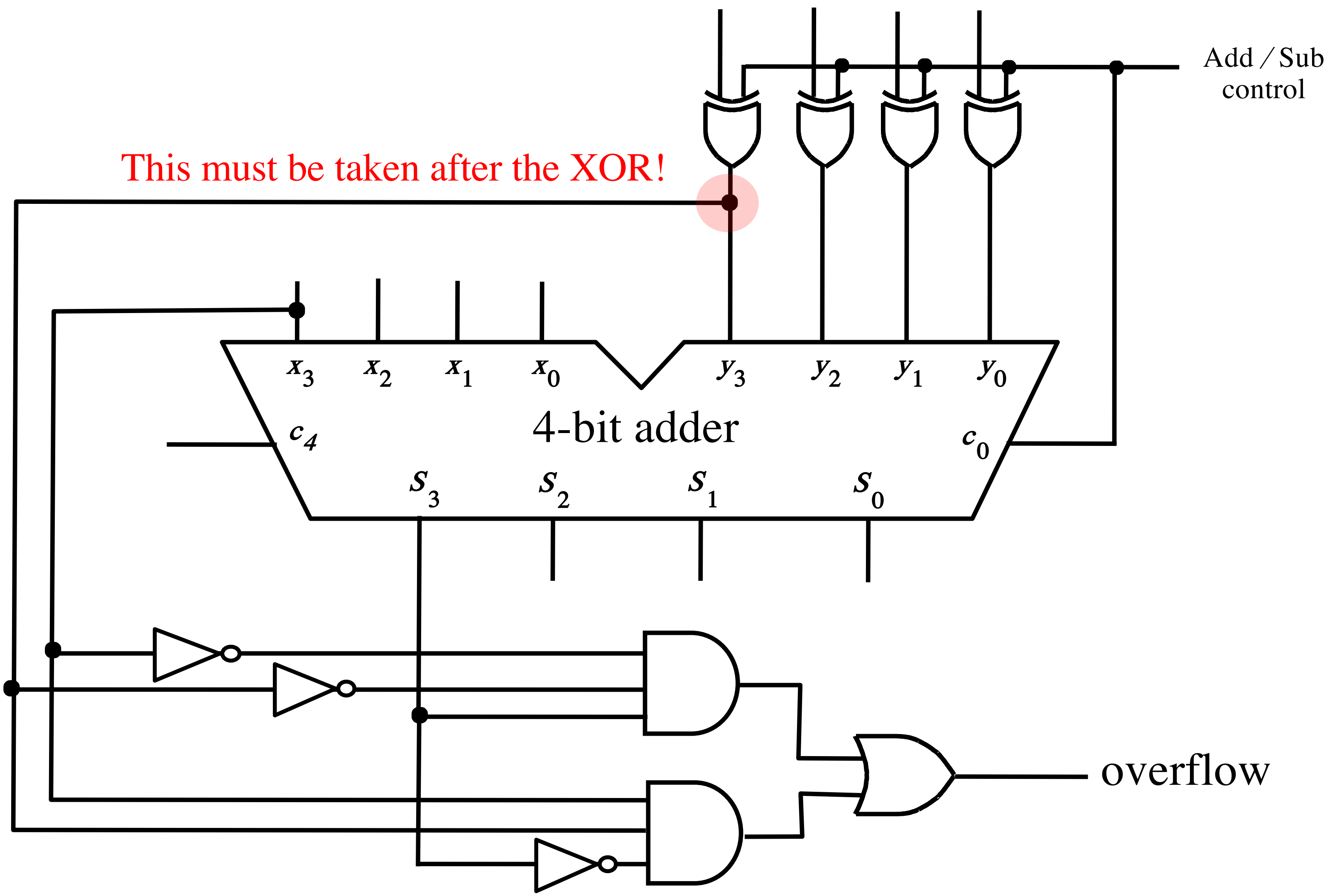
Overflow Detection



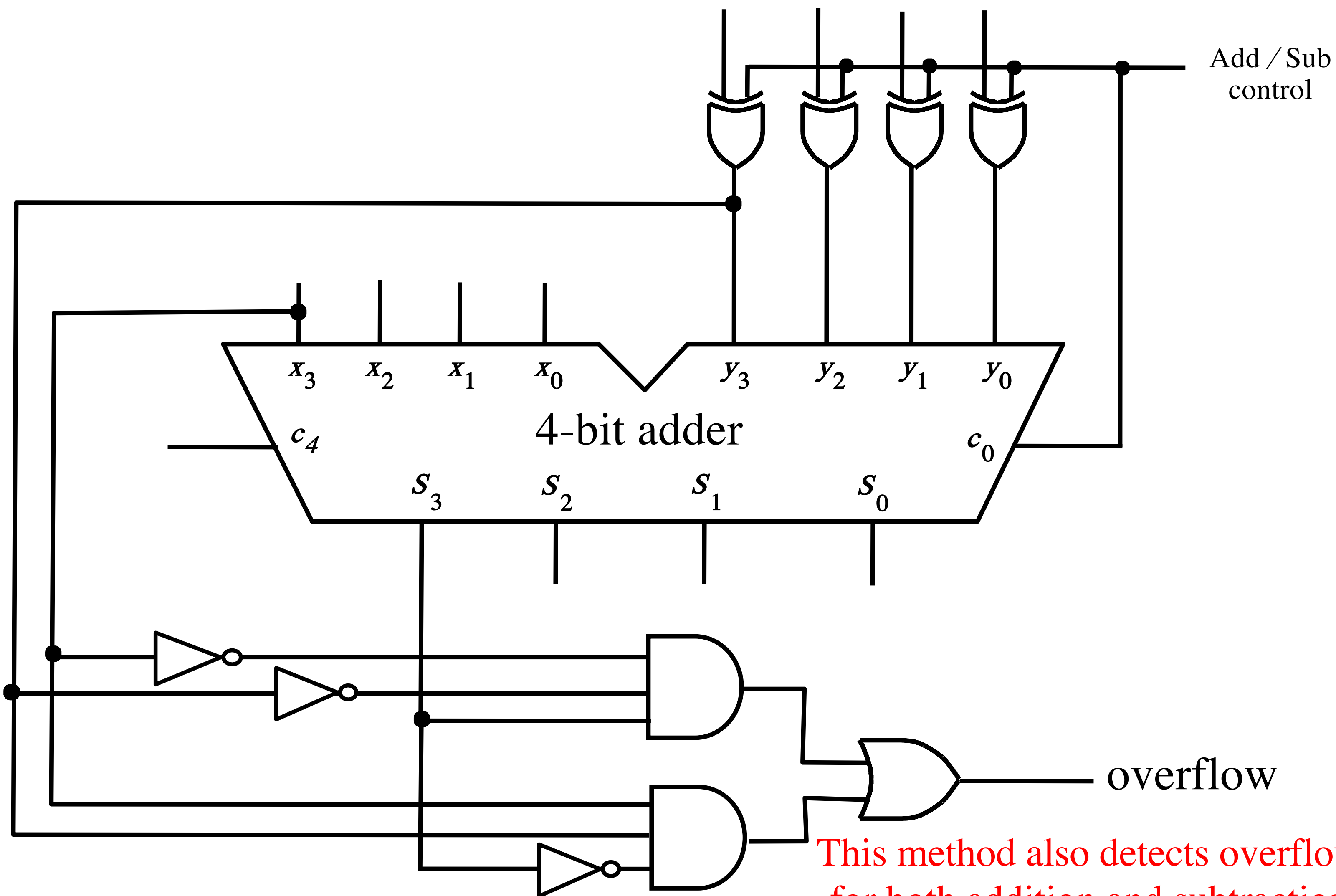
Overflow Detection



Overflow Detection



Overflow Detection



This method also detects overflow
for both addition and subtraction.

Questions?

THE END

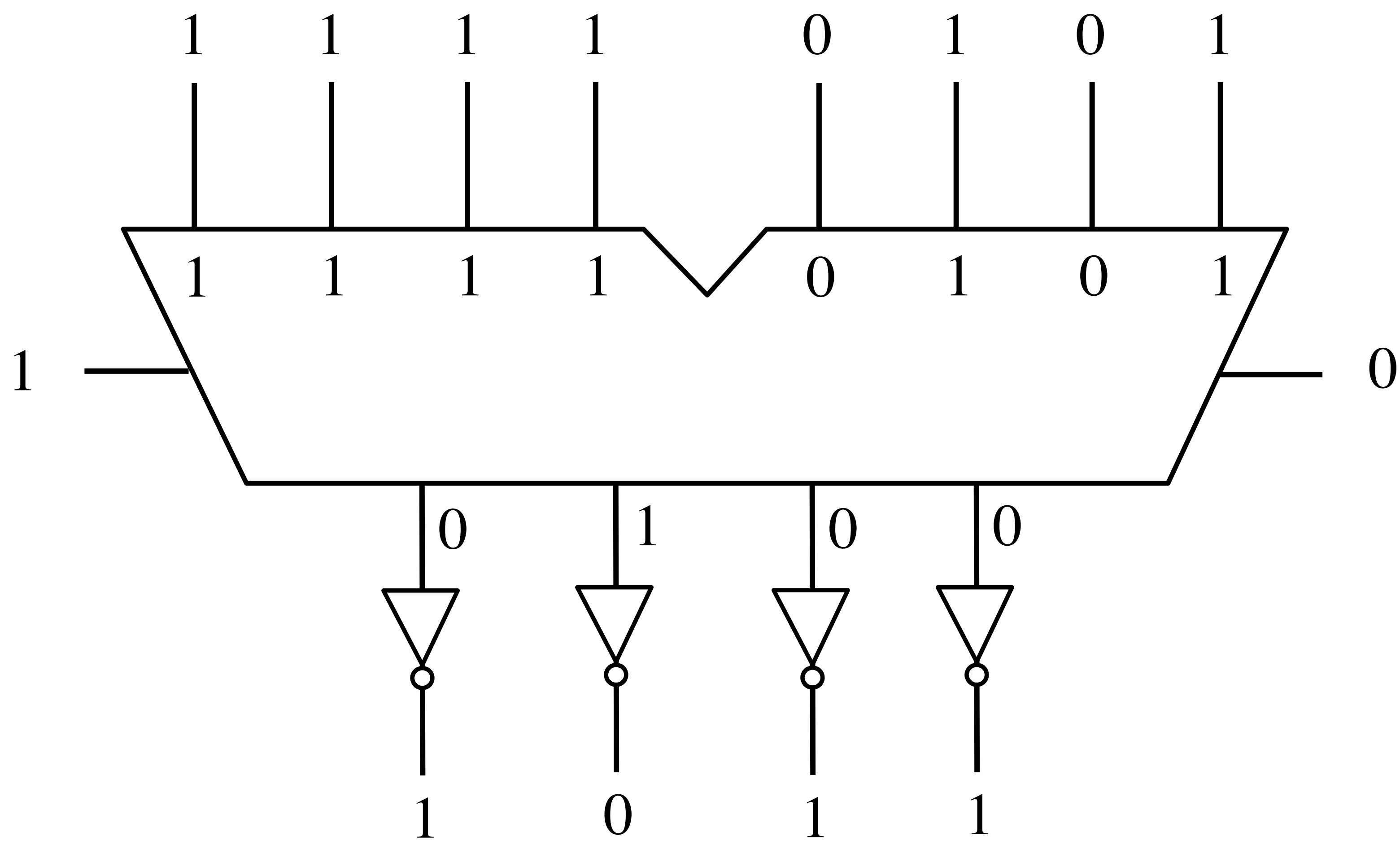
Additional Material

Alternative Circuit #3
(not used in practice)

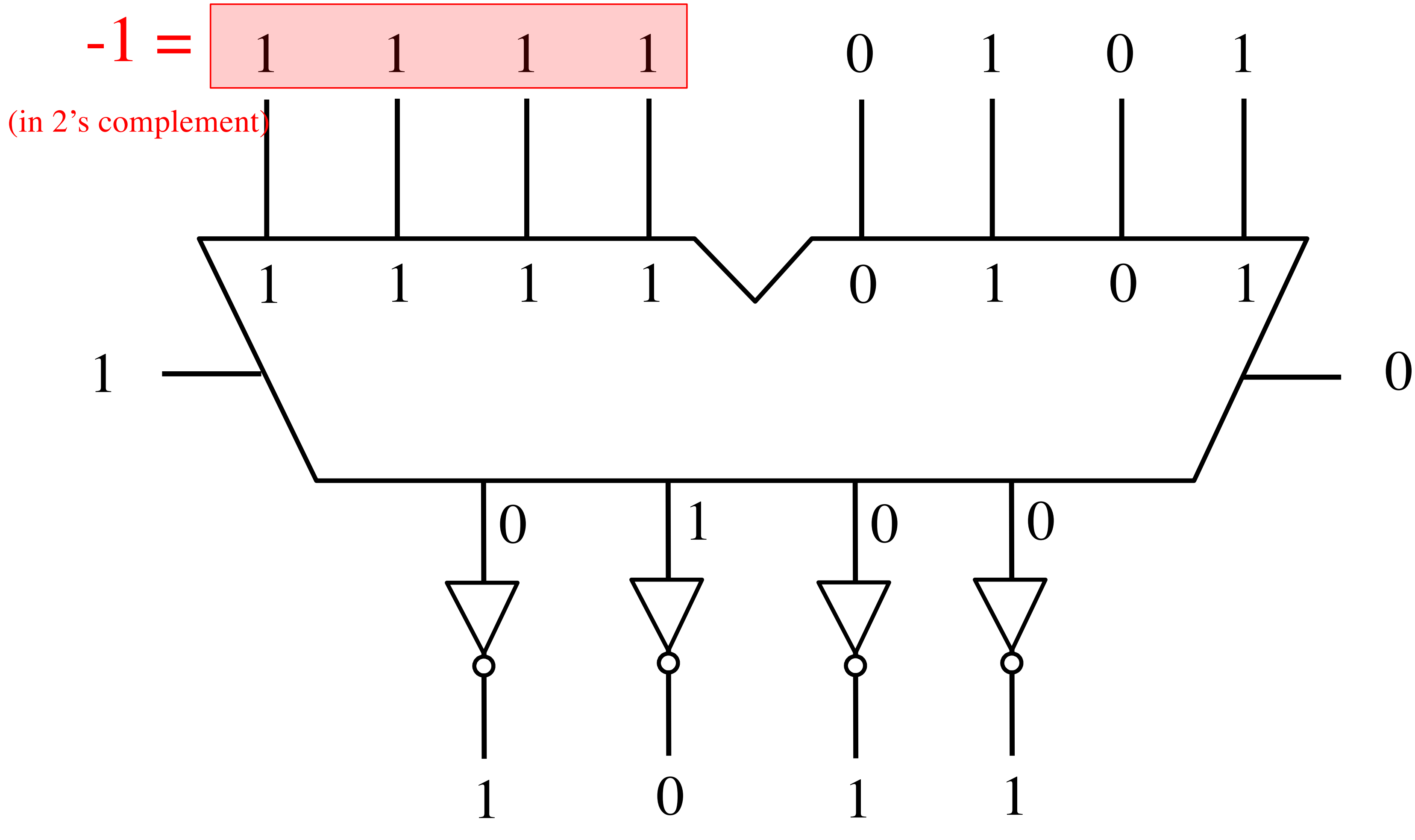
A former student came up with this circuit during a midterm exam!

Alternative Circuit #3
(not used in practice)

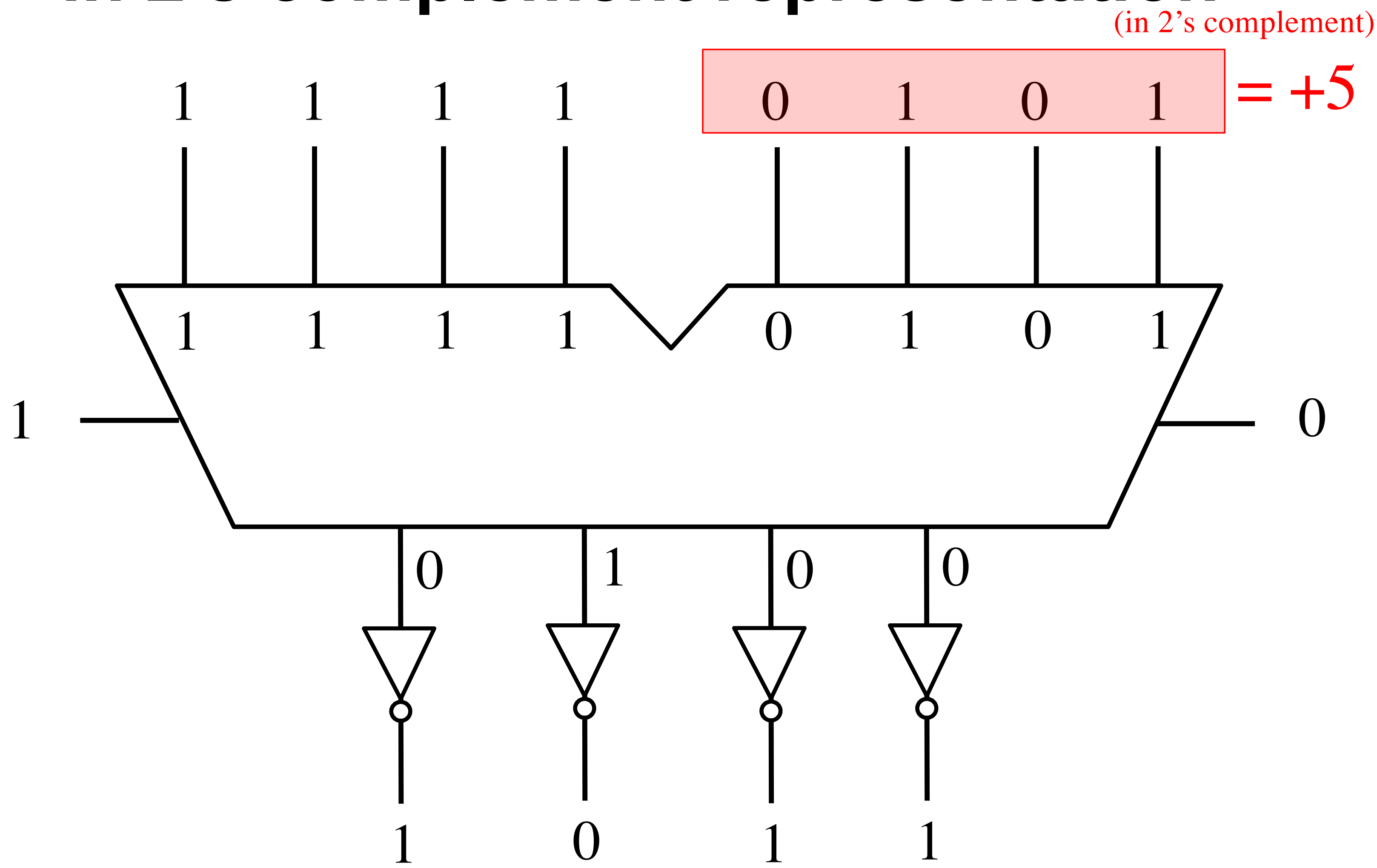
Circuit #3 for negating a number stored in 2's complement representation



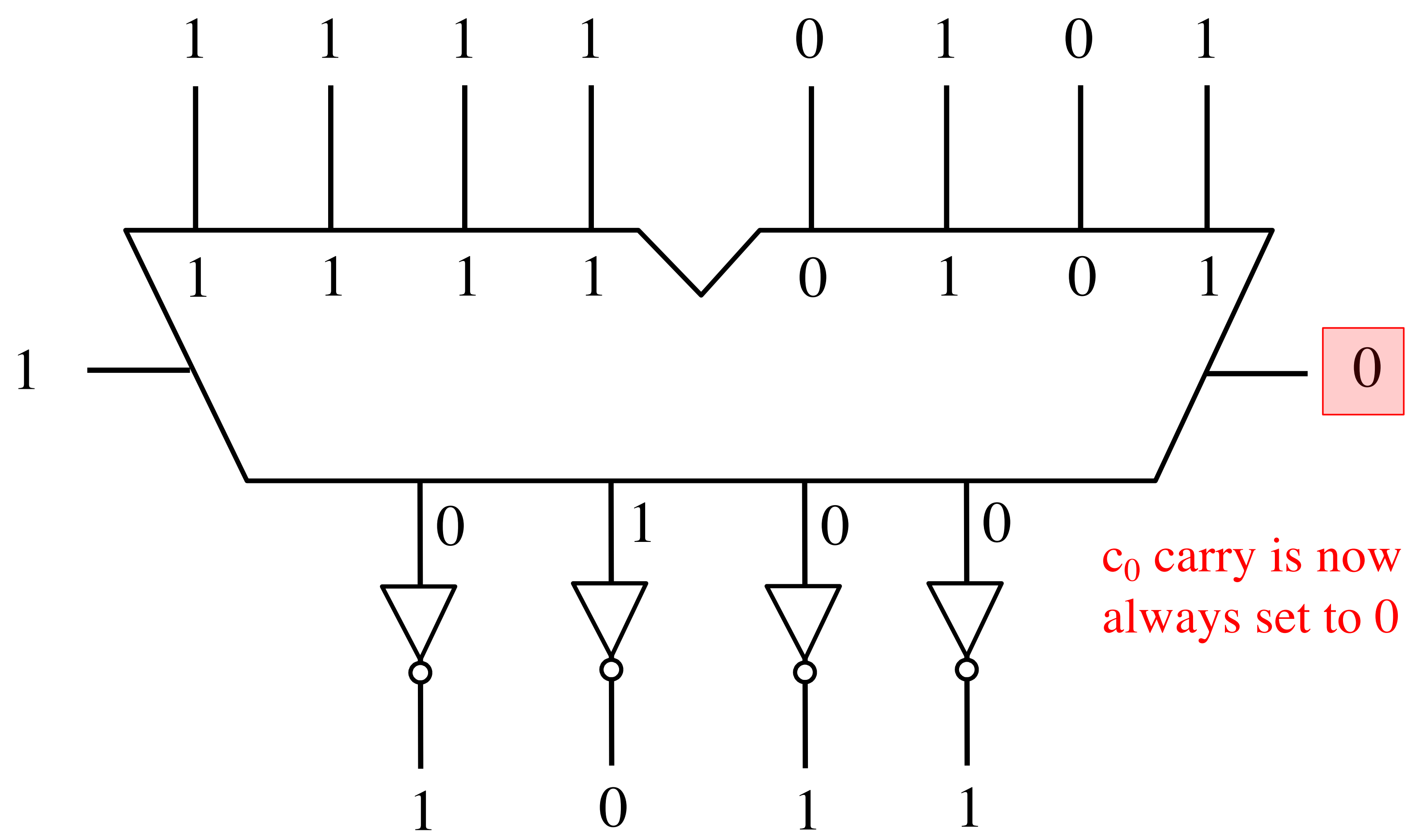
Circuit #3 for negating a number stored in 2's complement representation



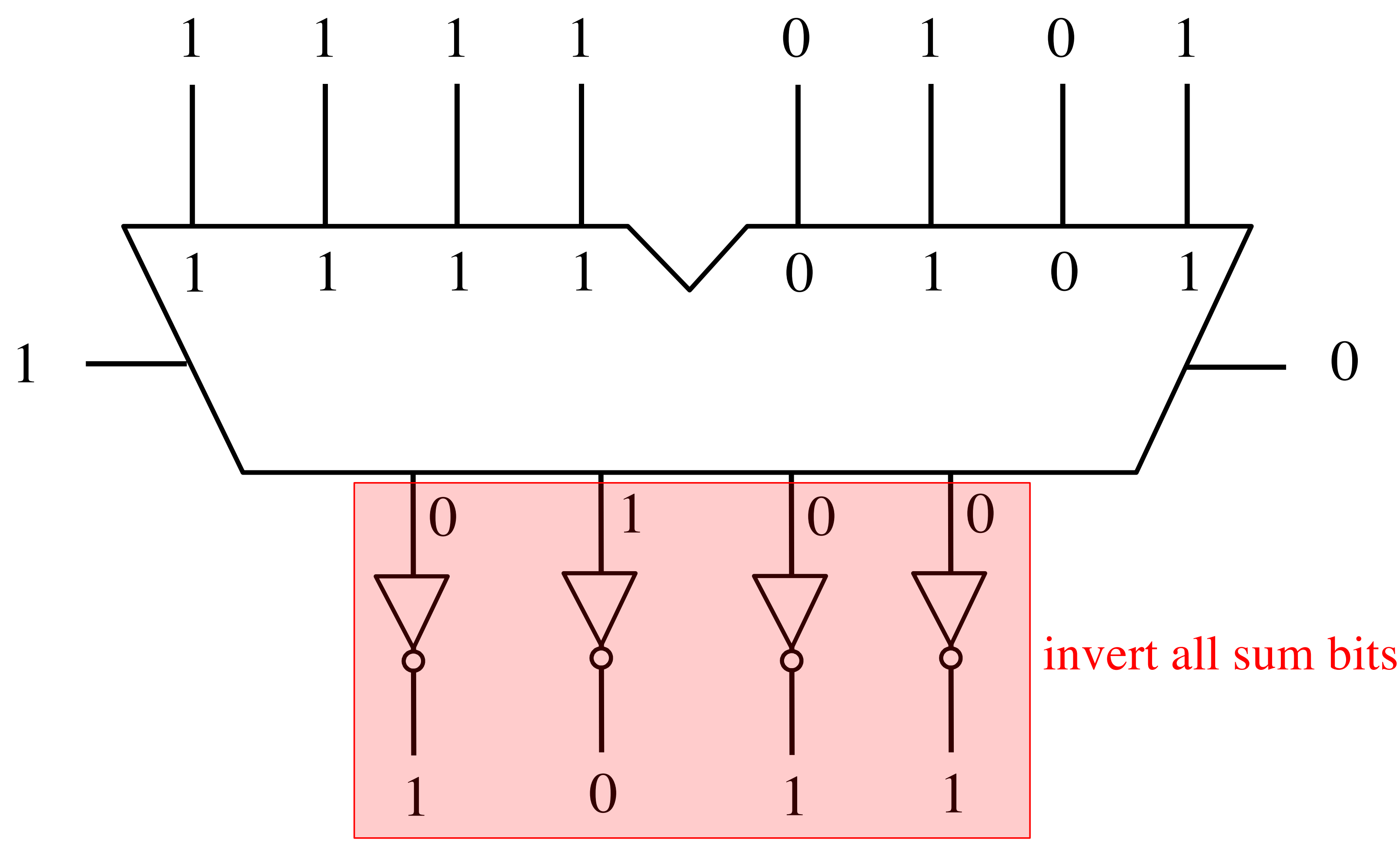
Circuit #3 for negating a number stored in 2's complement representation



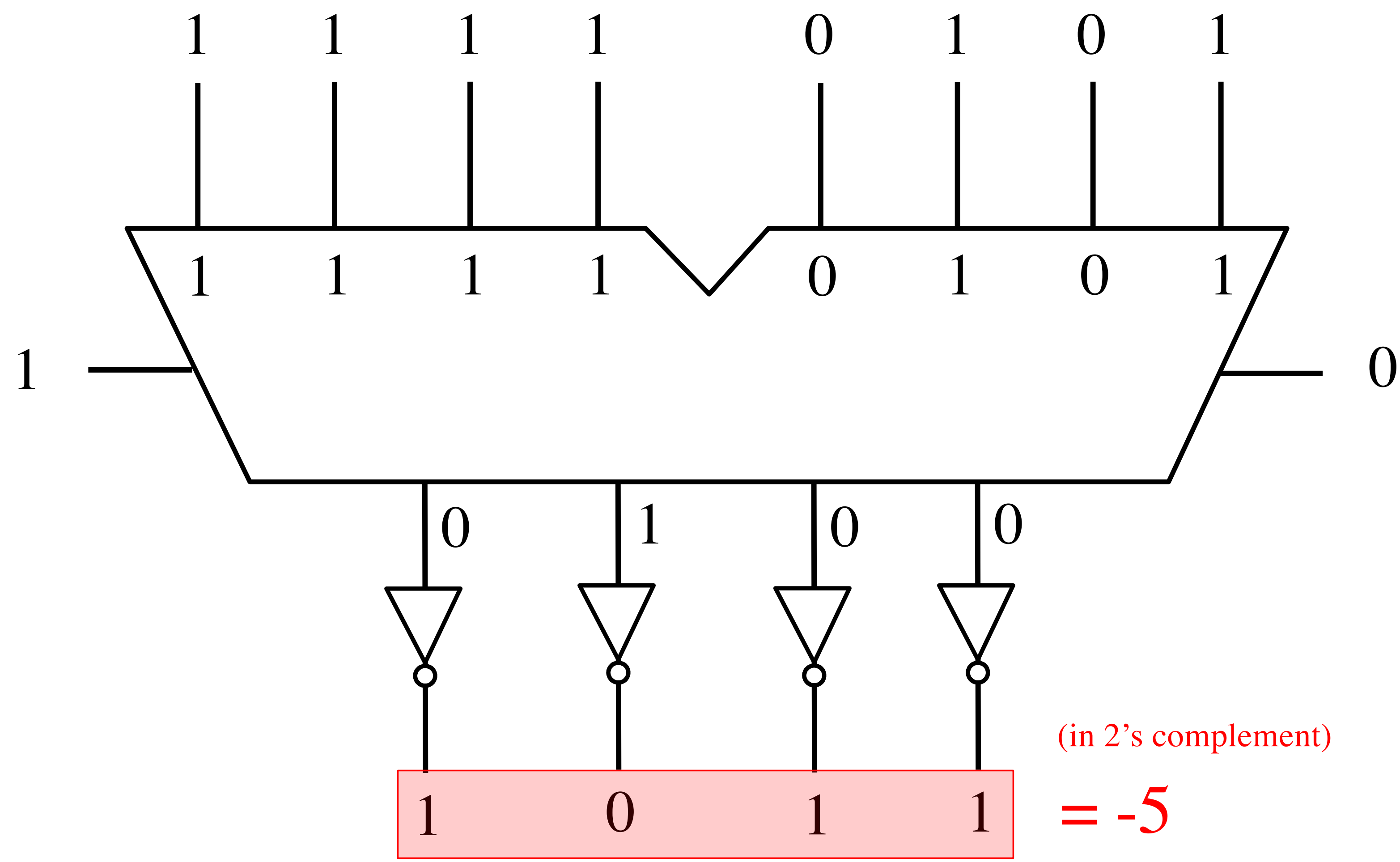
Circuit #3 for negating a number stored in 2's complement representation



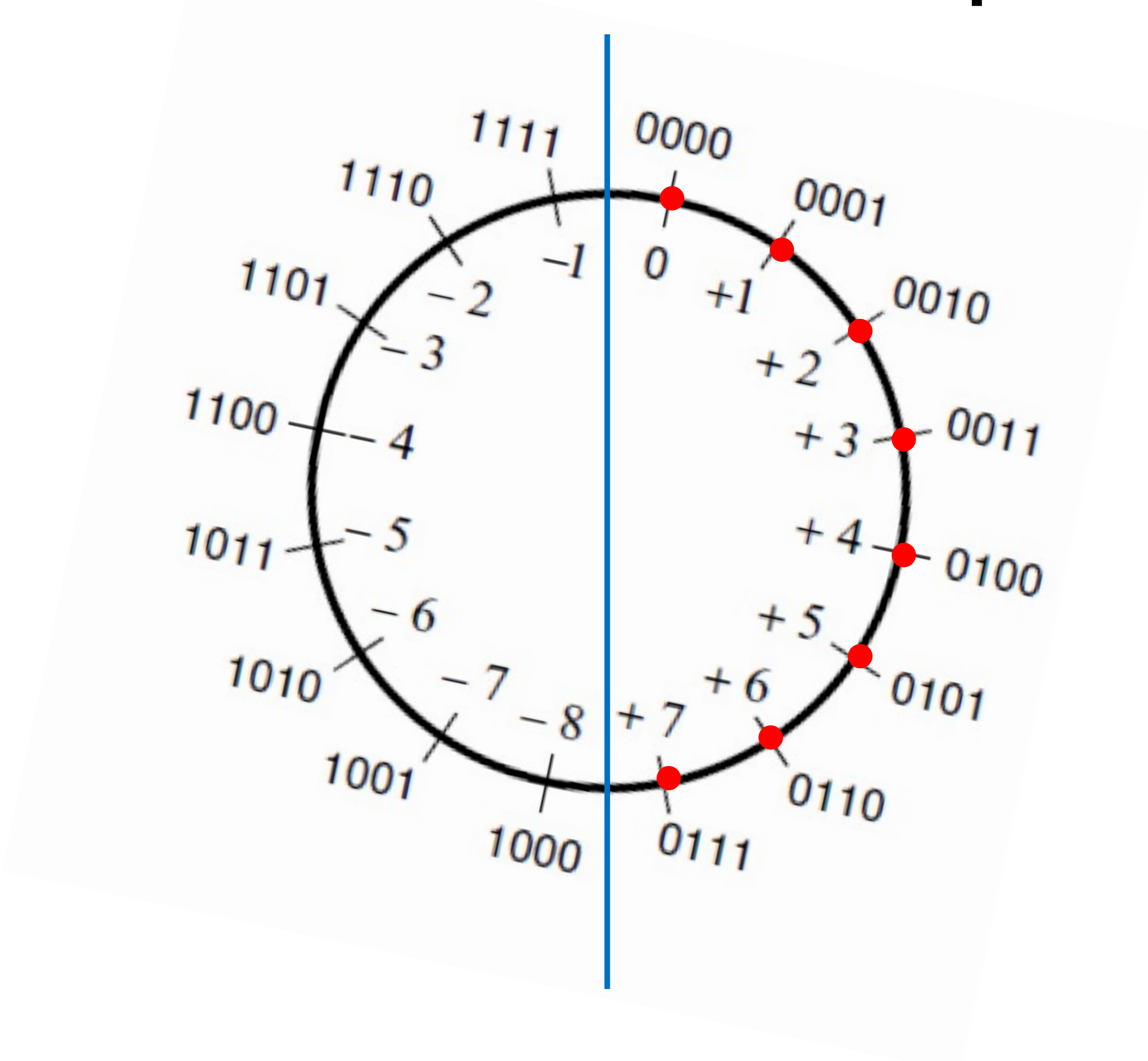
Circuit #3 for negating a number stored in 2's complement representation



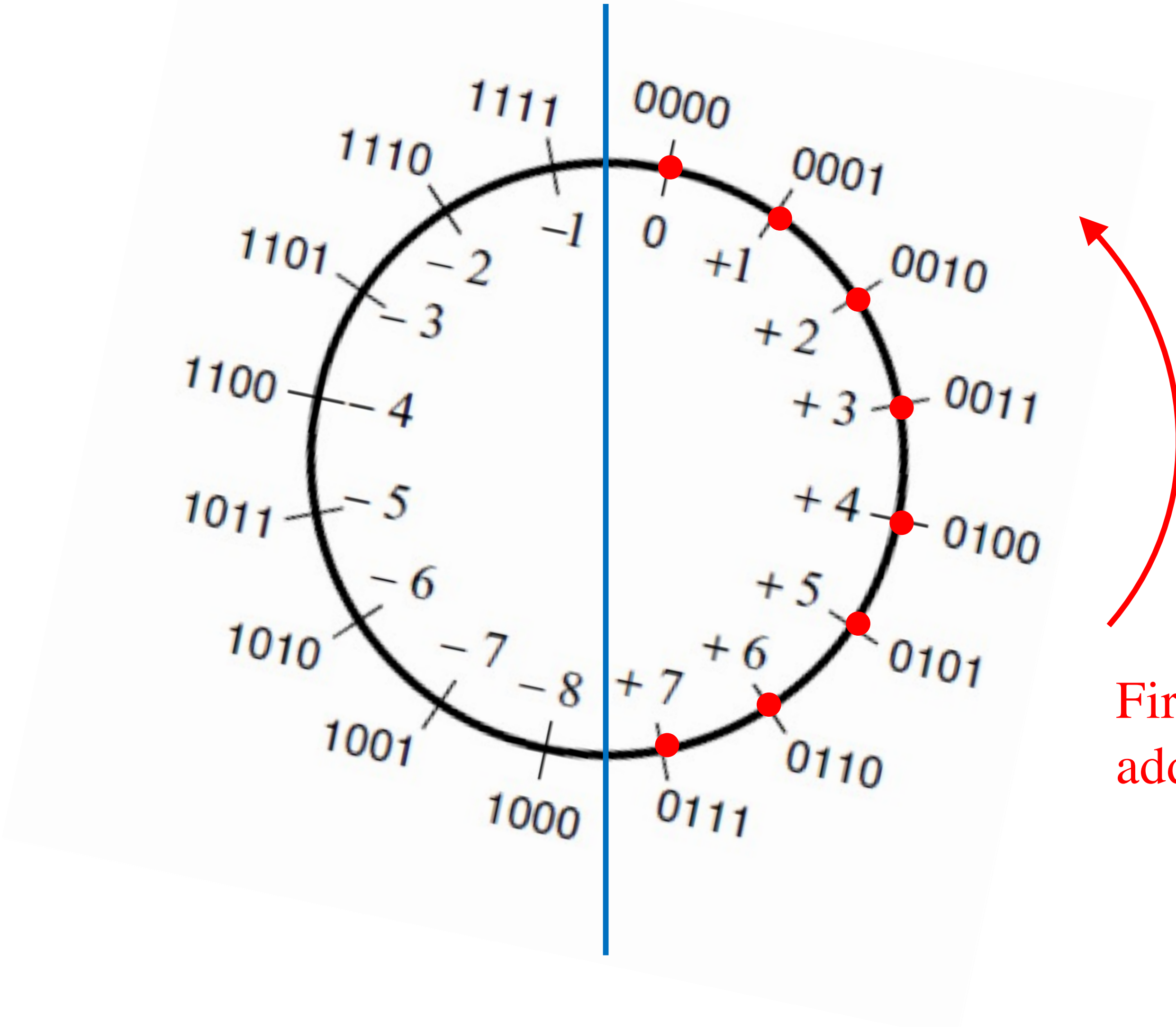
Circuit #3 for negating a number stored in 2's complement representation



The number circle for 2's complement

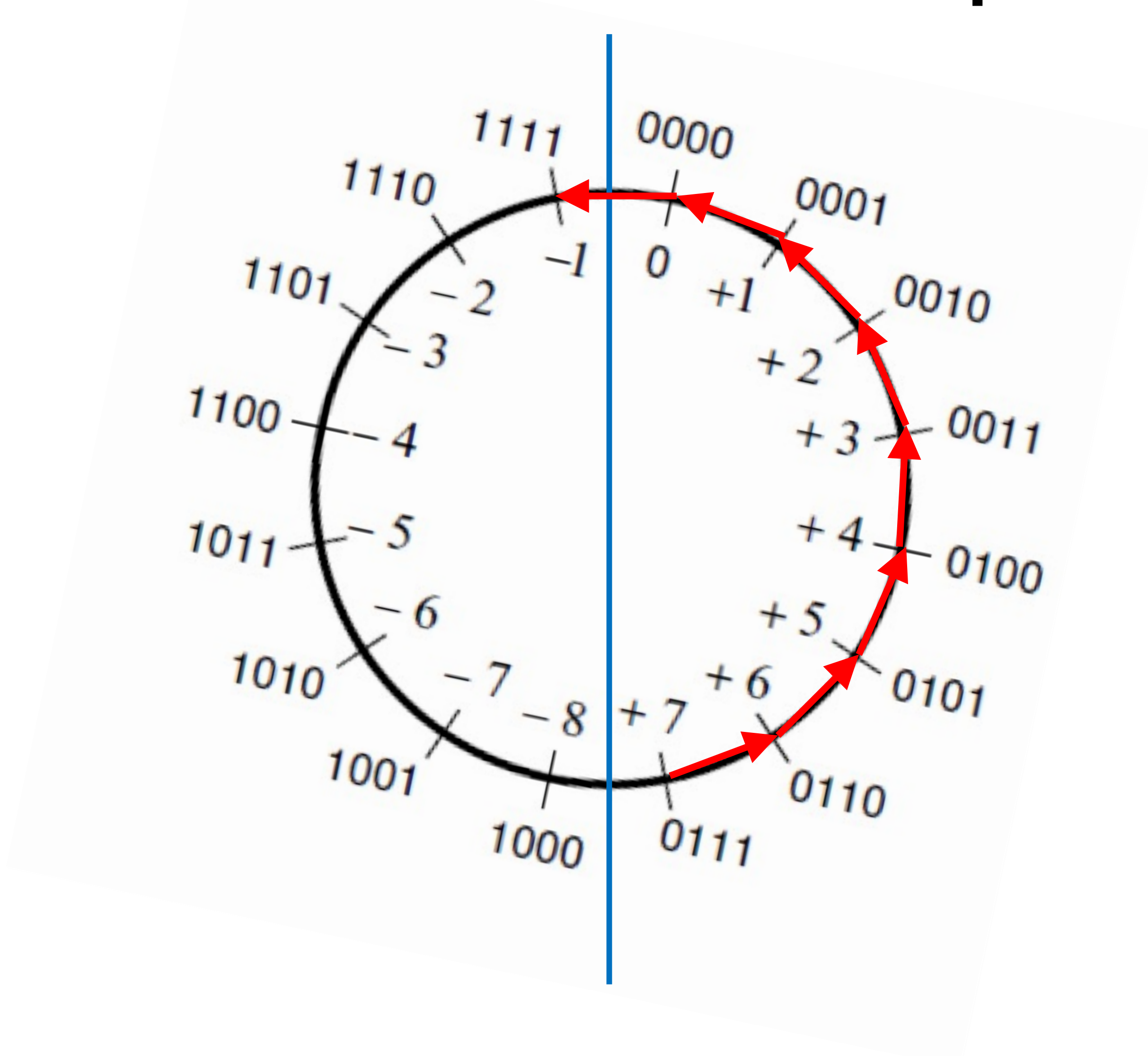


The number circle for 2's complement

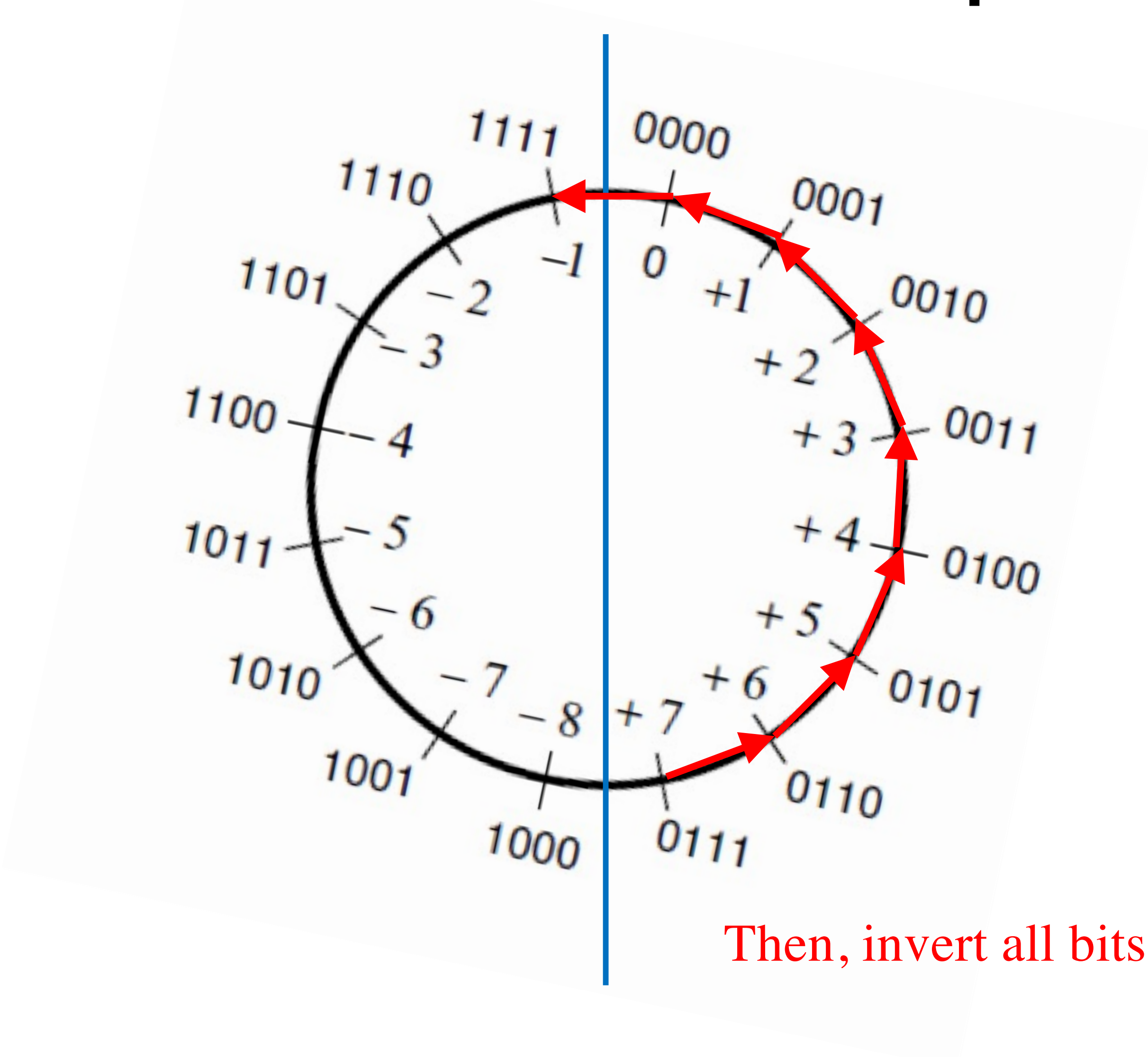


First,
add (-1) …

The number circle for 2's complement

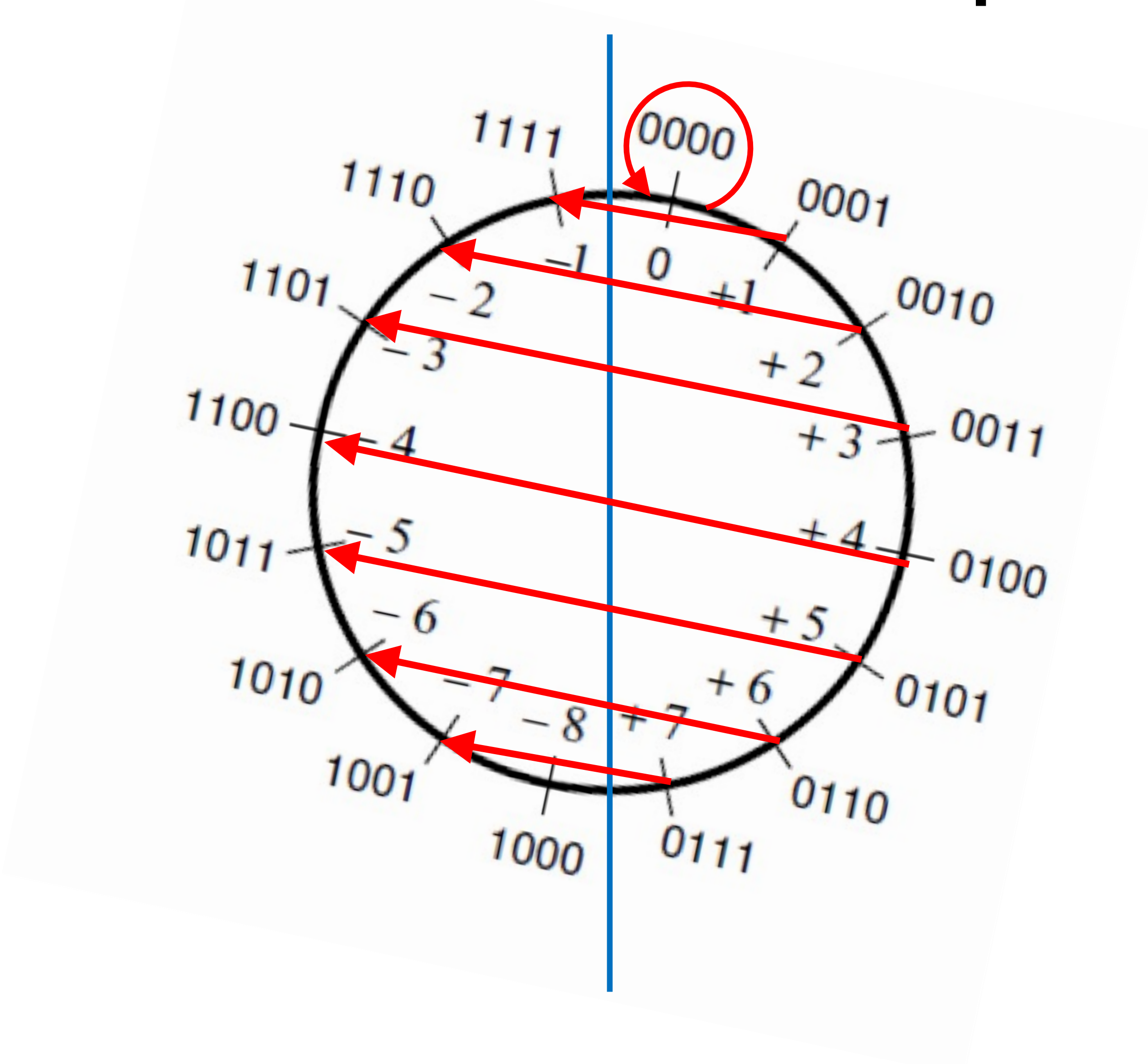


The number circle for 2's complement

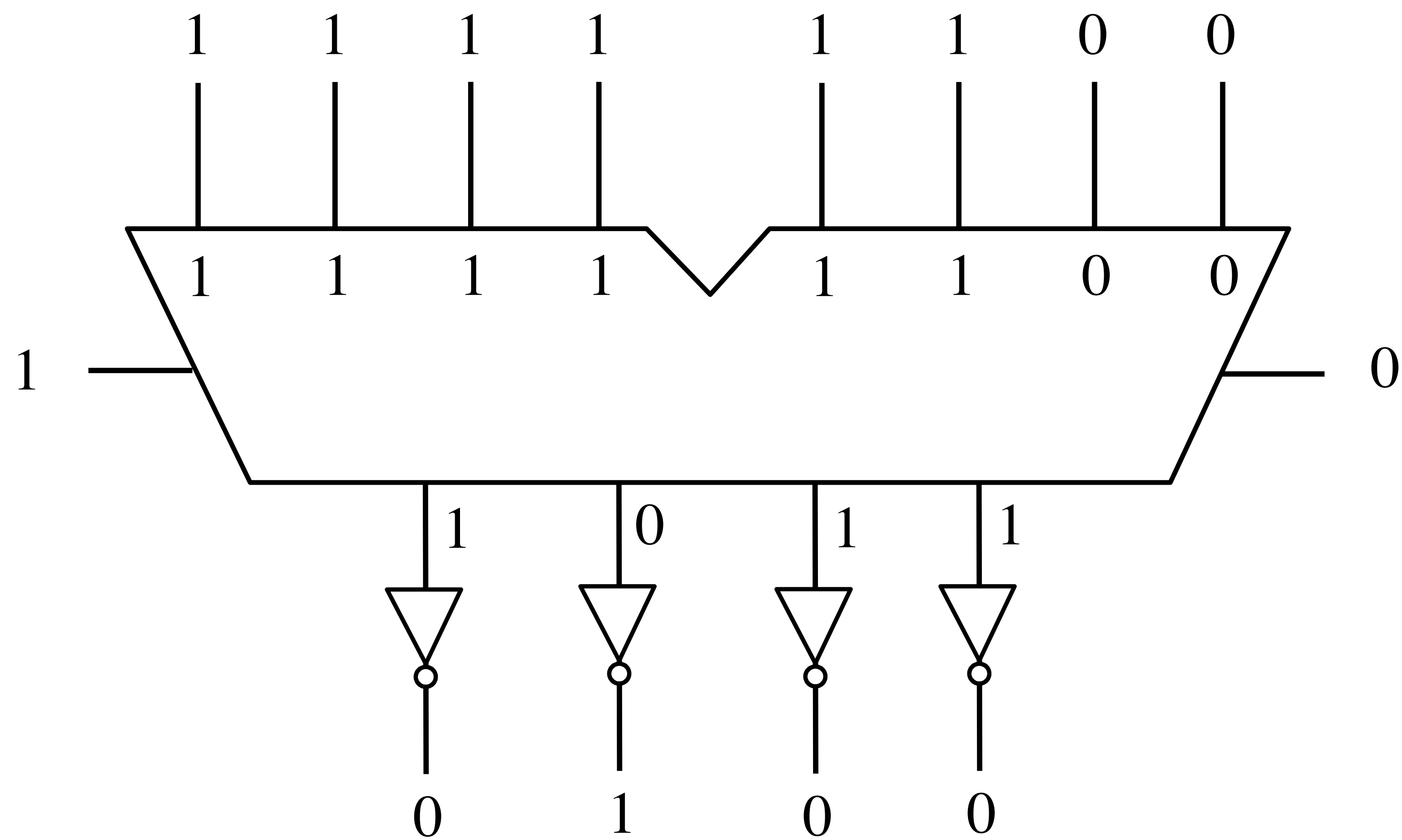


Then, invert all bits …

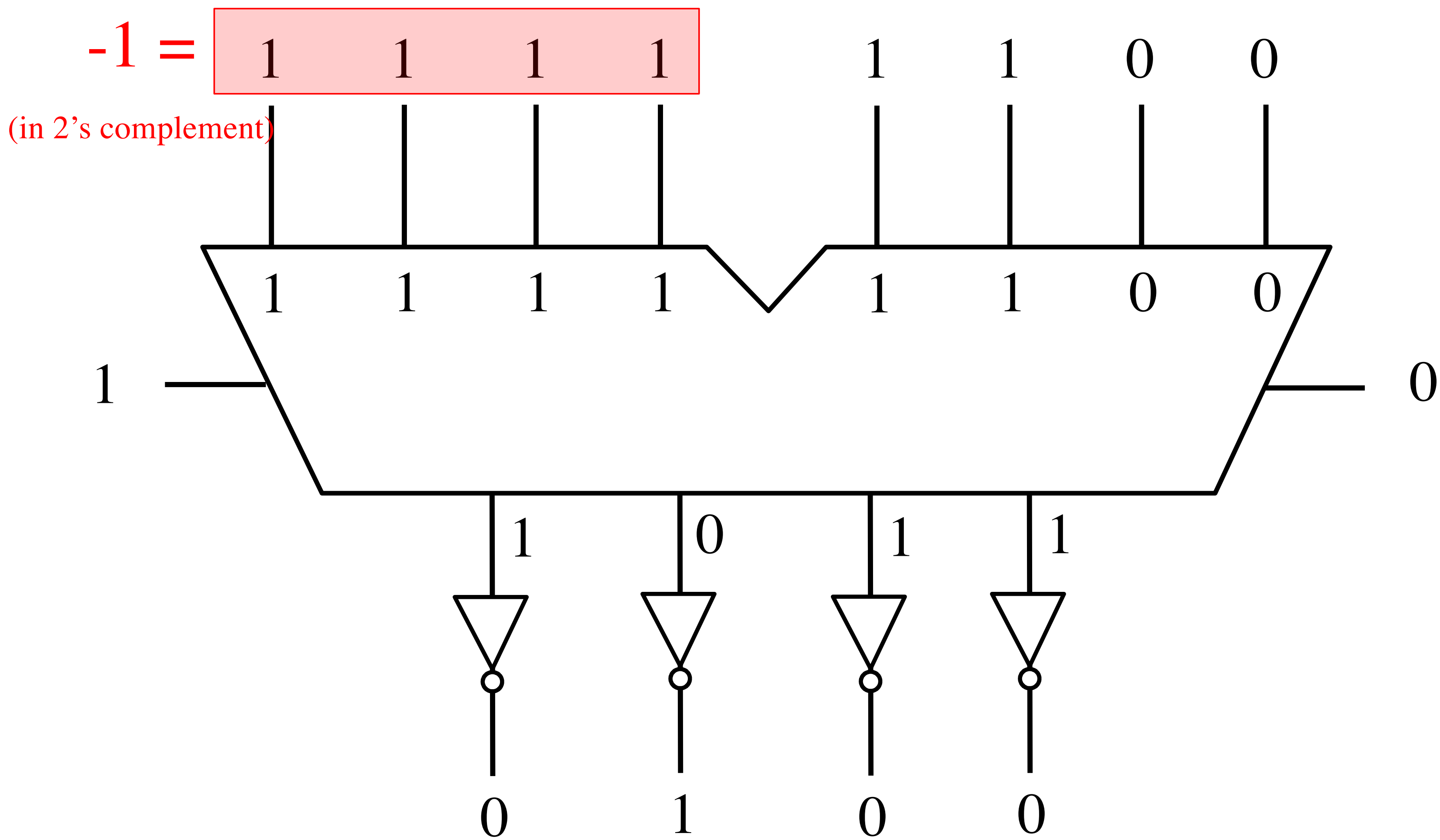
The number circle for 2's complement



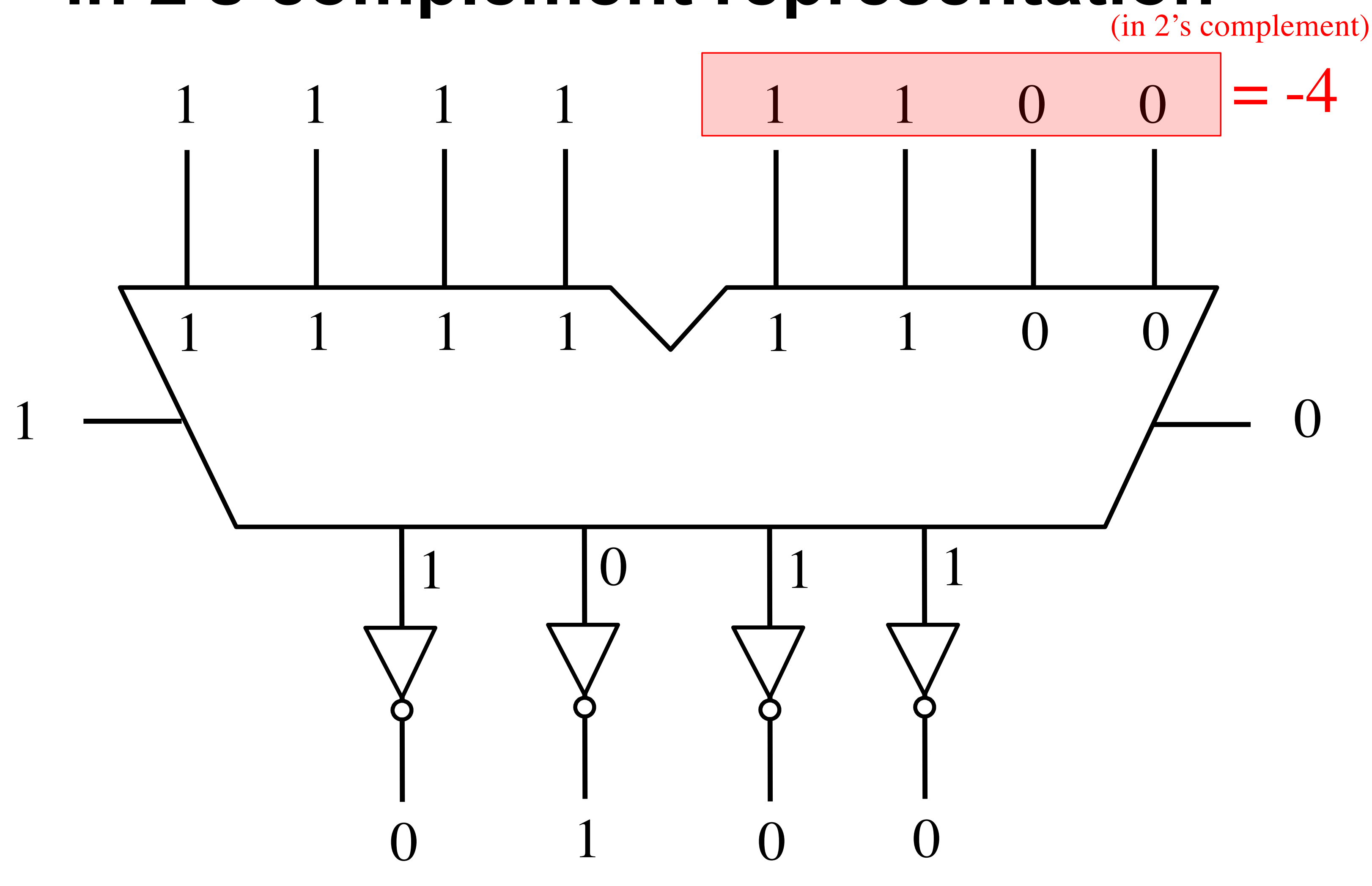
Circuit #3 for negating a number stored in 2's complement representation



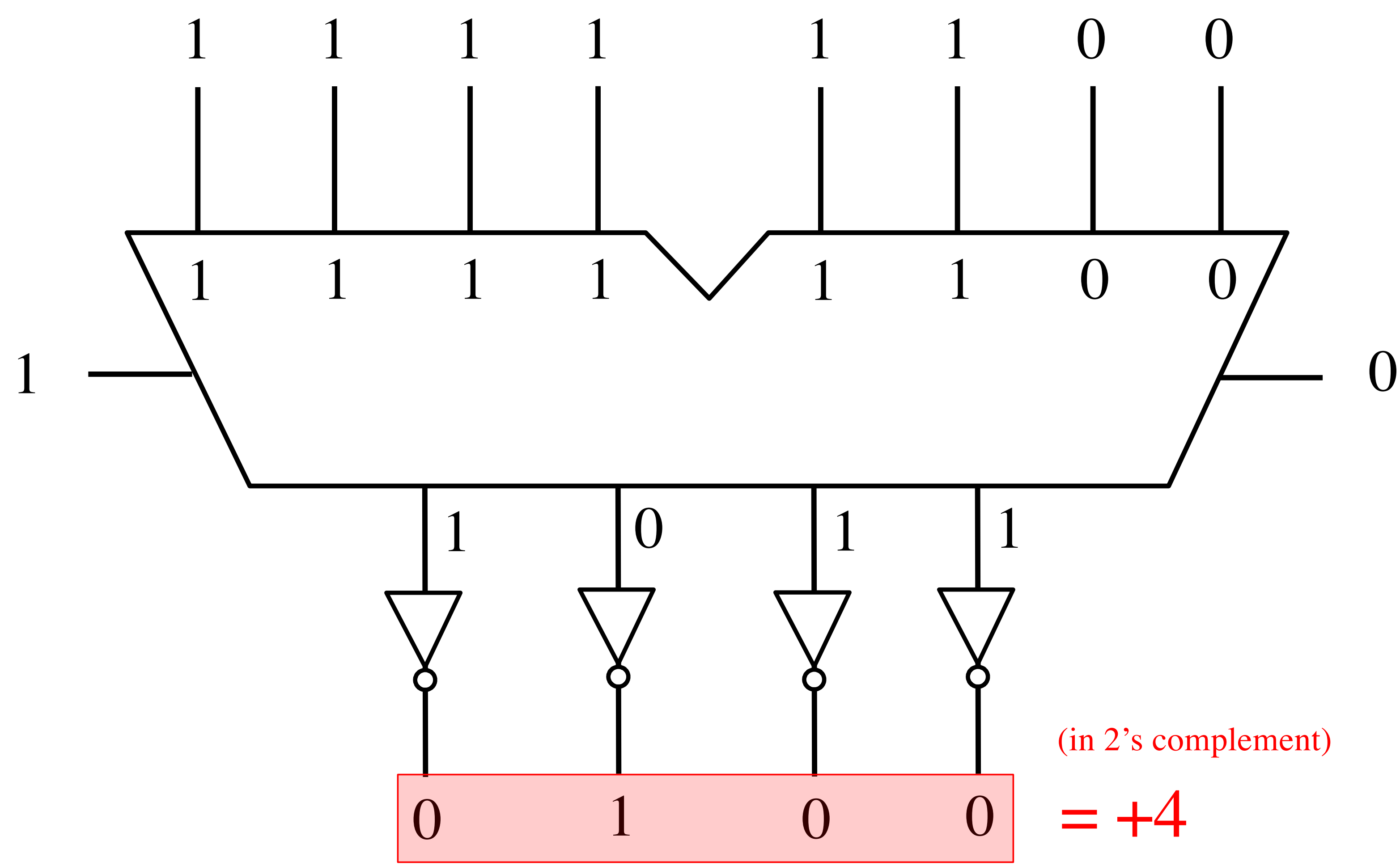
Circuit #3 for negating a number stored in 2's complement representation



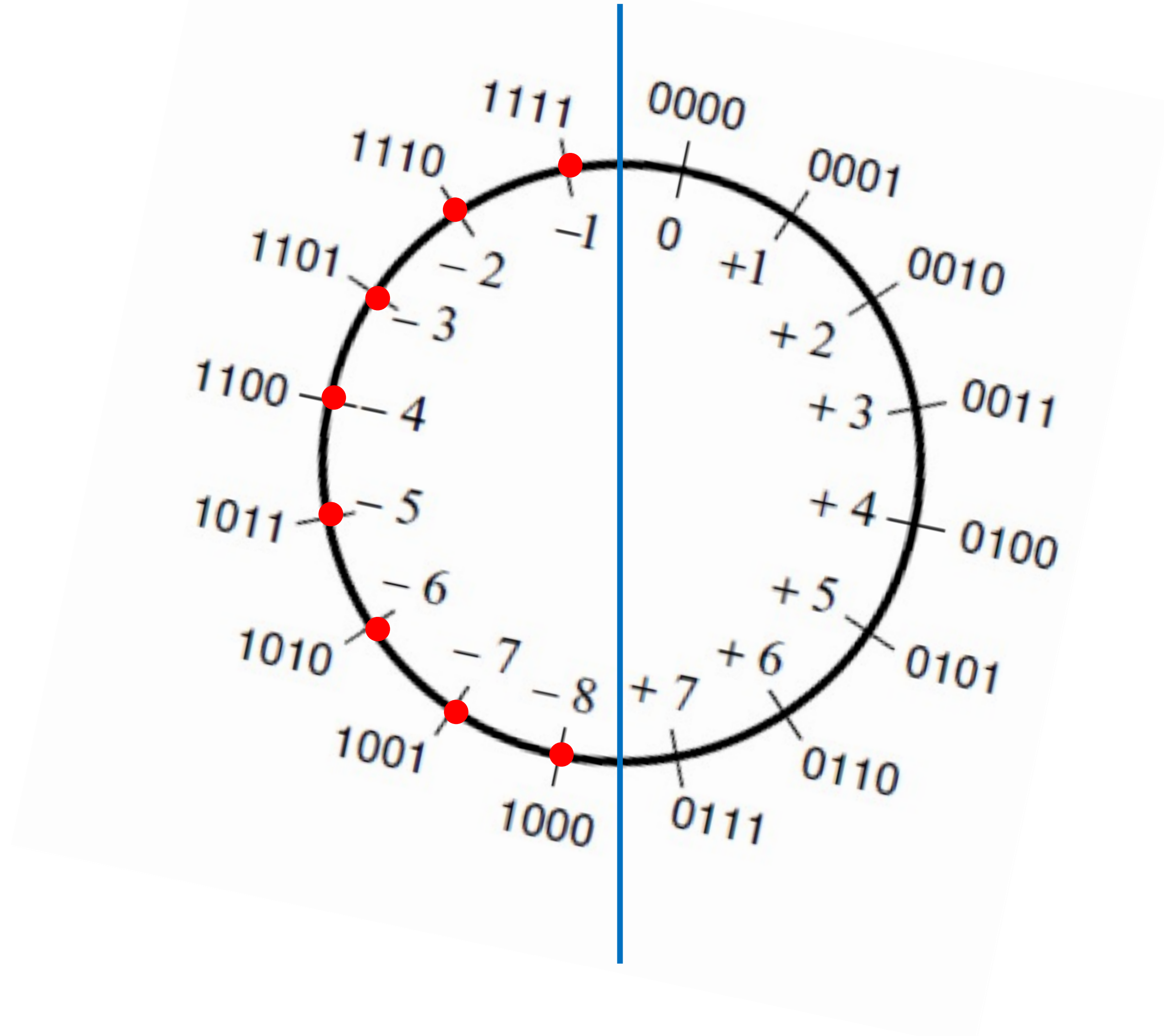
Circuit #3 for negating a number stored in 2's complement representation



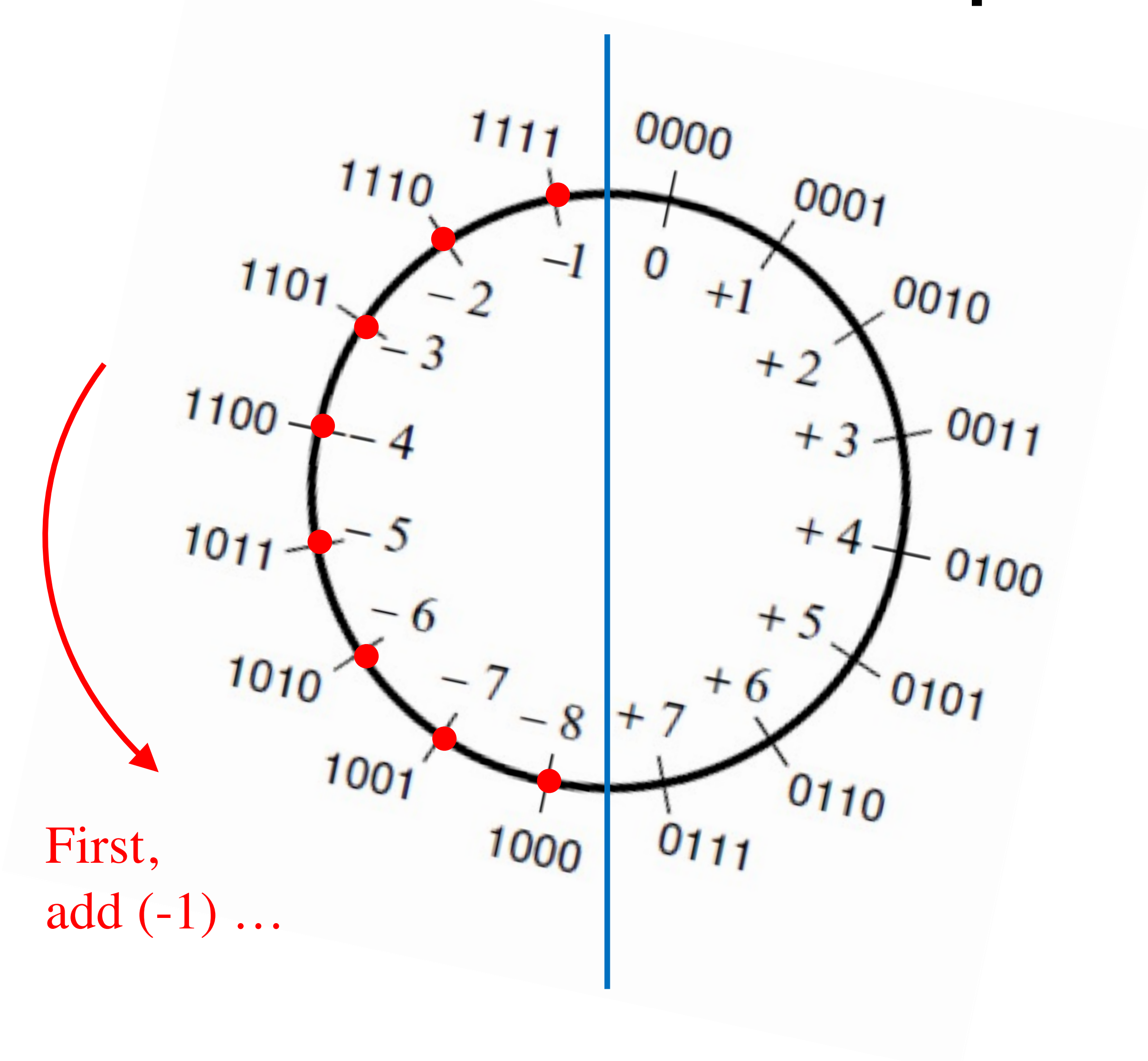
Circuit #3 for negating a number stored in 2's complement representation



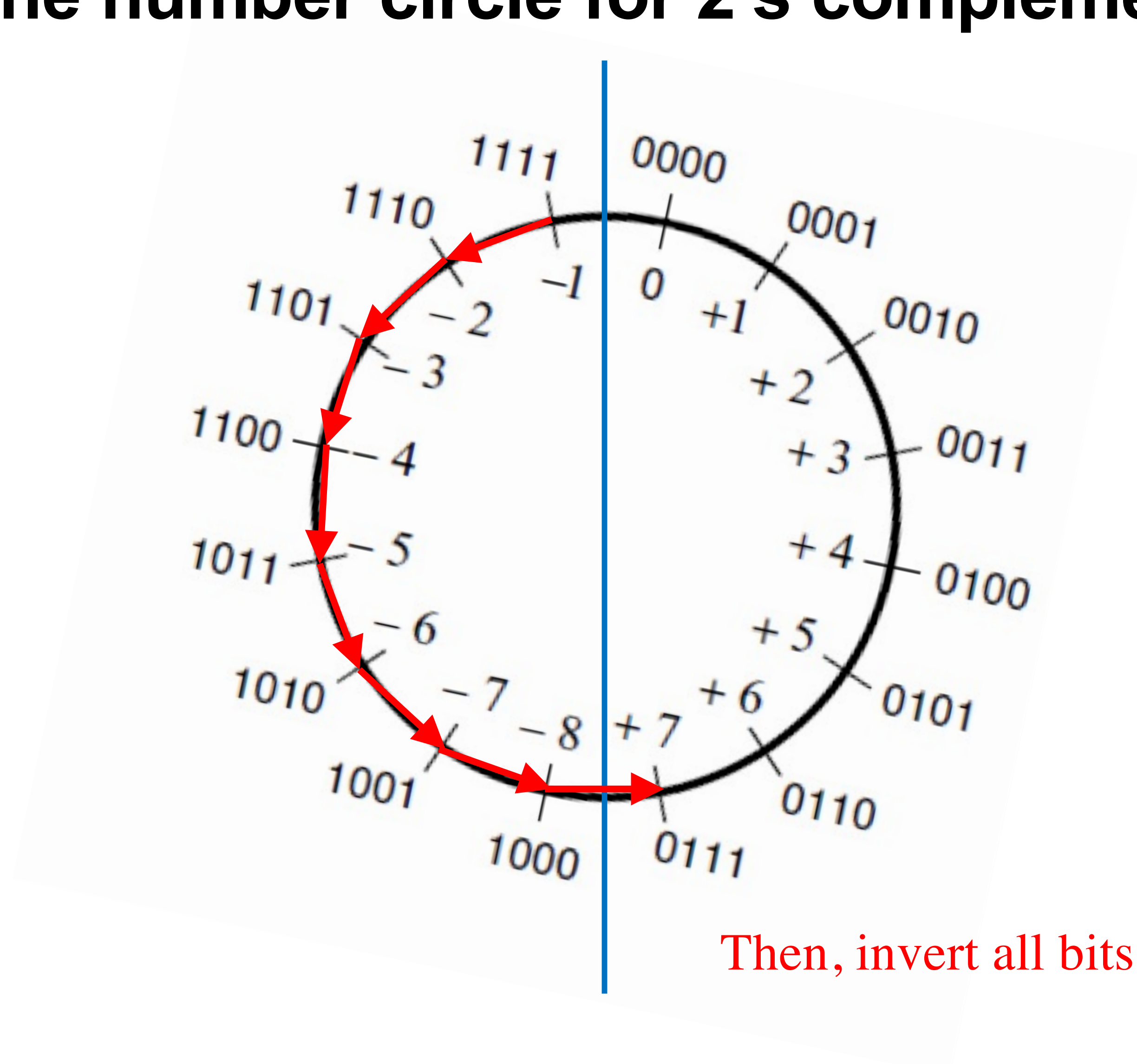
The number circle for 2's complement



The number circle for 2's complement



The number circle for 2's complement



Then, invert all bits …

The number circle for 2's complement