

CprE 281: Digital Logic

Instructor: Alexander Stoytchev

<http://www.ece.iastate.edu/~alexs/classes/>

Synthesis

Using AND, OR, and NOT Gates

Administrative Stuff

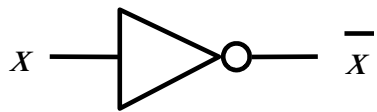
- **HW2 is due on Wednesday Sep 6 @ 10pm**
- **Please write clearly on the first page the following three things:**
 - **Your First and Last Name**
 - **Your Student ID Number**
 - **Your Lab Section Letter**
- **Submit on Canvas as *one* PDF file.**
- **Please orient your pages such that the text can be read without the need to rotate the page.**

Administrative Stuff

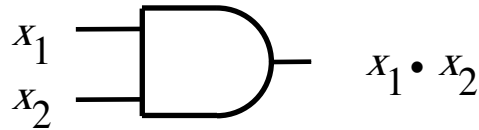
- **Next week we will start with Lab2**
- **Read the lab assignment and do the prelab at home.**
- **Complete the prelab on paper before you go to the lab. Otherwise you'll lose 20% of your grade for that lab.**

Quick Review

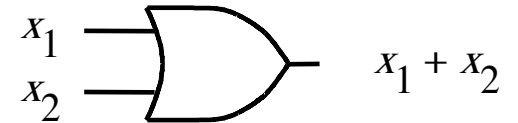
The Three Basic Logic Gates



NOT gate

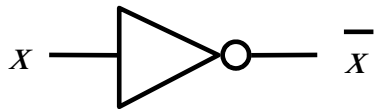


AND gate



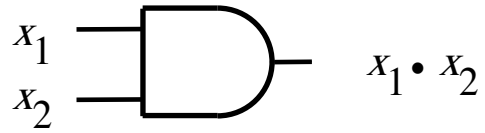
OR gate

Truth Table for NOT



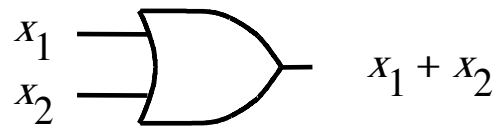
x	\bar{x}
0	1
1	0

Truth Table for AND



x_1	x_2	$x_1 \cdot x_2$
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table for OR



x_1	x_2	$x_1 + x_2$
0	0	0
0	1	1
1	0	1
1	1	1

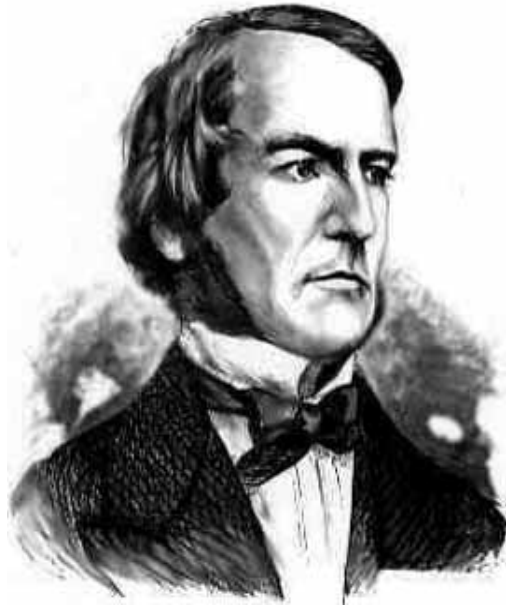
Truth Tables for AND and OR

x_1	x_2	x_1	x_2	$x_1 + x_2$
0	0	0	0	0
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

AND

OR

Boolean Algebra



George Boole
1815-1864

- **An algebraic structure consists of**
 - a set of elements $\{0, 1\}$
 - binary operators $\{+, \cdot\}$
 - and a unary operator $\{ ' \}$ or $\{ \bar{\quad} \}$ or $\{ \sim \}$
- **Introduced by George Boole in 1854**
- **An effective means of describing circuits built with switches**
- **A powerful tool that can be used for designing and analyzing logic circuits**

Different Notations for Negation

- All three of these mean “negate x”
 - x'
 - \bar{x}
 - $\sim x$

Operator Precedence

- In regular arithmetic and algebra, multiplication takes precedence over addition.
- This is also true in Boolean algebra.
- For example, $x + y \cdot z$ means multiply y by z and add the product to x .
- In other words, $x + y \cdot z$ is equal to $x + (y \cdot z)$, not $(x + y) \cdot z$.

The multiplication dot is optional

- In regular algebra, the multiplication operator is often omitted to shorten the equations.
- This is also true in Boolean algebra.
- Both of these mean the same thing:
 xy is equal to $x \cdot y$

Operator Precedence

(three different ways to write the same)

$$x_1 \cdot x_2 + \bar{x}_1 \cdot \bar{x}_2$$

$$(x_1 \cdot x_2) + ((\bar{x}_1) \cdot (\bar{x}_2))$$

$$x_1x_2 + \bar{x}_1\bar{x}_2$$

Operator Precedence

- **Negation of a single variable takes precedence over multiplication of that variable with another variable.**

- **For example,**

$\bar{A} B$ means negate A first and then multiply \bar{A} by B

Operator Precedence

- However, a horizontal bar over a product of two variables means that the negation is performed after the product is computed.

- For example,

$\overline{A B}$ means multiply A and B and then negate

Operator Precedence

- Note that these two expressions are different:

$\overline{A B}$ is not equal to $\overline{A} \overline{B}$

$\overline{A B}$ means multiply A and B and then negate

$\overline{A} \overline{B}$ means negate A and B separately and then multiply

Operator Precedence

- Note that these two expressions are different:

$\overline{A B}$ is not equal to $\overline{A} \overline{B}$

A	B	$\overline{A B}$
0	0	1
0	1	1
1	0	1
1	1	0

A	B	$\overline{A} \overline{B}$
0	0	1
0	1	0
1	0	0
1	1	0

DeMorgan's Theorem

$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

$$15b. \quad \overline{x + y} = \bar{x} \cdot \bar{y}$$

Proof of DeMorgan's theorem

$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

x	y	$x \cdot y$	$\overline{x \cdot y}$	\bar{x}	\bar{y}	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

$\underbrace{\hspace{15em}}_{\text{LHS}} \qquad \underbrace{\hspace{15em}}_{\text{RHS}}$

Proof of DeMorgan's theorem

$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

x	y	$x \cdot y$	$\overline{x \cdot y}$	\bar{x}	\bar{y}	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

$\underbrace{\hspace{10em}}_{\text{LHS}} \qquad \underbrace{\hspace{10em}}_{\text{RHS}}$

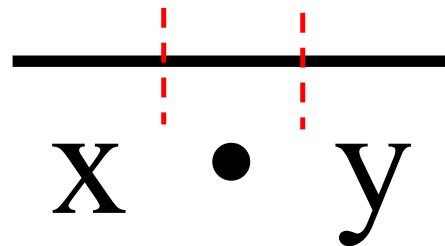
These two columns are equal. Therefore, the theorem is true.

How to remember DeMorgan's theorem

$$\overline{x \cdot y}$$

start with the
left-hand side

How to remember DeMorgan's theorem



divide the bar
into 3 equal parts

How to remember DeMorgan's theorem

$$\overline{X} \cdot \overline{y}$$

erase the
middle segment

How to remember DeMorgan's theorem

$$\overline{x} + \overline{y}$$

change the
product to a sum

How to remember DeMorgan's theorem

$$\overline{x} + \overline{y}$$

this is the
right-hand side



How to remember DeMorgan's theorem

$$\overline{x \cdot y} = \overline{x} + \overline{y}$$

Proof of the other DeMorgan's theorem

$$15b. \quad \overline{x + y} = \bar{x} \cdot \bar{y}$$

x	y	$x + y$	$\overline{x + y}$	\bar{x}	\bar{y}	$\bar{x} \cdot \bar{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

 LHS  RHS

Proof of the other DeMorgan's theorem

$$15b. \quad \overline{x + y} = \bar{x} \cdot \bar{y}$$

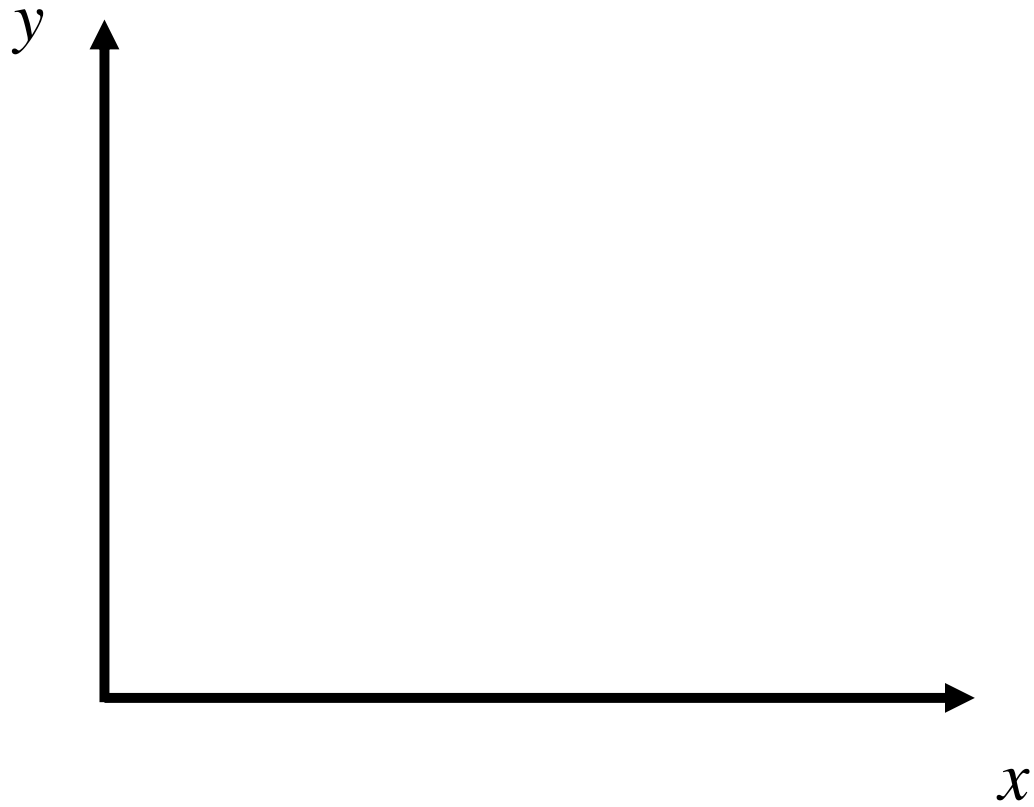
x	y	$x + y$	$\overline{x + y}$	\bar{x}	\bar{y}	$\bar{x} \cdot \bar{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

$\underbrace{\hspace{15em}}_{\text{LHS}} \qquad \underbrace{\hspace{15em}}_{\text{RHS}}$

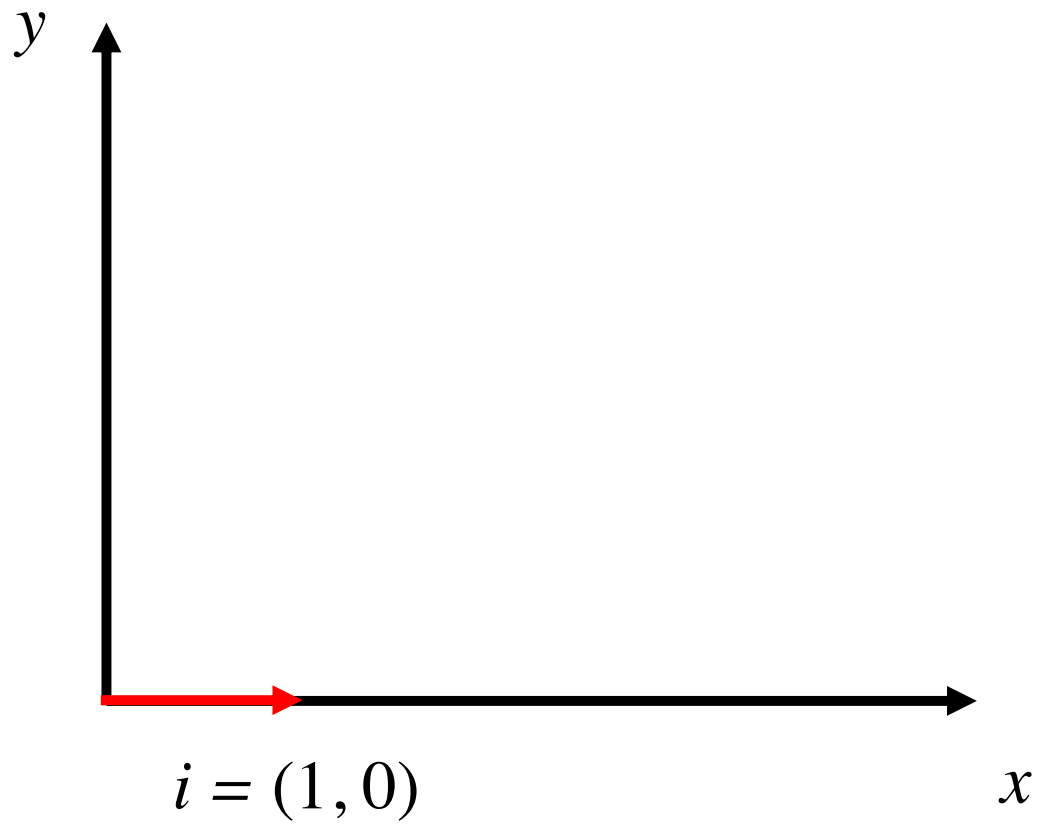
These two columns are equal. Therefore, the theorem is true.

A Short Digression

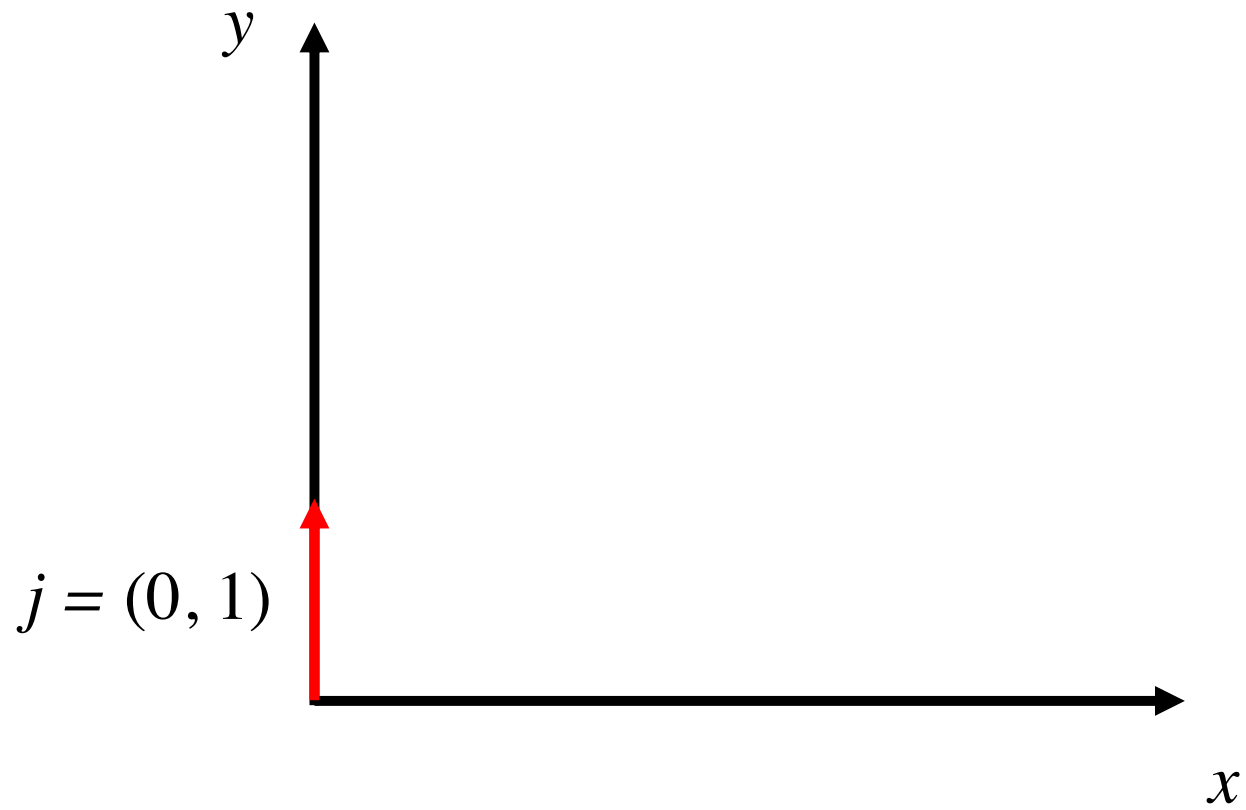
The 2D Plane



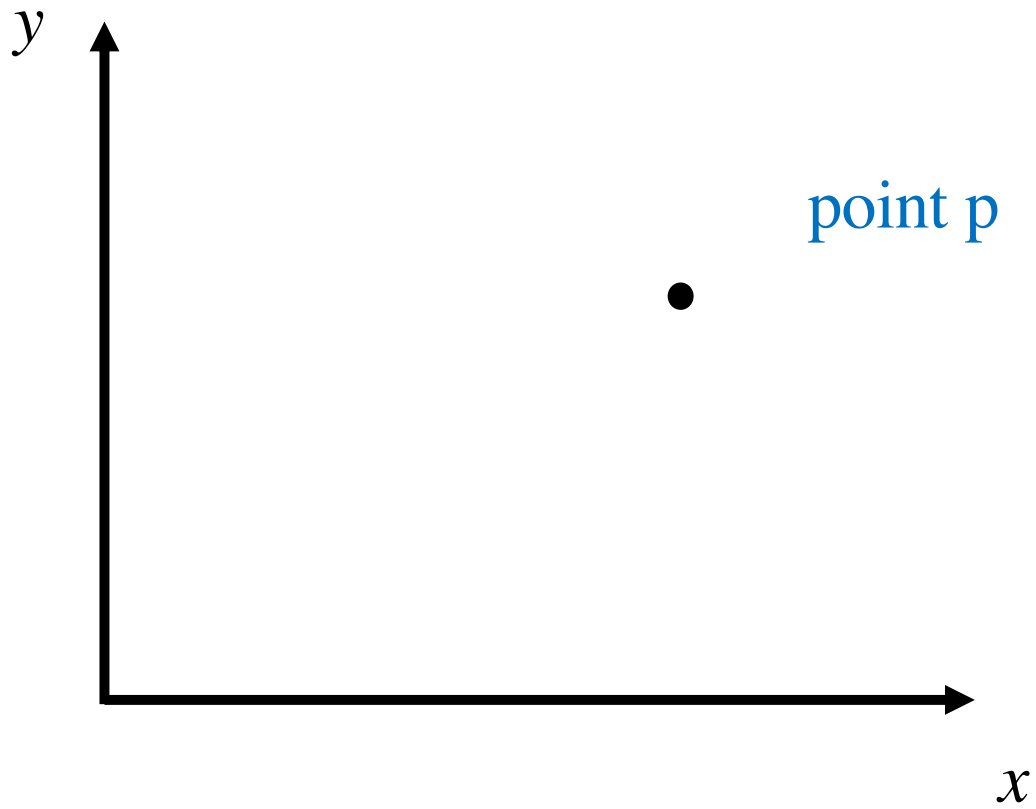
The 2D Plane



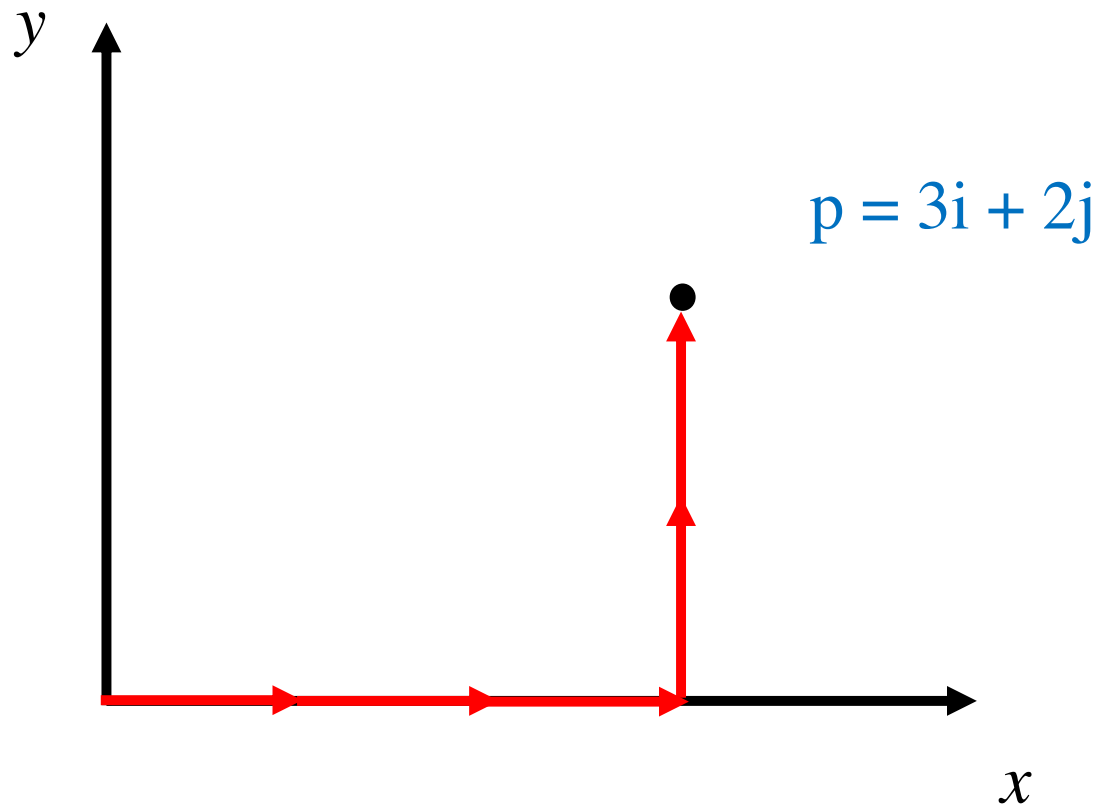
The 2D Plane



The 2D Plane



The 2D Plane



The unit vectors \mathbf{i} and \mathbf{j} form a basis

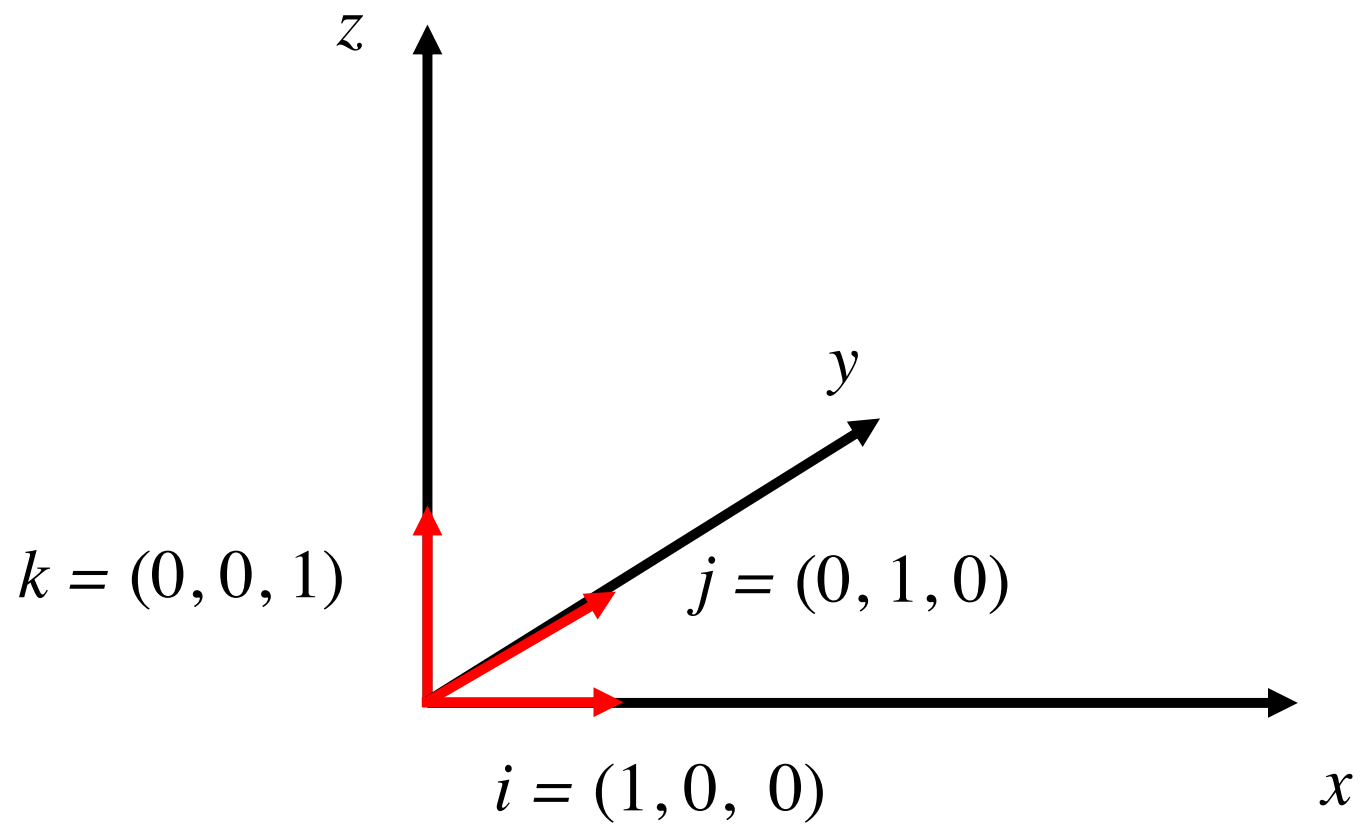
- Any point in the 2D plane can be represented as a linear combination of these two vectors.

$$\mathbf{i} = (1, 0)$$

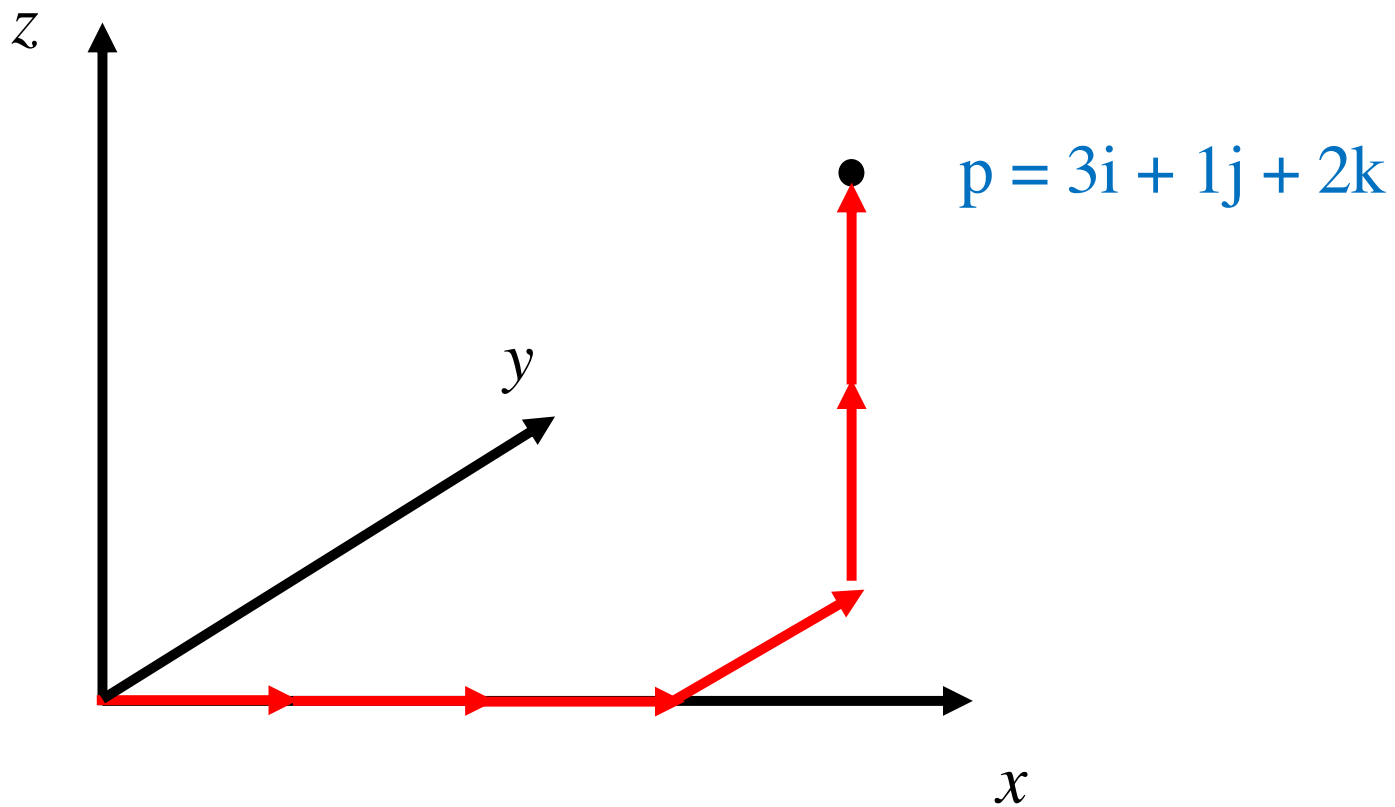
$$\mathbf{j} = (0, 1)$$

Note that there is only one 1 in each.

3D Space



3D Space



The 3D Basis

- In 3D we have i , j , and k

$$i = (1, 0, 0)$$

$$j = (0, 1, 0)$$

$$k = (0, 0, 1)$$

Note that there is only one 1 in each.

Any point in the 3D space can be represented as a linear combination of these three basis vectors.

The 4D Basis

- In 4D we have four vectors

$$\mathbf{x}^1 = (1, 0, 0, 0)$$

$$\mathbf{x}^2 = (0, 1, 0, 0)$$

$$\mathbf{x}^3 = (0, 0, 1, 0)$$

$$\mathbf{x}^4 = (0, 0, 0, 1)$$

Note that there is only one 1 in each.

Any point in this 4D space can be represented as a linear combination of these four basis vectors.

Basis Functions (for two variables)

The Four Basis Functions

x	y	f₀₀
0	0	1
0	1	0
1	0	0
1	1	0

$f_{00}(x, y)$

x	y	f₀₁
0	0	0
0	1	1
1	0	0
1	1	0

$f_{01}(x, y)$

x	y	f₁₀
0	0	0
0	1	0
1	0	1
1	1	0

$f_{10}(x, y)$

x	y	f₁₁
0	0	0
0	1	0
1	0	0
1	1	1

$f_{11}(x, y)$

The Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$f_{00}(x, y)$

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$f_{01}(x, y)$

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$f_{10}(x, y)$

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$f_{11}(x, y)$

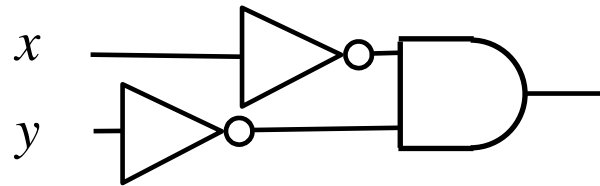
The Four Basis Functions

x	y	$f_{00}(x, y)$	$f_{01}(x, y)$	$f_{10}(x, y)$	$f_{11}(x, y)$
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

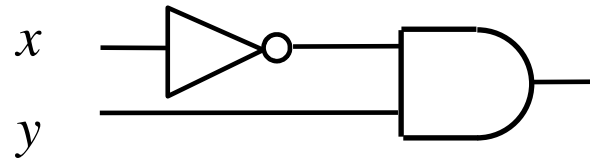
The Four Basis Functions

x	y	$\overline{x} \overline{y}$	$\overline{x} y$	$x \overline{y}$	$x y$
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

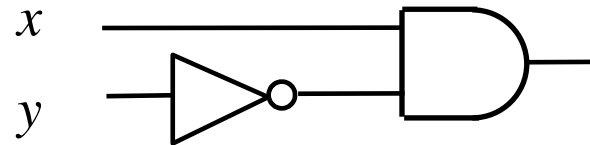
Circuits for the four basis functions



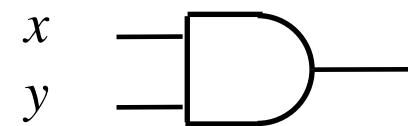
$$f_{00}(x, y) = \bar{x} \bar{y}$$



$$f_{01}(x, y) = \bar{x} y$$



$$f_{10}(x, y) = x \bar{y}$$



$$f_{11}(x, y) = x y$$

The Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$$f_{00}(x, y) = \bar{x} \bar{y}$$

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$$f_{01}(x, y) = \bar{x} y$$

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$$f_{10}(x, y) = x \bar{y}$$

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{11}(x, y) = x y$$

The Four Basis Functions

x	y	f ₀₀
0	0	1
0	1	0
1	0	0
1	1	0

x	y	f ₀₁
0	0	0
0	1	1
1	0	0
1	1	0

x	y	f ₁₀
0	0	0
0	1	0
1	0	1
1	1	0

x	y	f ₁₁
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{00}(x, y) = \overline{x} \overline{y}$$

$$f_{01}(x, y) = \overline{x} y$$

$$f_{10}(x, y) = x \overline{y}$$

$$f_{11}(x, y) = x y$$

Negate the variable if the corresponding subscript of f is 0.

The Four Basis Functions

x	y	f ₀₀
0	0	1
0	1	0
1	0	0
1	1	0

x	y	f ₀₁
0	0	0
0	1	1
1	0	0
1	1	0

x	y	f ₁₀
0	0	0
0	1	0
1	0	1
1	1	0

x	y	f ₁₁
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{00}(x, y) = \bar{x} \bar{y}$$

$$f_{01}(x, y) = \bar{x} y$$

$$f_{10}(x, y) = x \bar{y}$$

$$f_{11}(x, y) = x y$$

Negate the variable if the corresponding subscript of f is 0.

The Four Basis Functions

x	y	f ₀₀
0	0	1
0	1	0
1	0	0
1	1	0

x	y	f ₀₁
0	0	0
0	1	1
1	0	0
1	1	0

x	y	f ₁₀
0	0	0
0	1	0
1	0	1
1	1	0

x	y	f ₁₁
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{00}(x, y) = \bar{x} \bar{y}$$

$$f_{01}(x, y) = \bar{x} y$$

$$f_{10}(x, y) = x \bar{y}$$

$$f_{11}(x, y) = x y$$

Negate the variable if the corresponding subscript of f is 0.

minterms
(an alternative name for
the set of basis functions)

The Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$$f_{00}(x, y) = \bar{x} \bar{y}$$

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$$f_{01}(x, y) = \bar{x} y$$

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$$f_{10}(x, y) = x \bar{y}$$

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{11}(x, y) = x y$$

The Four Basis Functions (alternative names)

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$$f_{00}(x, y) = \bar{x} \bar{y}$$

m_0

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$$f_{01}(x, y) = \bar{x} y$$

m_1

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$$f_{10}(x, y) = x \bar{y}$$

m_2

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{11}(x, y) = x y$$

m_3

The Four Basis Functions (minterms)

x	y	m_0
0	0	1
0	1	0
1	0	0
1	1	0

$$f_{00}(x, y) = \bar{x} \bar{y}$$

m_0

x	y	m_1
0	0	0
0	1	1
1	0	0
1	1	0

$$f_{01}(x, y) = \bar{x} y$$

m_1

x	y	m_2
0	0	0
0	1	0
1	0	1
1	1	0

$$f_{10}(x, y) = x \bar{y}$$

m_2

x	y	m_3
0	0	0
0	1	0
1	0	0
1	1	1

$$f_{11}(x, y) = x y$$

m_3

The Four Basis Functions (minterms)

x	y	$\bar{x}\bar{y}$	$\bar{x}y$	$x\bar{y}$	xy
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

Expressions for the minterms

$$m_0 = \bar{x} \bar{y}$$

$$m_1 = \bar{x} y$$

$$m_2 = x \bar{y}$$

$$m_3 = x y$$

Expressions for the minterms

$$0 \quad 0 \quad m_0 = \bar{x} \bar{y}$$

$$0 \quad 1 \quad m_1 = \bar{x} y$$

$$1 \quad 0 \quad m_2 = x \bar{y}$$

$$1 \quad 1 \quad m_3 = x y$$

The bars coincide
with the 0's
in the binary expansion
of the minterm sub-index

Expressions for the minterms

$$0 \quad 0 \quad m_0 = \bar{x} \bar{y}$$

$$0 \quad 1 \quad m_1 = \bar{x} y$$

$$1 \quad 0 \quad m_2 = x \bar{y}$$

$$1 \quad 1 \quad m_3 = x y$$

The bars coincide
with the 0's
in the binary expansion
of the minterm sub-index

Function Synthesis Example (with two variables)

Synthesize the Following Function

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

1) Split the function into a sum of 4 functions

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

1) Split the function into a sum of 4 functions

x_1	x_2	$f(x_1, x_2)$	$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1	1	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	1	0
1	1	1	0	0	0	1

$$f(x_1, x_2) = 1 \cdot f_{00} + 1 \cdot f_{01} + 0 \cdot f_{10} + 1 \cdot f_{11}$$

2) Write the expressions for all four

x_1	x_2	$f(x_1, x_2)$	$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1	1	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	1	0
1	1	1	0	0	0	1

$$f(x_1, x_2) = \underbrace{1 \cdot f_{00}} + \underbrace{1 \cdot f_{01}} + \underbrace{0 \cdot f_{10}} + \underbrace{1 \cdot f_{11}}$$

2) Write the expressions for all four

x_1	x_2	$f(x_1, x_2)$	$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1	1	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	1	0
1	1	1	0	0	0	1

$$f(x_1, x_2) = \underbrace{1 \cdot f_{00}}_{\bar{x}_1 \bar{x}_2} + \underbrace{1 \cdot f_{01}}_{\bar{x}_1 x_2} + \underbrace{0 \cdot f_{10}}_0 + \underbrace{1 \cdot f_{11}}_{x_1 x_2}$$

3) Then just add them together

x_1	x_2	$f(x_1, x_2)$	$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1	1	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	1	0
1	1	1	0	0	0	1

$$f(x_1, x_2) = \underbrace{1 \cdot f_{00}} + \underbrace{1 \cdot f_{01}} + \underbrace{0 \cdot f_{10}} + \underbrace{1 \cdot f_{11}}$$

$$f(x_1, x_2) = \bar{x}_1 \bar{x}_2 + \bar{x}_1 x_2 + 0 + x_1 x_2$$

3) Then just add them together

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

$$f(x_1, x_2) = \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + 0 + x_1x_2$$

A function to be synthesized

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

**Let's look at it row by row.
How can we express the last row?**

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

**Let's look at it row by row.
How can we express the last row?**

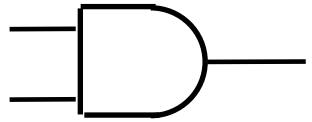
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

x_1x_2

**Let's look at it row by row.
How can we express the last row?**

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

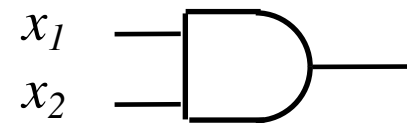
x_1
 x_2



The diagram shows a truth table for a function $f(x_1, x_2)$. The last row, where $x_1=1$ and $x_2=1$ and $f=1$, is highlighted with a green box. To the right of this box is an AND gate symbol with two inputs labeled x_1 and x_2 .

What about this row?

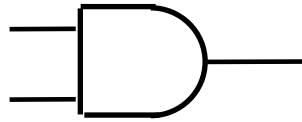
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



What about this row?

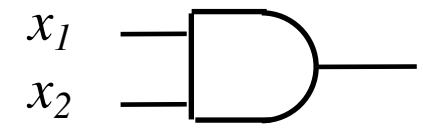
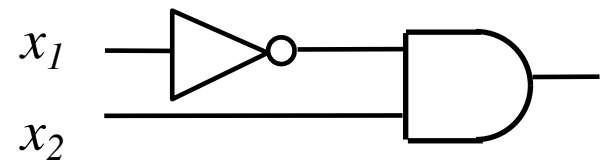
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

$\bar{x}_1 x_2$

x_1
 x_2 

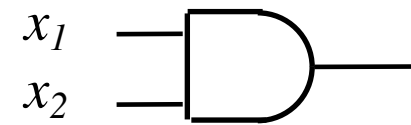
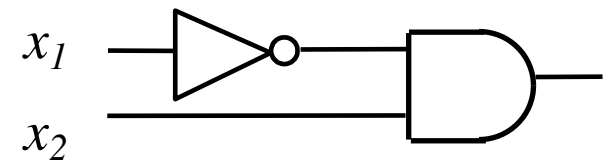
What about this row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



What about the first row?

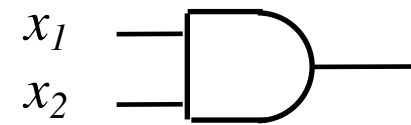
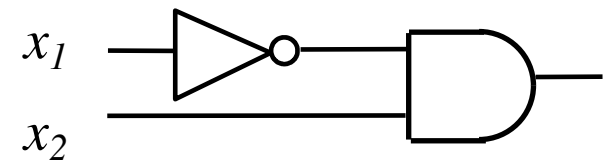
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



What about the first row?

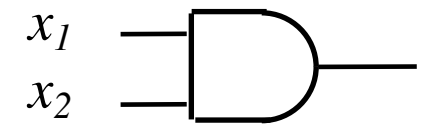
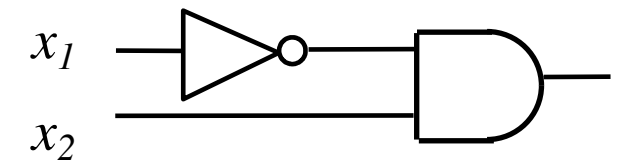
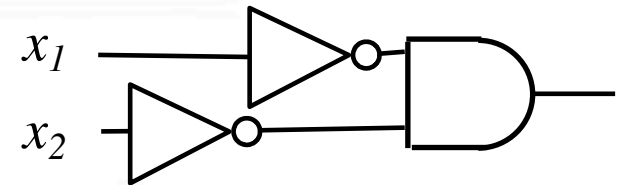
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

$$\bar{x}_1 \bar{x}_2$$



What about the first row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



Finally, what about the zero?

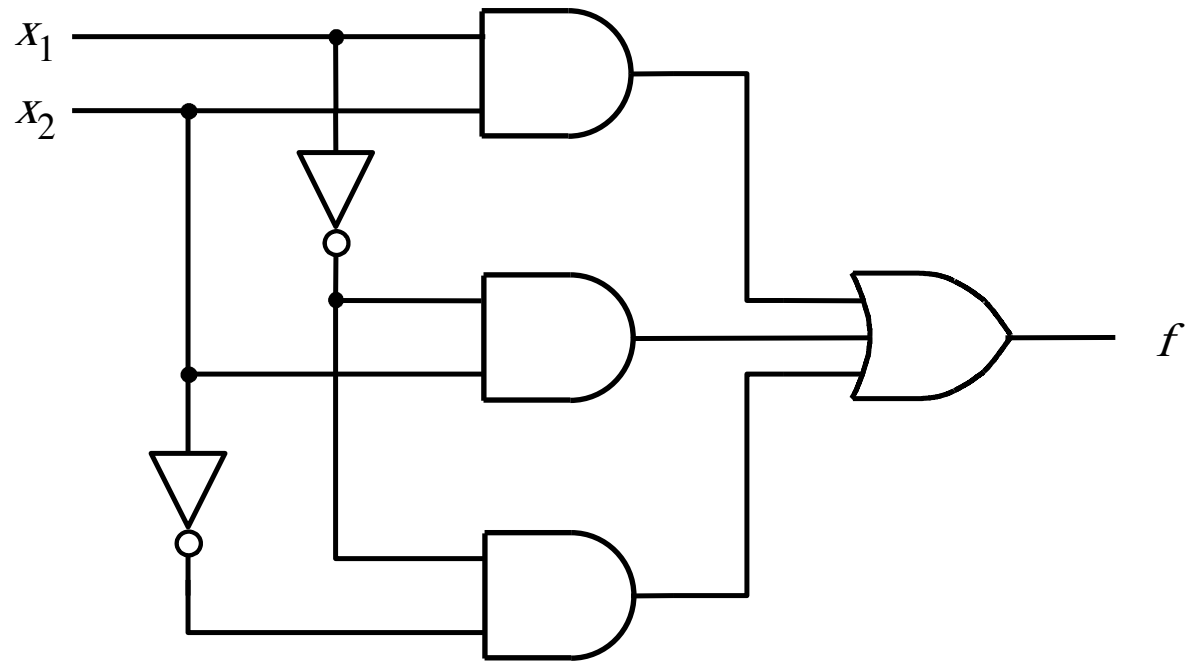
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

The truth table above shows the function $f(x_1, x_2)$ for all combinations of x_1 and x_2 . The output is 1 for all combinations except when $x_1 = 1$ and $x_2 = 0$, where the output is 0. This 0 is highlighted in a green box.

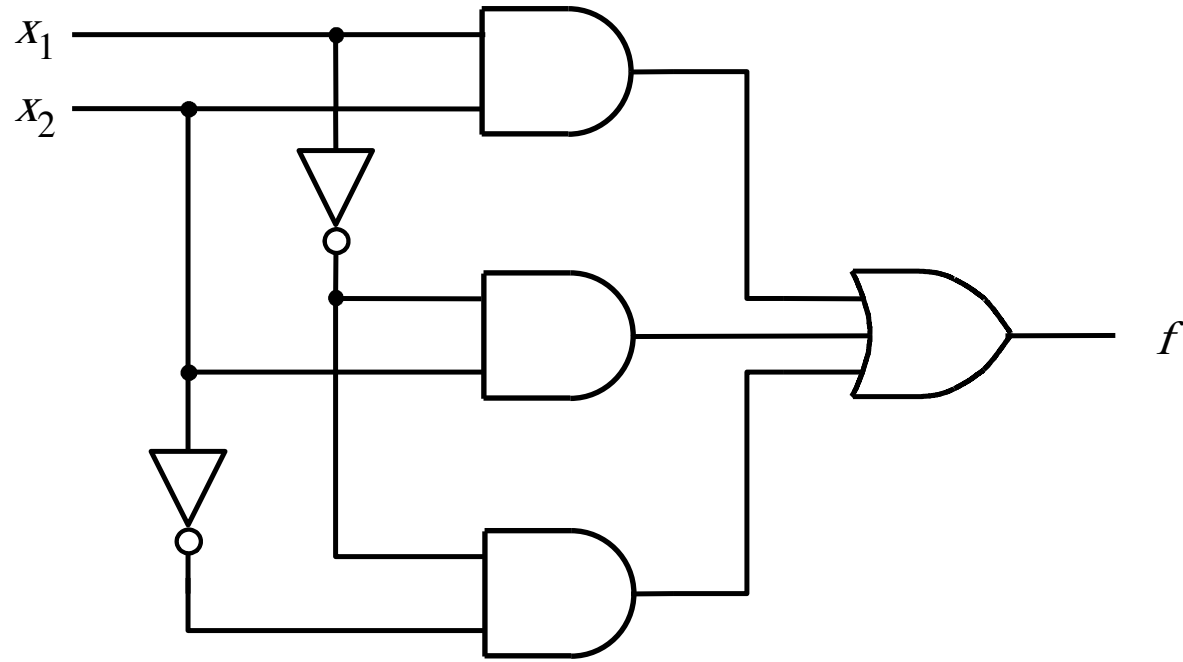
Three logic circuit diagrams are shown to the right of the truth table, illustrating different ways to implement the function $f(x_1, x_2)$:

- The top diagram shows x_1 and x_2 connected to two NOT gates. The outputs of these NOT gates are connected to the inputs of an AND gate. This circuit implements the function $f(x_1, x_2) = \overline{x_1} \cdot \overline{x_2}$.
- The middle diagram shows x_1 connected to a NOT gate, and x_2 connected directly to the other input of an AND gate. This circuit implements the function $f(x_1, x_2) = \overline{x_1} \cdot x_2$.
- The bottom diagram shows x_1 and x_2 connected directly to the inputs of an AND gate. This circuit implements the function $f(x_1, x_2) = x_1 \cdot x_2$.

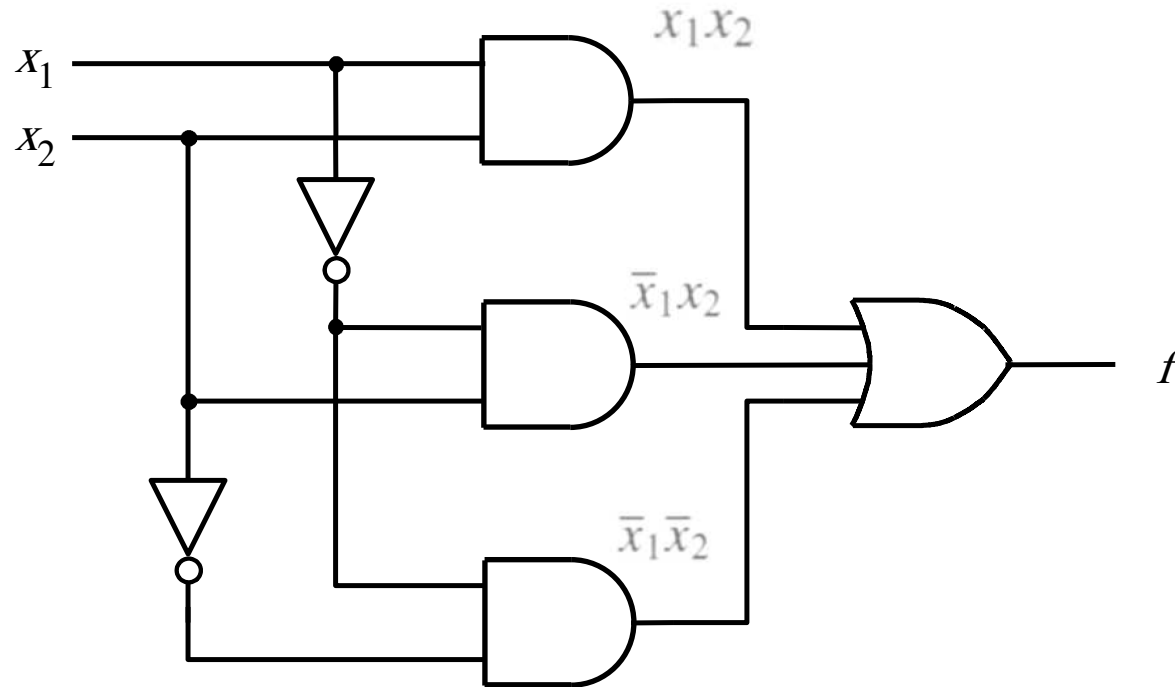
Putting it all together



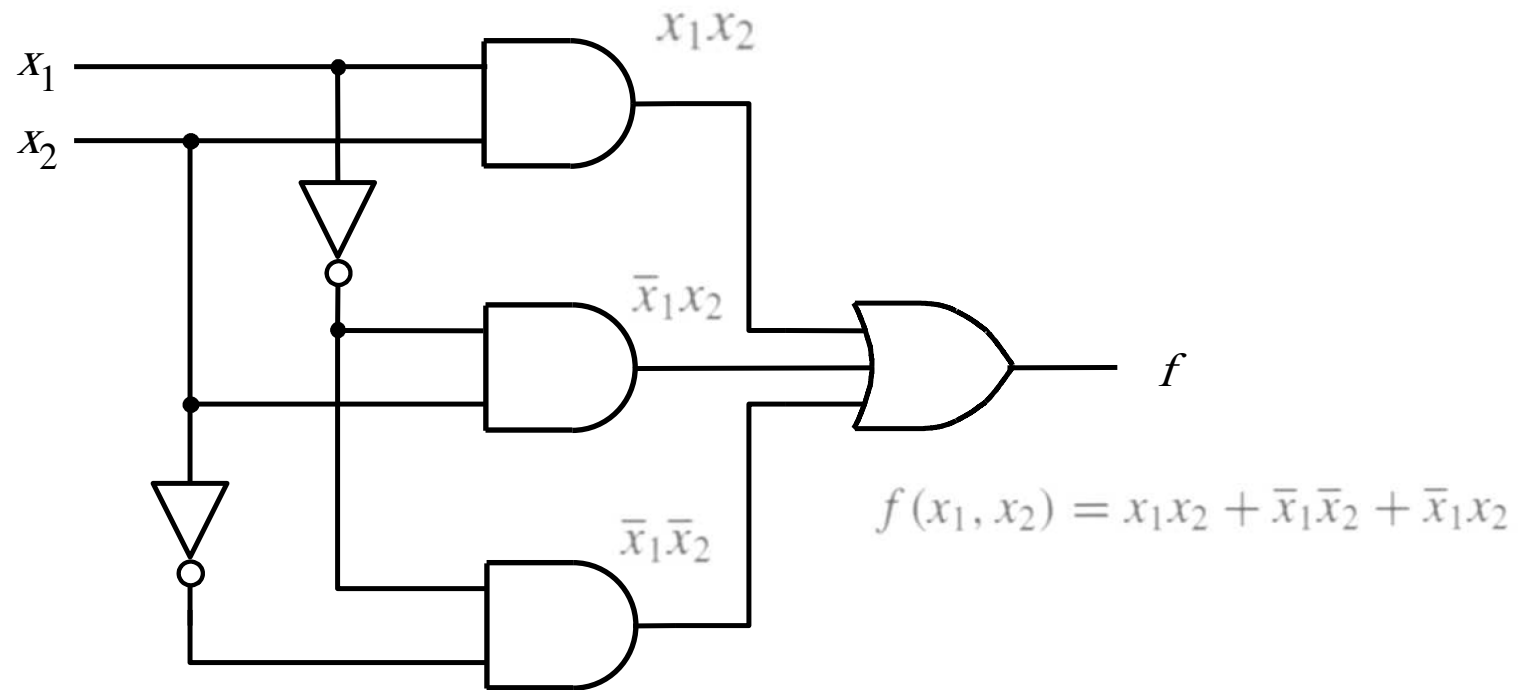
Let's verify that this circuit implements correctly the target truth table



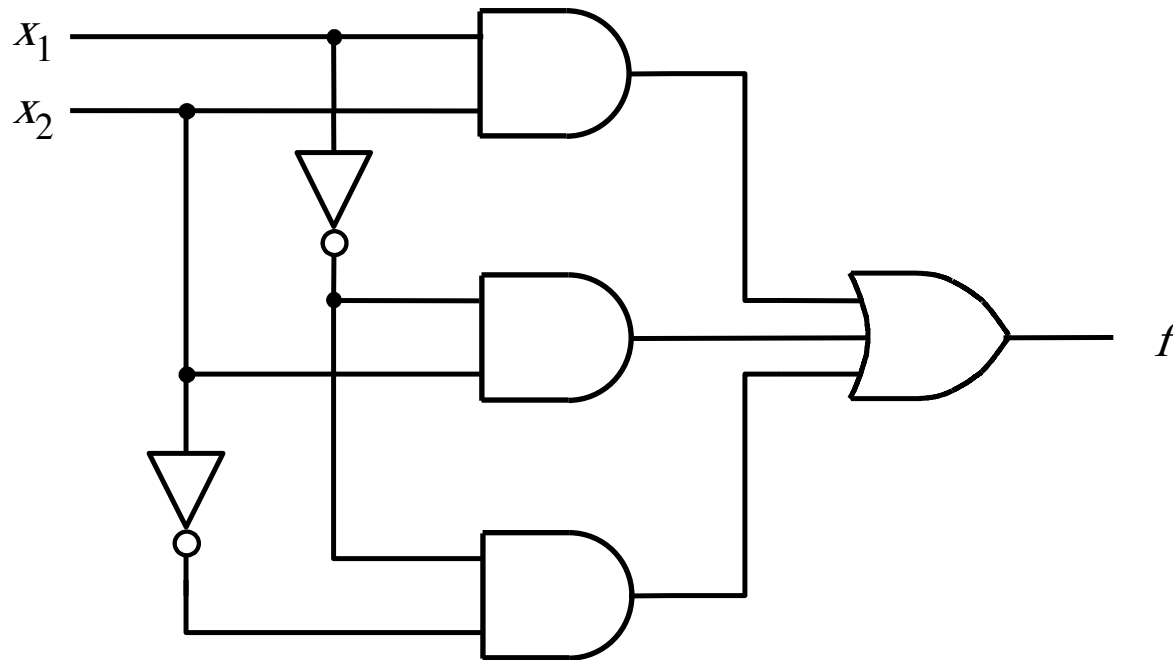
Let's verify that this circuit implements correctly the target truth table



Putting it all together



Canonical Sum-Of-Products (SOP)



$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

Summary of This Procedure

- **Get the truth table of the function**
- **Form a product term (AND gate) for each row of the table for which the function is 1**
- **Each product term contains all input variables**
- **In each row, if $x_i = 1$ enter it as x_i , otherwise use \bar{x}_i**
- **Sum all of these products (OR gate) to get the function**

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

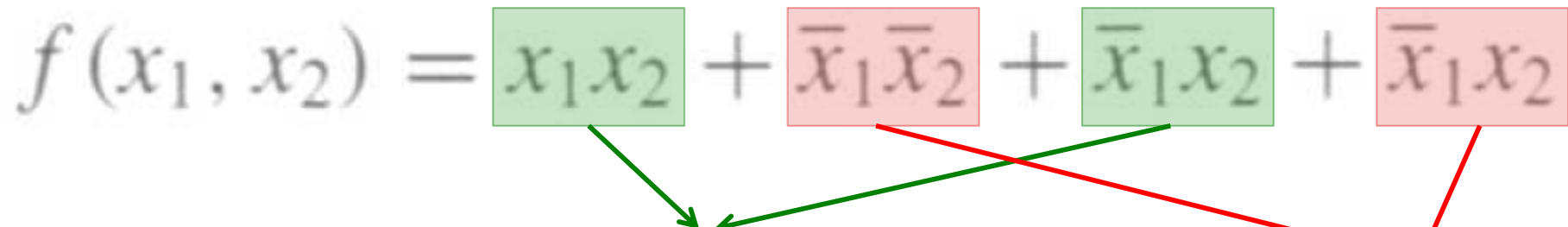
replicate
this term

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1x_2$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

group
these terms

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1x_2$$


$$f(x_1, x_2) = (x_1 + \bar{x}_1)x_2 + \bar{x}_1(\bar{x}_2 + x_2)$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1x_2$$

These two terms are trivially equal to 1

$$f(x_1, x_2) = (x_1 + \bar{x}_1)x_2 + \bar{x}_1(\bar{x}_2 + x_2)$$

$$f(x_1, x_2) = 1 \cdot x_2 + \bar{x}_1 \cdot 1$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1x_2$$

$$f(x_1, x_2) = (x_1 + \bar{x}_1)x_2 + \bar{x}_1(\bar{x}_2 + x_2)$$

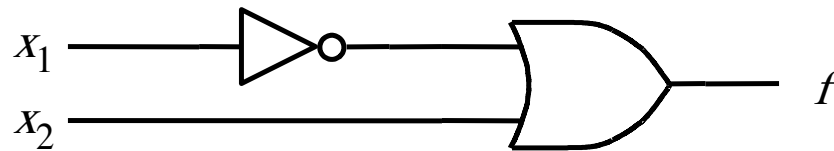
$$f(x_1, x_2) = \boxed{1} \cdot x_2 + \bar{x}_1 \cdot \boxed{1}$$

Drop the 1's

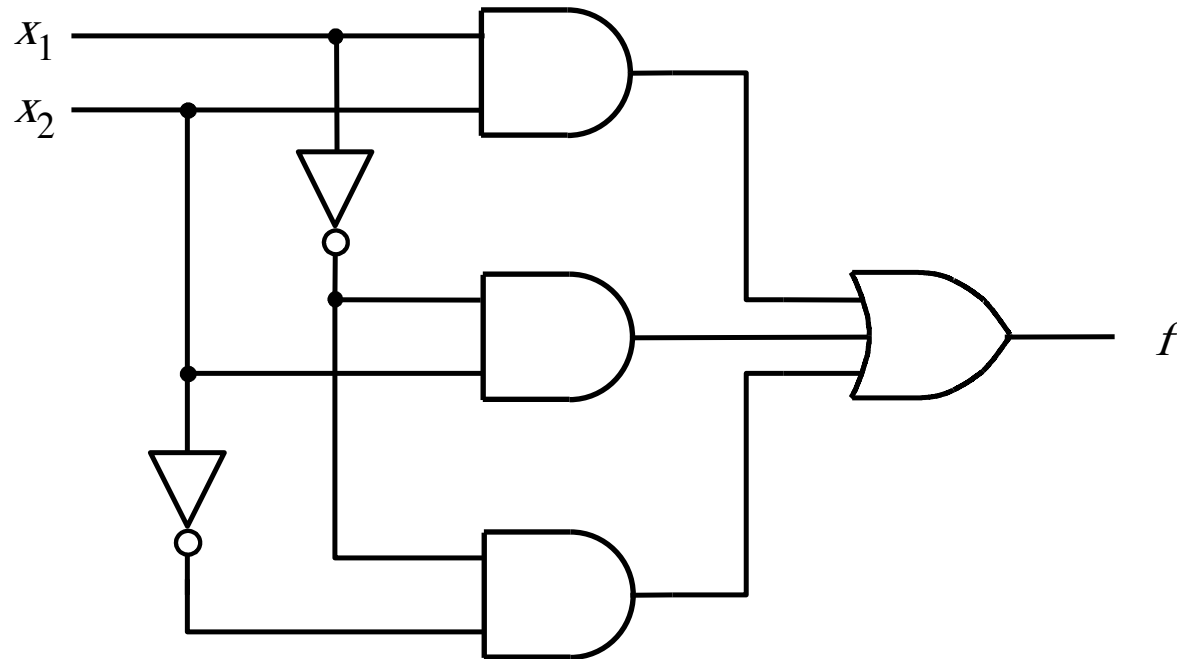
$$f(x_1, x_2) = x_2 + \bar{x}_1$$

Minimal-cost realization

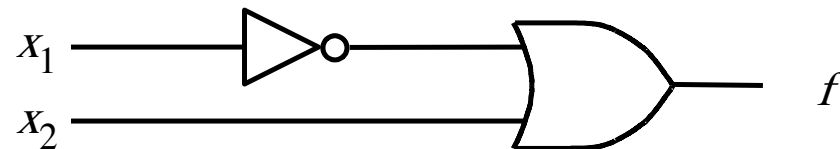
$$f(x_1, x_2) = x_2 + \bar{x}_1$$



Two implementations for the same function



(a) Canonical sum-of-products



(b) Minimal-cost realization

Basis Functions / minterms (for three variables)

Expressions for the minterms

$$m_0 = \bar{x} \bar{y} \bar{z}$$

$$m_1 = \bar{x} \bar{y} z$$

$$m_2 = \bar{x} y \bar{z}$$

$$m_3 = \bar{x} y z$$

$$m_4 = x \bar{y} \bar{z}$$

$$m_5 = x \bar{y} z$$

$$m_6 = x y \bar{z}$$

$$m_7 = x y z$$

Expressions for the minterms

$$0 \ 0 \ 0 \quad m_0 = \bar{x} \bar{y} \bar{z}$$

$$0 \ 0 \ 1 \quad m_1 = \bar{x} \bar{y} z$$

$$0 \ 1 \ 0 \quad m_2 = \bar{x} y \bar{z}$$

$$0 \ 1 \ 1 \quad m_3 = \bar{x} y z$$

$$1 \ 0 \ 0 \quad m_4 = x \bar{y} \bar{z}$$

$$1 \ 0 \ 1 \quad m_5 = x \bar{y} z$$

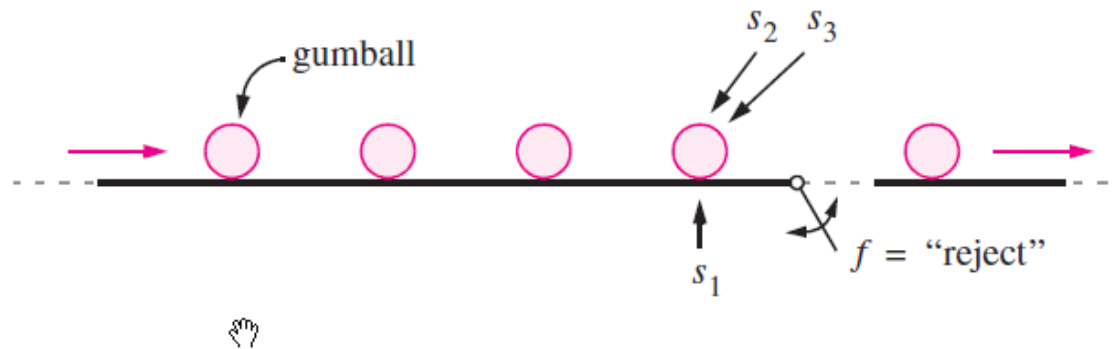
$$1 \ 1 \ 0 \quad m_6 = x y \bar{z}$$

$$1 \ 1 \ 1 \quad m_7 = x y z$$

The bars coincide
with the 0's
in the binary expansion
of the minterm sub-index

Function Synthesis Example (with three variables)

Let's look at another problem



(a) Conveyor and sensors

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(b) Truth table

Let's look at another problem

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Let's look at another problem

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\bar{s}_1 \bar{s}_2 s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1 \bar{s}_2 s_3$
1	1	0	1	$s_1 s_2 \bar{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
<u>0</u>	<u>0</u>	0	0	
0	0	1	1	$\bar{s}_1 \bar{s}_2 s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1 \bar{s}_2 s_3$
1	1	0	1	$s_1 s_2 \bar{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
<u>0</u>	<u>0</u>	0	0	
0	0	1	1	$\bar{s}_1 \bar{s}_2 s_3$
0	1	0	0	
<u>0</u>	1	1	1	$\bar{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1 \bar{s}_2 s_3$
1	1	0	1	$s_1 s_2 \bar{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
<u>0</u>	<u>0</u>	0	0	
0	0	1	1	$\bar{s}_1 \bar{s}_2 s_3$
0	1	0	0	
<u>0</u>	1	1	1	$\bar{s}_1 s_2 s_3$
1	0	0	0	
1	<u>0</u>	1	1	$s_1 \bar{s}_2 s_3$
1	1	0	1	$s_1 s_2 \bar{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
<u>0</u>	<u>0</u>	0	0	
0	0	1	1	<u>\bar{s}_1</u> <u>\bar{s}_2</u> s_3
0	1	0	0	
<u>0</u>	1	1	1	<u>\bar{s}_1</u> s_2 s_3
1	0	0	0	
1	<u>0</u>	1	1	s_1 <u>\bar{s}_2</u> s_3
1	1	<u>0</u>	1	s_1 s_2 <u>\bar{s}_3</u>
1	1	1	1	s_1 s_2 s_3

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\bar{s}_1\bar{s}_2s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1s_2s_3$
1	0	0	0	
1	0	1	1	$s_1\bar{s}_2s_3$
1	1	0	1	$s_1s_2\bar{s}_3$
1	1	1	1	$s_1s_2s_3$

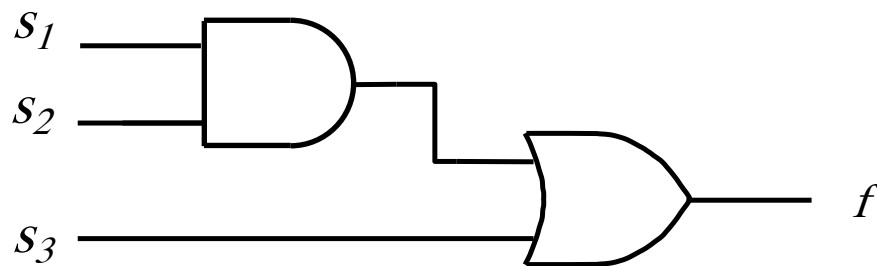
$$f = \bar{s}_1\bar{s}_2s_3 + \bar{s}_1s_2s_3 + s_1\bar{s}_2s_3 + s_1s_2\bar{s}_3 + s_1s_2s_3$$

Let's look at another problem (minimization)

$$\begin{aligned} f &= \bar{s}_1 \bar{s}_2 s_3 + \bar{s}_1 s_2 s_3 + s_1 \bar{s}_2 s_3 + s_1 s_2 s_3 + s_1 s_2 \bar{s}_3 + s_1 s_2 s_3 \\ &= \bar{s}_1 s_3 (\bar{s}_2 + s_2) + s_1 s_3 (\bar{s}_2 + s_2) + s_1 s_2 (\bar{s}_3 + s_3) \\ &= \bar{s}_1 s_3 + s_1 s_3 + s_1 s_2 \\ &= s_3 + s_1 s_2 \end{aligned}$$

Let's look at another problem (minimization)

$$\begin{aligned} f &= \bar{s}_1\bar{s}_2s_3 + \bar{s}_1s_2s_3 + s_1\bar{s}_2s_3 + s_1s_2s_3 + s_1s_2\bar{s}_3 + s_1s_2s_3 \\ &= \bar{s}_1s_3(\bar{s}_2 + s_2) + s_1s_3(\bar{s}_2 + s_2) + s_1s_2(\bar{s}_3 + s_3) \\ &= \bar{s}_1s_3 + s_1s_3 + s_1s_2 \\ &= s_3 + s_1s_2 \end{aligned}$$



Maxterms

(an alternative set of basis functions)

The Four Maxterms

x	y	M_0
0	0	0
0	1	1
1	0	1
1	1	1

$M_0(x, y)$

x	y	M_1
0	0	1
0	1	0
1	0	1
1	1	1

$M_1(x, y)$

x	y	M_2
0	0	1
0	1	1
1	0	0
1	1	1

$M_2(x, y)$

x	y	M_3
0	0	1
0	1	1
1	0	1
1	1	0

$M_3(x, y)$

The Four Maxterms

x	y	M_0
0	0	0
0	1	1
1	0	1
1	1	1

$M_0(x, y)$

x	y	M_1
0	0	1
0	1	0
1	0	1
1	1	1

$M_1(x, y)$

x	y	M_2
0	0	1
0	1	1
1	0	0
1	1	1

$M_2(x, y)$

x	y	M_3
0	0	1
0	1	1
1	0	1
1	1	0

$M_3(x, y)$

The Four Maxterms

x	y		$M_0(x, y)$	$M_1(x, y)$	$M_2(x, y)$	$M_3(x, y)$
0	0		0	1	1	1
0	1		1	0	1	1
1	0		1	1	0	1
1	1		1	1	1	0

The Four Maxterms

x	y	$x + y$	$x + \bar{y}$	$\bar{x} + y$	$\bar{x} + \bar{y}$
0	0	0	1	1	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0

Expressions for the Maxterms

$$M_0 = x + y$$

$$M_1 = x + \bar{y}$$

$$M_2 = \bar{x} + y$$

$$M_3 = \bar{x} + \bar{y}$$

Expressions for the Maxterms

$$M_0 = x + y$$

$$M_1 = x + \bar{y}$$

$$M_2 = \bar{x} + y$$

$$M_3 = \bar{x} + \bar{y}$$

Note that these are now sums, not products.

Expressions for the Maxterms

$$0 \quad 0 \quad M_0 = x + y$$

$$0 \quad 1 \quad M_1 = x + \bar{y}$$

$$1 \quad 0 \quad M_2 = \bar{x} + y$$

$$1 \quad 1 \quad M_3 = \bar{x} + \bar{y}$$

The bars coincide
with the 1's
in the binary expansion
of the maxterm sub-index

Expressions for the Maxterms

$$0 \ 0 \quad M_0 = x + y$$

$$0 \ 1 \quad M_1 = x + \bar{y}$$

$$1 \ 0 \quad M_2 = \bar{x} + y$$

$$1 \ 1 \quad M_3 = \bar{x} + \bar{y}$$

The bars coincide
with the 1's
in the binary expansion
of the maxterm sub-index

Minterms and Maxterms

Row number	x_1	x_2	Minterm	Maxterm
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	$M_0 = x_1 + x_2$
1	0	1	$m_1 = \bar{x}_1x_2$	$M_1 = x_1 + \bar{x}_2$
2	1	0	$m_2 = x_1\bar{x}_2$	$M_2 = \bar{x}_1 + x_2$
3	1	1	$m_3 = x_1x_2$	$M_3 = \bar{x}_1 + \bar{x}_2$

Minterms and Maxterms

Row number	x_1	x_2	Minterm	Maxterm
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	$M_0 = x_1 + x_2$
1	0	1	$m_1 = \bar{x}_1x_2$	$M_1 = x_1 + \bar{x}_2$
2	1	0	$m_2 = x_1\bar{x}_2$	$M_2 = \bar{x}_1 + x_2$
3	1	1	$m_3 = x_1x_2$	$M_3 = \bar{x}_1 + \bar{x}_2$

Use these for
Sum-of-Products
Minimization
(1's of the function)

Use these for
Product-of-Sums
Minimization
(0's of the function)

Minterms and Maxterms

Row number	x_1	x_2	Minterm	Maxterm
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	$M_0 = x_1 + x_2$
1	0	1	$m_1 = \bar{x}_1x_2$	$M_1 = x_1 + \bar{x}_2$
2	1	0	$m_2 = x_1\bar{x}_2$	$M_2 = \bar{x}_1 + x_2$
3	1	1	$m_3 = x_1x_2$	$M_3 = \bar{x}_1 + \bar{x}_2$

Use these for
Sum-of-Products
 Minimization
 (1's of the function)

Use these for
Product-of-Sums
 Minimization
 (0's of the function)

Sum-of-Products Form

(uses the **ones** of the function)

Sum-of-Products Form

(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	0
1	0	1	$m_1 = \bar{x}_1x_2$	0
2	1	0	$m_2 = x_1\bar{x}_2$	0
3	1	1	$m_3 = x_1x_2$	1

Sum-of-Products Form

(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	0
1	0	1	$m_1 = \bar{x}_1x_2$	0
2	1	0	$m_2 = x_1\bar{x}_2$	0
3	1	1	$m_3 = x_1x_2$	1

Sum-of-Products Form

(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1 \bar{x}_2$	0
1	0	1	$m_1 = \bar{x}_1 x_2$	0
2	1	0	$m_2 = x_1 \bar{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

$$f(x_1, x_2) = m_3 = x_1 x_2$$

(In this case there is just one product and there is no need for a sum)

Another Example

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	1
1	0	1	$m_1 = \bar{x}_1x_2$	1
2	1	0	$m_2 = x_1\bar{x}_2$	0
3	1	1	$m_3 = x_1x_2$	1

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	1
1	0	1	$m_1 = \bar{x}_1x_2$	1
2	1	0	$m_2 = x_1\bar{x}_2$	0
3	1	1	$m_3 = x_1x_2$	1

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \bar{x}_1\bar{x}_2$	1
1	0	1	$m_1 = \bar{x}_1x_2$	1
2	1	0	$m_2 = x_1\bar{x}_2$	0
3	1	1	$m_3 = x_1x_2$	1

$$\begin{aligned}f &= m_0 \cdot 1 + m_1 \cdot 1 + m_2 \cdot 0 + m_3 \cdot 1 \\&= m_0 + m_1 + m_3 \\&= \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + x_1x_2\end{aligned}$$

Product-of-Sums Form

(uses the **zeros** of the function)

Product-of-Sums Form (for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	1
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

Product-of-Sums Form (for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	1
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

Product-of-Sums Form (for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	1
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

$$f(x_1, x_2) = M_0 = x_1 + x_2$$

(In this case there is just one sum and there is no need for a product)

Another Example

Product-of-Sums Form (for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	0
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

Product-of-Sums Form (for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	0
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

Product-of-Sums Form (for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	0
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

$$f(x_1, x_2) = M_0 \cdot M_2 = (x_1 + x_2) \cdot (\bar{x}_1 + x_2)$$

Yet Another Example

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1
1	0	1	$M_1 = x_1 + \bar{x}_2$	1
2	1	0	$M_2 = \bar{x}_1 + x_2$	0
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1

We need to minimize using the zeros of the function f .
 But let's first minimize the inverse of f , i.e., \bar{f} .

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x_2}$	1	0
2	1	0	$M_2 = \overline{x_1} + x_2$	0	1
3	1	1	$M_3 = \overline{x_1} + \overline{x_2}$	1	0

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x_2}$	1	0
2	1	0	$M_2 = \overline{x_1} + x_2$	0	1
3	1	1	$M_3 = \overline{x_1} + \overline{x_2}$	1	0

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

$$\begin{aligned}\overline{f}(x_1, x_2) &= m_2 \\ &= x_1 \overline{x}_2\end{aligned}$$

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\bar{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1	0
2	1	0	$M_2 = \bar{x}_1 + x_2$	0	1
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1	0

$$\begin{aligned} \bar{\bar{f}} &= f = \overline{x_1 \bar{x}_2} \\ &= \bar{x}_1 + x_2 \end{aligned}$$

$$\begin{aligned} \bar{f}(x_1, x_2) &= m_2 \\ &= x_1 \bar{x}_2 \end{aligned}$$

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\bar{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \bar{x}_2$	1	0
2	1	0	$M_2 = \bar{x}_1 + x_2$	0	1
3	1	1	$M_3 = \bar{x}_1 + \bar{x}_2$	1	0

$$\begin{aligned} \bar{f} = f = \overline{x_1 \bar{x}_2} & & \bar{f}(x_1, x_2) = m_2 \\ & = \bar{x}_1 + x_2 & = x_1 \bar{x}_2 \end{aligned}$$

$$f = \bar{m}_2 = M_2$$

minterms
(for three variables)

Expressions for the minterms

$$m_0 = \bar{x} \bar{y} \bar{z}$$

$$m_1 = \bar{x} \bar{y} z$$

$$m_2 = \bar{x} y \bar{z}$$

$$m_3 = \bar{x} y z$$

$$m_4 = x \bar{y} \bar{z}$$

$$m_5 = x \bar{y} z$$

$$m_6 = x y \bar{z}$$

$$m_7 = x y z$$

Expressions for the minterms

$$0 \ 0 \ 0 \quad m_0 = \bar{x} \bar{y} \bar{z}$$

$$0 \ 0 \ 1 \quad m_1 = \bar{x} \bar{y} z$$

$$0 \ 1 \ 0 \quad m_2 = \bar{x} y \bar{z}$$

$$0 \ 1 \ 1 \quad m_3 = \bar{x} y z$$

$$1 \ 0 \ 0 \quad m_4 = x \bar{y} \bar{z}$$

$$1 \ 0 \ 1 \quad m_5 = x \bar{y} z$$

$$1 \ 1 \ 0 \quad m_6 = x y \bar{z}$$

$$1 \ 1 \ 1 \quad m_7 = x y z$$

The bars coincide
with the 0's
in the binary expansion
of the minterm sub-index

Maxterms (for three variables)

Expressions for the Maxterms

$$M_0 = x + y + z$$

$$M_1 = x + y + \bar{z}$$

$$M_2 = x + \bar{y} + z$$

$$M_3 = x + \bar{y} + \bar{z}$$

$$M_4 = \bar{x} + y + z$$

$$M_5 = \bar{x} + y + \bar{z}$$

$$M_6 = \bar{x} + \bar{y} + z$$

$$M_7 = \bar{x} + \bar{y} + \bar{z}$$

Expressions for the Maxterms

$$0 \ 0 \ 0 \quad M_0 = x + y + z$$

$$0 \ 0 \ 1 \quad M_1 = x + y + \bar{z}$$

$$0 \ 1 \ 0 \quad M_2 = x + \bar{y} + z$$

$$0 \ 1 \ 1 \quad M_3 = x + \bar{y} + \bar{z}$$

$$1 \ 0 \ 0 \quad M_4 = \bar{x} + y + z$$

$$1 \ 0 \ 1 \quad M_5 = \bar{x} + y + \bar{z}$$

$$1 \ 1 \ 0 \quad M_6 = \bar{x} + \bar{y} + z$$

$$1 \ 1 \ 1 \quad M_7 = \bar{x} + \bar{y} + \bar{z}$$

The bars coincide
with the 1's
in the binary expansion
of the maxterm sub-index

minterms and Maxterms (for three variables)

minterms and Maxterms

$$m_0 = \bar{x} \bar{y} \bar{z}$$

$$m_1 = \bar{x} \bar{y} z$$

$$m_2 = \bar{x} y \bar{z}$$

$$m_3 = \bar{x} y z$$

$$m_4 = x \bar{y} \bar{z}$$

$$m_5 = x \bar{y} z$$

$$m_6 = x y \bar{z}$$

$$m_7 = x y z$$

$$M_0 = x + y + z$$

$$M_1 = x + y + \bar{z}$$

$$M_2 = x + \bar{y} + z$$

$$M_3 = x + \bar{y} + \bar{z}$$

$$M_4 = \bar{x} + y + z$$

$$M_5 = \bar{x} + y + \bar{z}$$

$$M_6 = \bar{x} + \bar{y} + z$$

$$M_7 = \bar{x} + \bar{y} + \bar{z}$$

minterms and Maxterms (with three variables)

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \bar{x}_1\bar{x}_2\bar{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \bar{x}_1\bar{x}_2x_3$	$M_1 = x_1 + x_2 + \bar{x}_3$
2	0	1	0	$m_2 = \bar{x}_1x_2\bar{x}_3$	$M_2 = x_1 + \bar{x}_2 + x_3$
3	0	1	1	$m_3 = \bar{x}_1x_2x_3$	$M_3 = x_1 + \bar{x}_2 + \bar{x}_3$
4	1	0	0	$m_4 = x_1\bar{x}_2\bar{x}_3$	$M_4 = \bar{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\bar{x}_2x_3$	$M_5 = \bar{x}_1 + x_2 + \bar{x}_3$
6	1	1	0	$m_6 = x_1x_2\bar{x}_3$	$M_6 = \bar{x}_1 + \bar{x}_2 + x_3$
7	1	1	1	$m_7 = x_1x_2x_3$	$M_7 = \bar{x}_1 + \bar{x}_2 + \bar{x}_3$

Examples with three-variable functions

A three-variable function

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3$$

Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

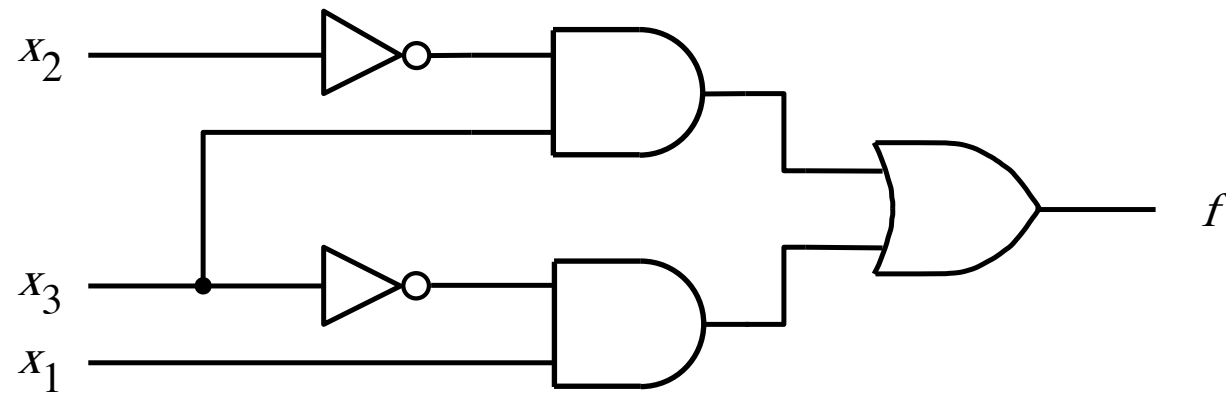
$$f(x_1, x_2, x_3) = \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3$$

$$f = (\bar{x}_1 + x_1)\bar{x}_2x_3 + x_1(\bar{x}_2 + x_2)\bar{x}_3$$

$$= 1 \cdot \bar{x}_2x_3 + x_1 \cdot 1 \cdot \bar{x}_3$$

$$= \bar{x}_2x_3 + x_1\bar{x}_3$$

Sum-of-products realization of this function



$$f = \overline{x_2} x_3 + x_1 \overline{x_3}$$

A three-variable function

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f = \overline{m_0 + m_2 + m_3 + m_7}$$

$$= \overline{m_0} \cdot \overline{m_2} \cdot \overline{m_3} \cdot \overline{m_7}$$

$$= M_0 \cdot M_2 \cdot M_3 \cdot M_7$$

$$= (x_1 + x_2 + x_3)(x_1 + \bar{x}_2 + x_3)(x_1 + \bar{x}_2 + \bar{x}_3)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)$$

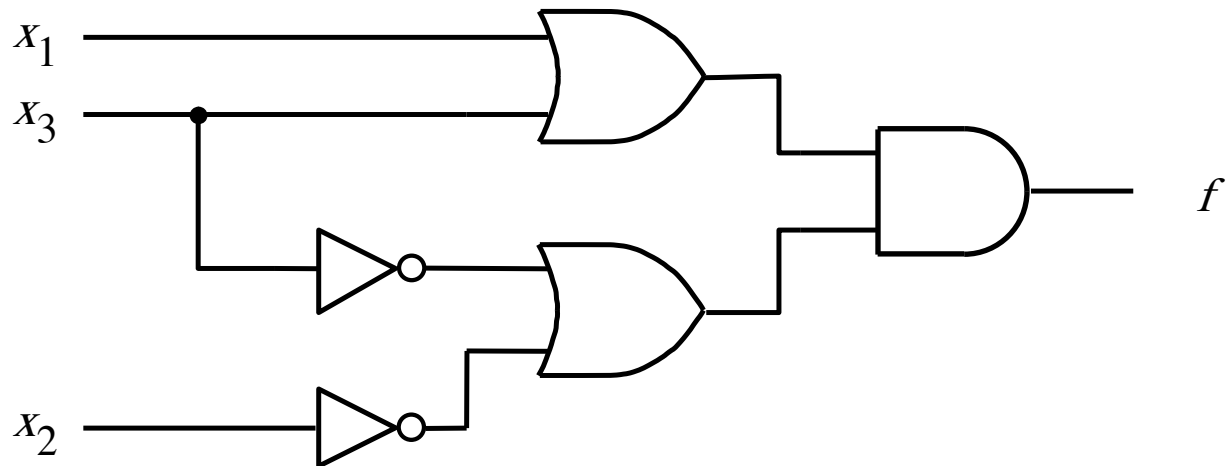
Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f = ((x_1 + x_3) + x_2)((x_1 + x_3) + \bar{x}_2)(x_1 + (\bar{x}_2 + \bar{x}_3))(\bar{x}_1 + (\bar{x}_2 + \bar{x}_3))$$

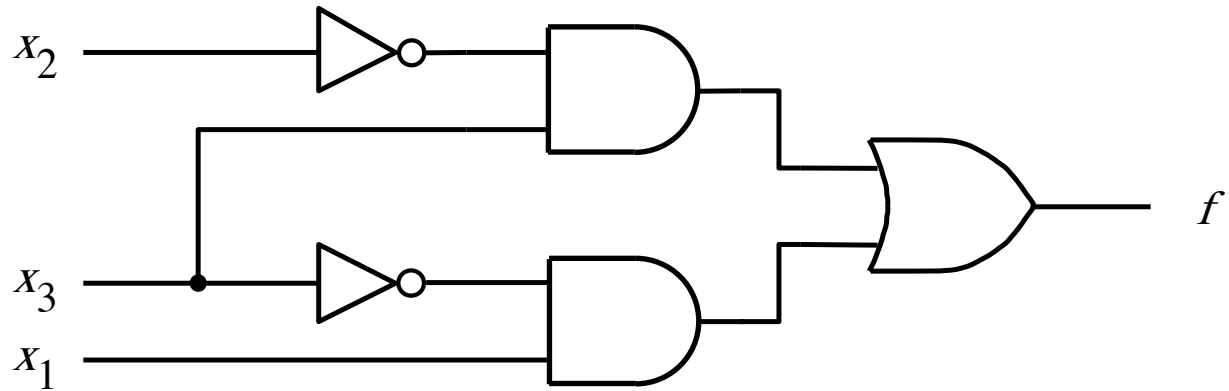
$$f = (x_1 + x_3)(\bar{x}_2 + \bar{x}_3)$$

Product-of-sums realization of this function

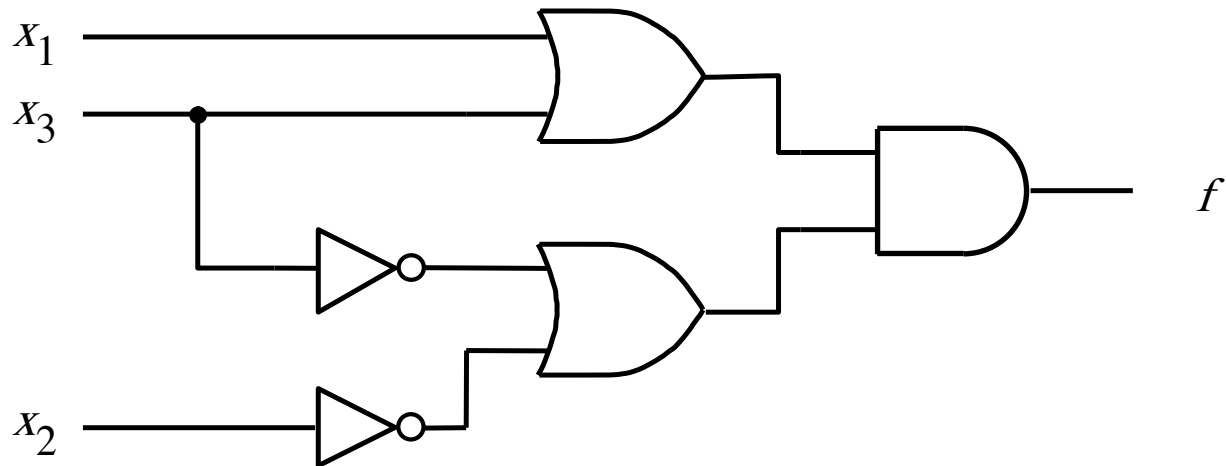


$$f = (x_1 + x_3) \cdot (\overline{x_2} + \overline{x_3})$$

Two realizations of this function



(a) A minimal sum-of-products realization



(b) A minimal product-of-sums realization

[Figure 2.24 from the textbook]

Shorthand Notation for SOP

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \sum (m_1, m_4, m_5, m_6)$$

or

$$f(x_1, x_2, x_3) = \sum m(1, 4, 5, 6)$$

Shorthand Notation for POS

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \Pi(M_0, M_2, M_3, M_7)$$

or

$$f(x_1, x_2, x_3) = \Pi M(0, 2, 3, 7)$$

Shorthand Notation

- **Sum-of-Products (SOP)**

$$f(x_1, x_2, x_3) = \sum (m_1, m_4, m_5, m_6)$$

or

$$f(x_1, x_2, x_3) = \sum m(1, 4, 5, 6)$$

- **Product-of-Sums (POS)**

$$f(x_1, x_2, x_3) = \Pi (M_0, M_2, M_3, M_7)$$

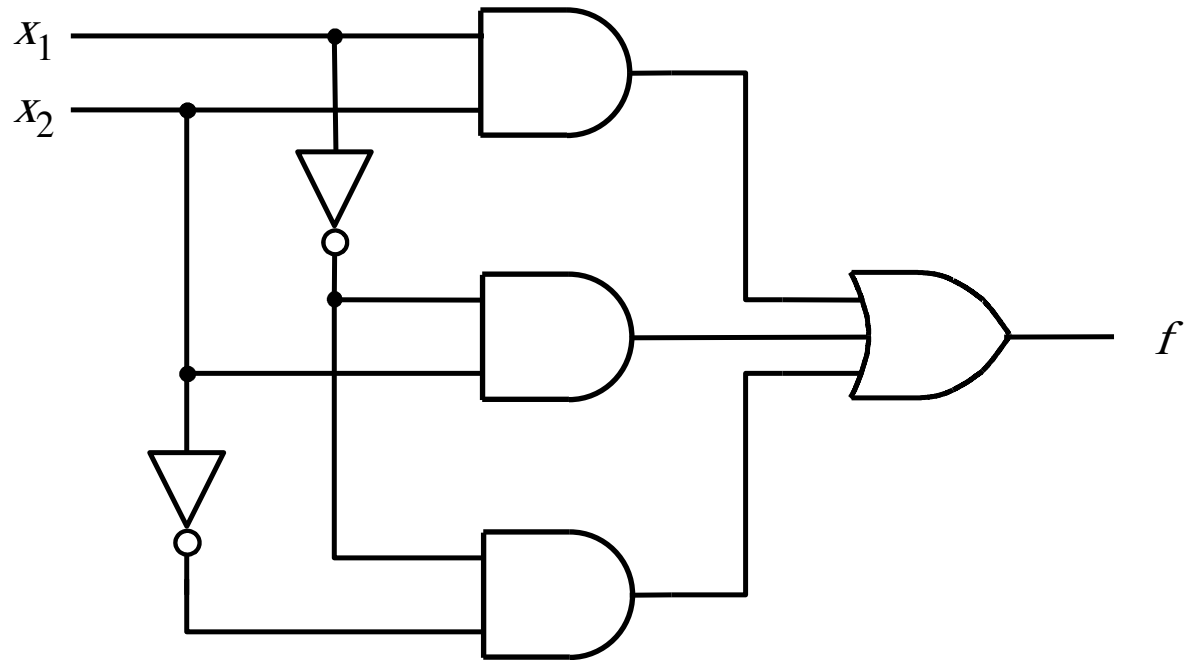
or

$$f(x_1, x_2, x_3) = \Pi M (0, 2, 3, 7)$$

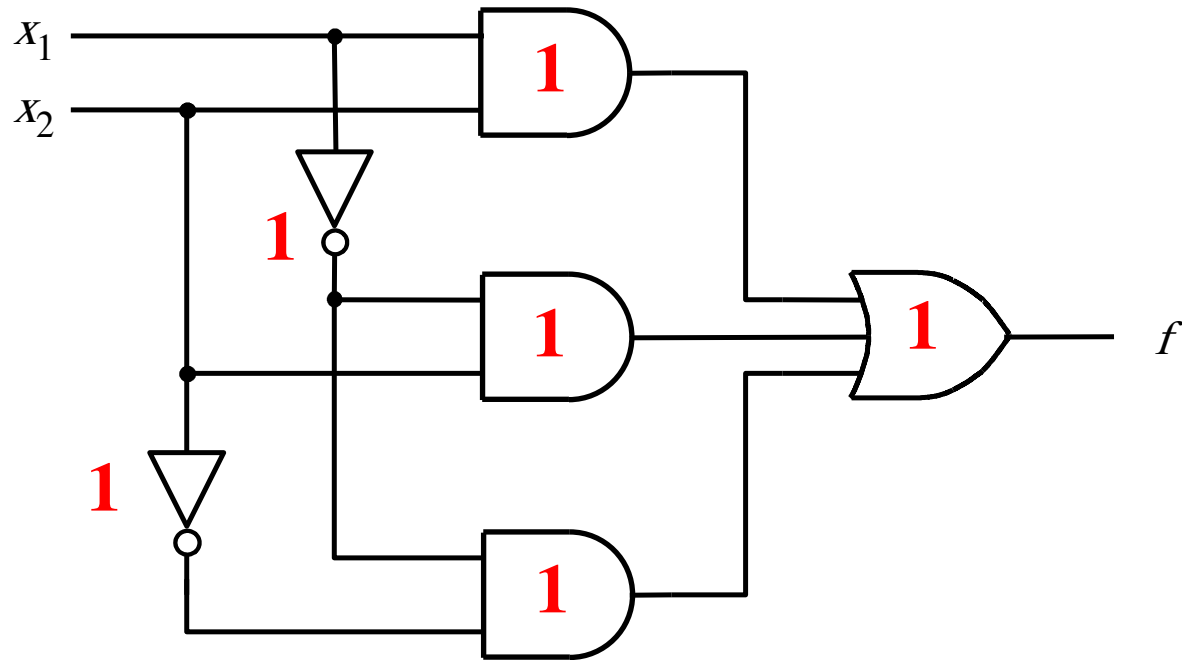
The Cost of a Circuit

- **Count all gates**
- **Count all inputs/wires to the gates**
- **Add the two partial counts. That is the cost.**

What is the cost of this circuit?

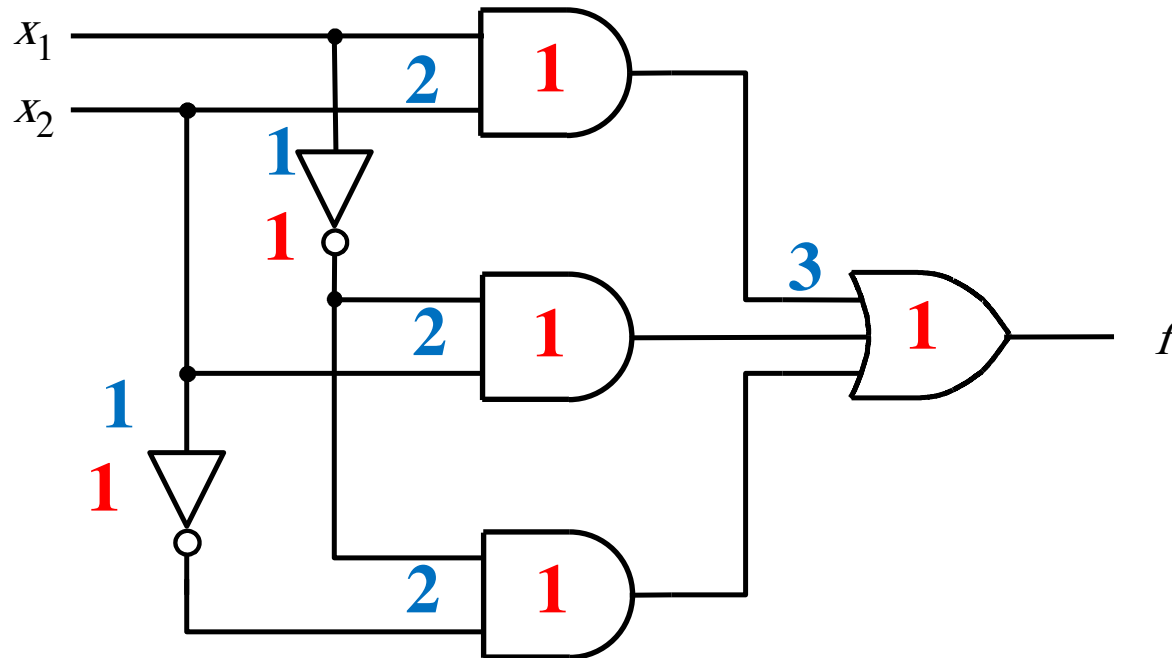


What is the cost of this circuit?



There are **6 gates**.

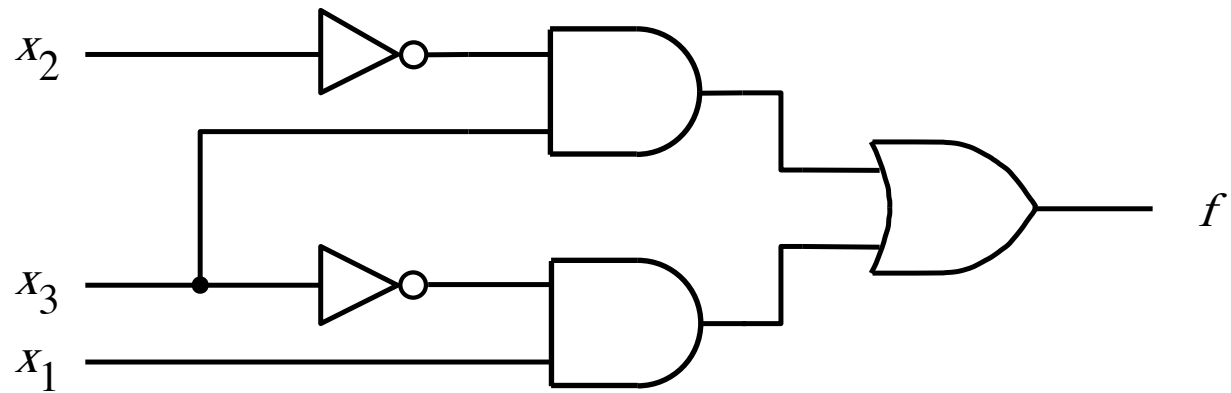
What is the cost of this circuit?



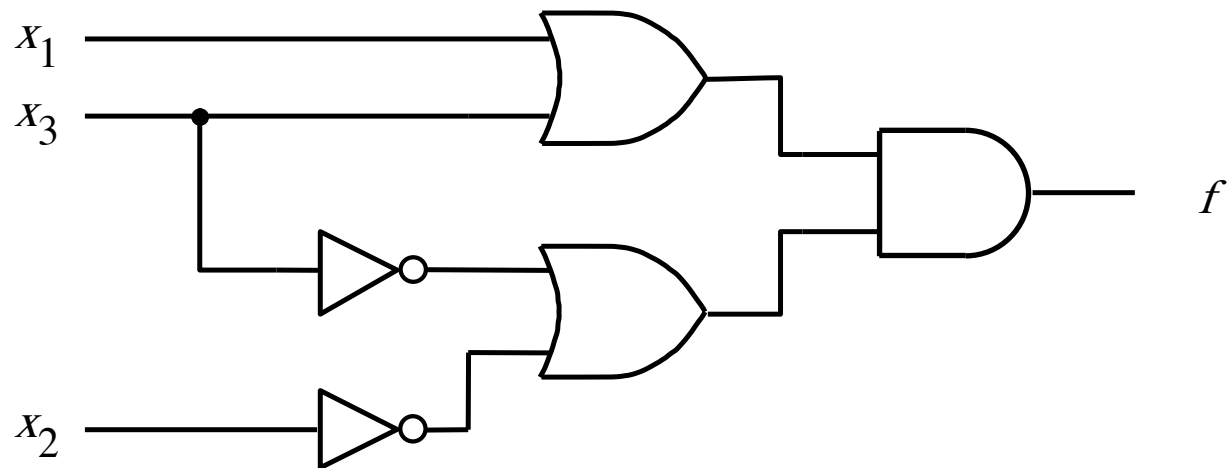
There are **6 gates** and **11 inputs**.

The total cost is **17**.

What is the cost of each circuit?



(a) A minimal sum-of-products realization



(b) A minimal product-of-sums realization

Questions?

THE END