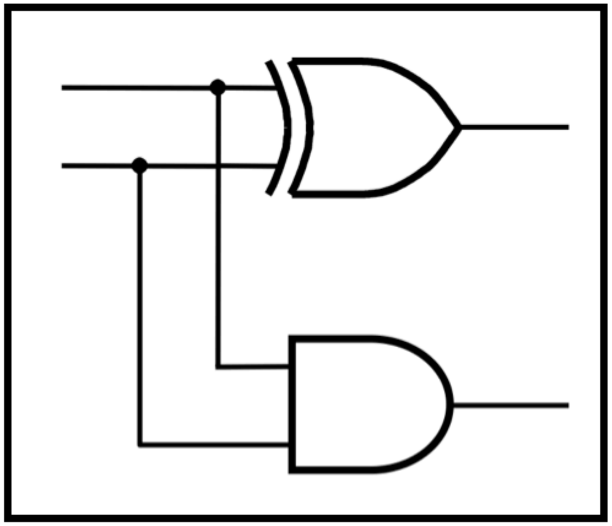


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Boolean Algebra

CprE 281: Digital Logic
Iowa State University, Ames, IA

Administrative Stuff

- HW1 is due today

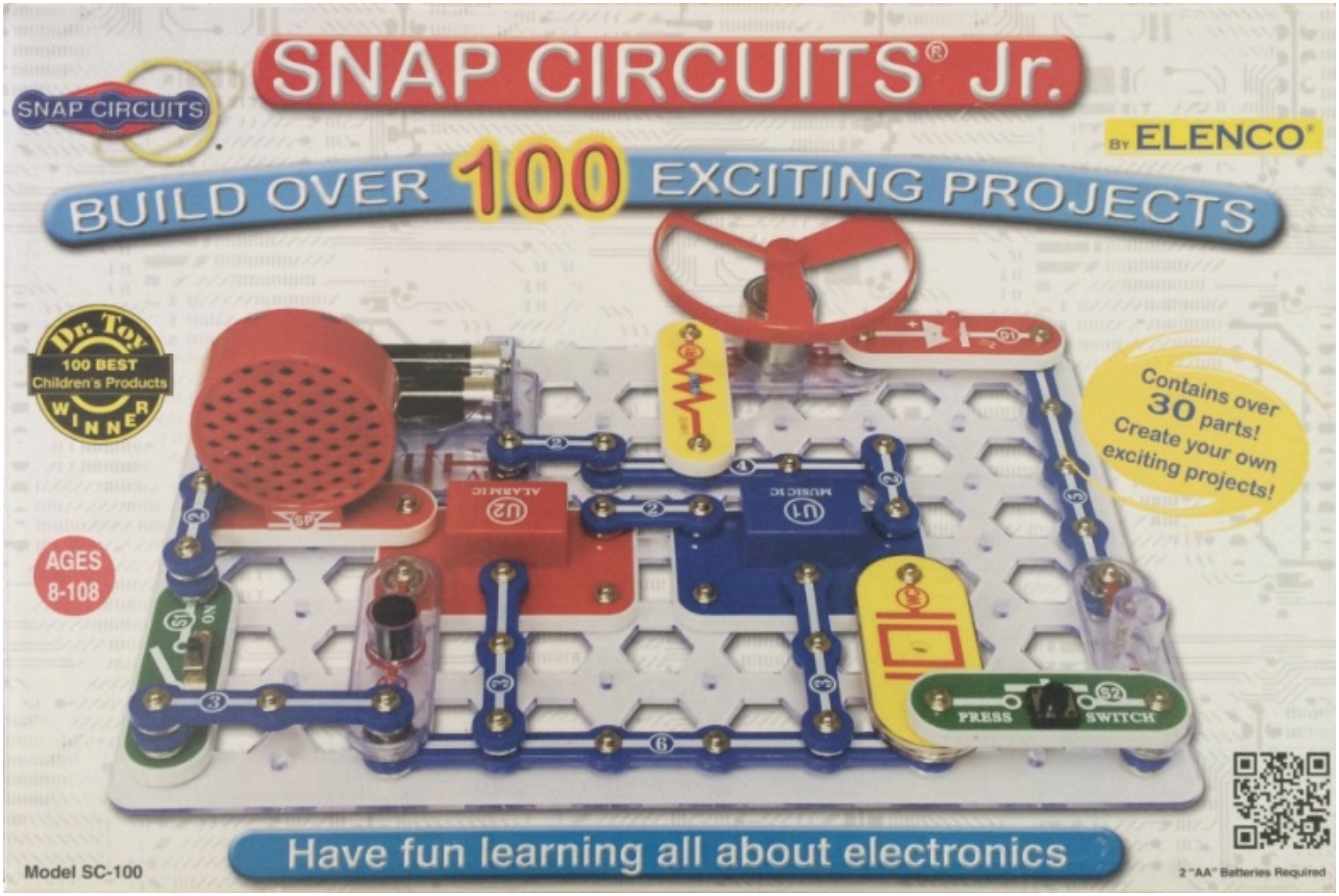
Administrative Stuff

- HW2 is out

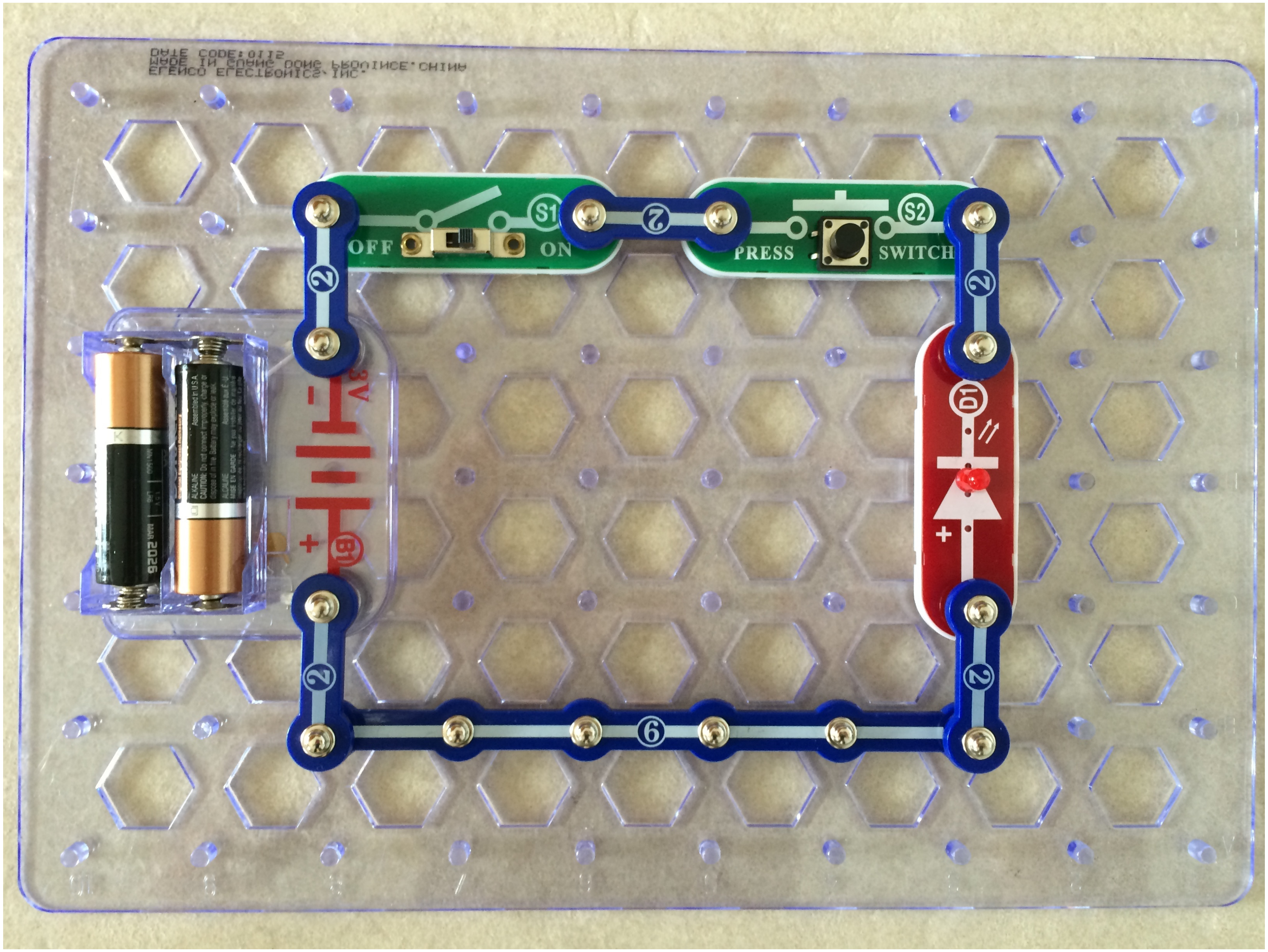
- It is due on Monday Aug 31 @ 4pm.

- Submit it on Canvas before the start of the lecture

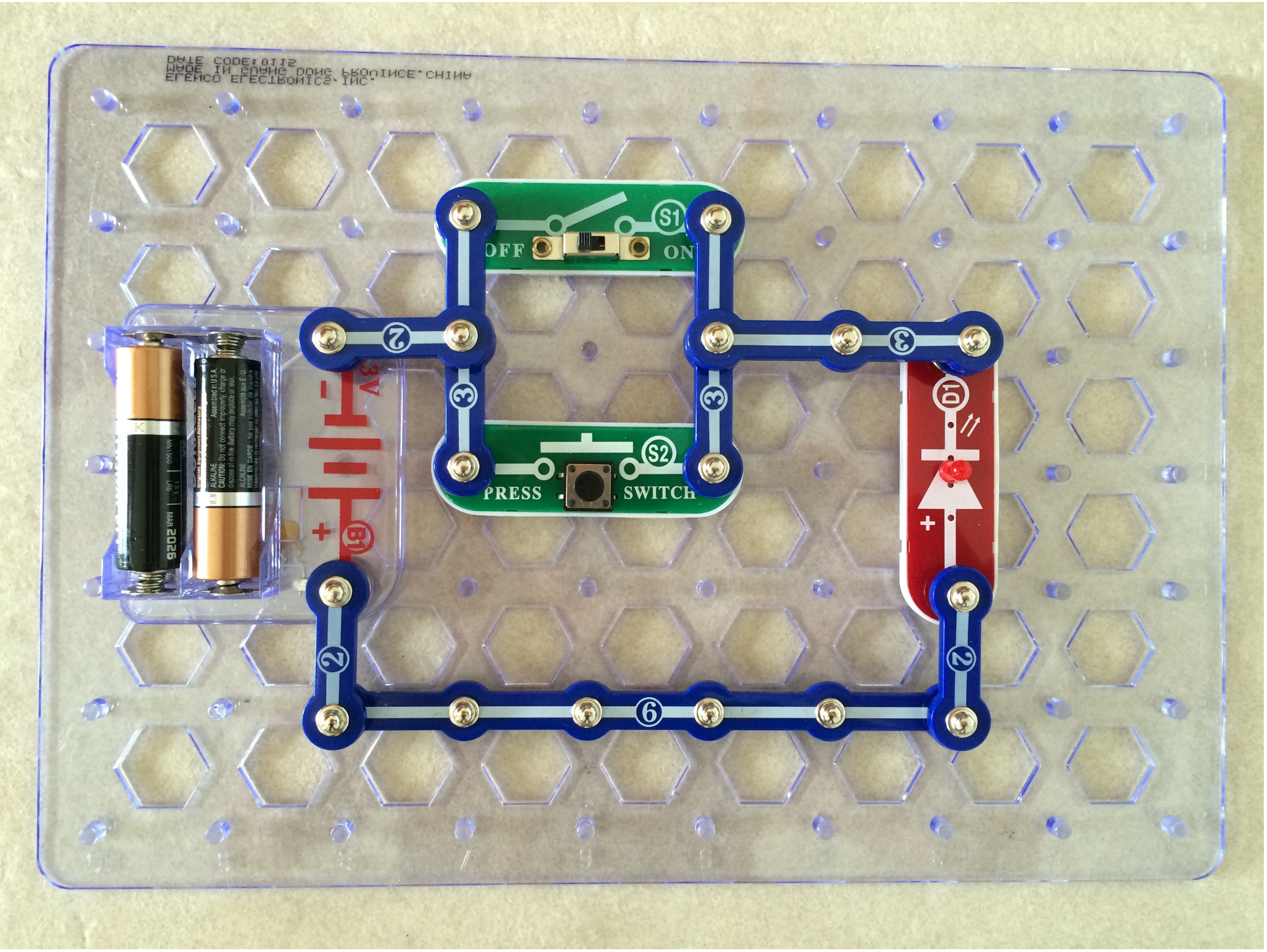
Did you play with this toy?



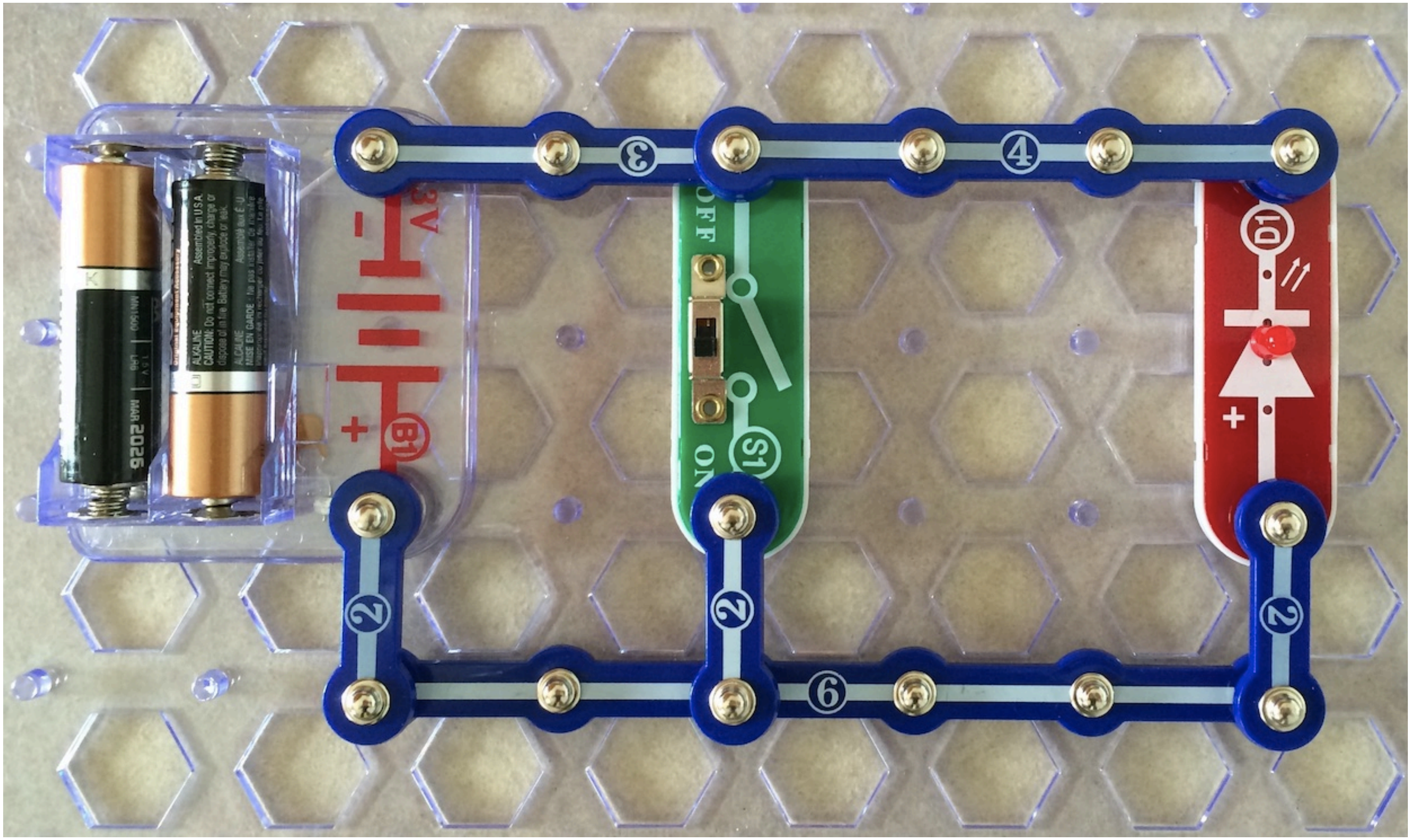
AND Gate



OR Gate

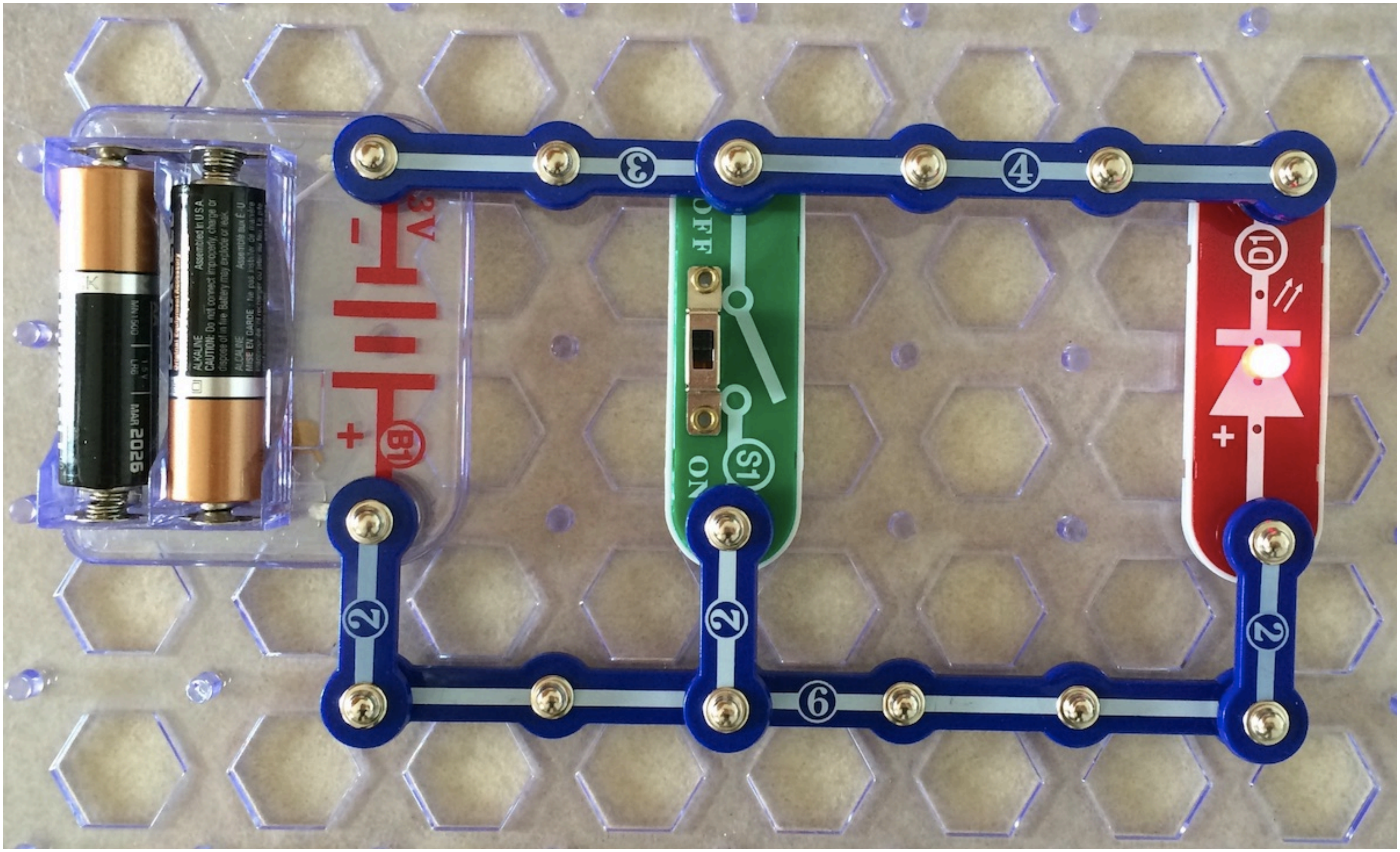


NOT Gate
(the switch is ON but the light is OFF)

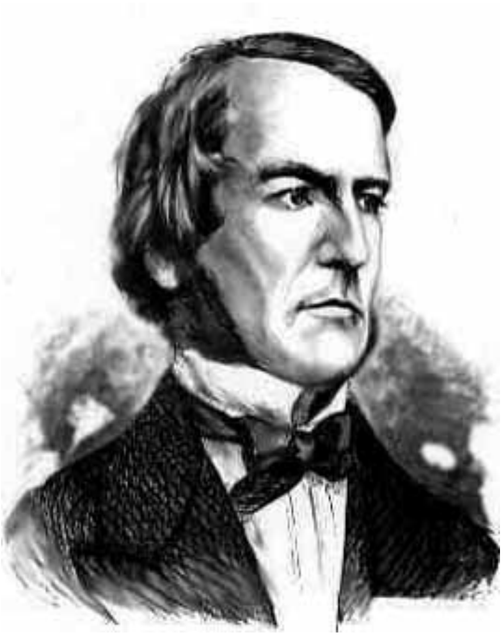


NOT Gate

(the switch is OFF but the light is ON)



Boolean Algebra



George Boole
1815-1864

- **An algebraic structure consists of**
    - **a set of elements {0, 1}**
    - **binary operators {+, •}**
    - **and a unary operator { ’ } or { ‾ } or { ~ }**

- **Introduced by George Boole in 1854**

- **An effective means of describing circuits built with switches**
- **A powerful tool that can be used for designing and analyzing logic circuits**

Axioms of Boolean Algebra

1a. $0 \bullet 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \bullet 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \bullet 1 = 1 \bullet 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

4a. If $x=0$, then $\overline{x} = 1$

4b. If $x=1$, then $\overline{x} = 0$

Single-Variable Theorems

5a. $x \bullet 0 = 0$
5b. $x + 1 = 1$

6a. $x \bullet 1 = x$
6b. $x + 0 = x$

7a. $x \bullet x = x$
7b. $x + x = x$

8a. $x \bullet \overline{x} = 0$
8b. $x + \overline{x} = 1$

9. $\overline{\overline{x}} = x$

Two- and Three-Variable Properties

10a. $x \bullet y = y \bullet x$ **Commutative**

10b. $x + y = y + x$

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$ **Associative**

11b. $x + (y + z) = (x + y) + z$

12a. $x \bullet (y + z) = x \bullet y + x \bullet z$ **Distributive**

12b. $x + y \bullet z = (x + y) \bullet (x + z)$

13a. $x + x \bullet y = x$ **Absorption**

13b. $x \bullet (x + y) = x$

Two- and Three-Variable Properties

14a. $x \bullet y + x \bullet \overline{y} = x$ **Combining**
14b. $(x + y) \bullet (x + \overline{y}) = x$

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$ **DeMorgan's**
15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$ **theorem**

16a. $x + \overline{x} \bullet y = x + y$
16b. $x \bullet (\overline{x} + y) = x \bullet y$

17a. $x \bullet y + y \bullet z + \overline{x} \bullet z = x \bullet y + \overline{x} \bullet z$ **Consensus**
17b. $(x+y) \bullet (y+z) \bullet (\overline{x}+z) = (x+y) \bullet (\overline{x}+z)$

Now, let’s prove all of these

The First Four are Axioms (i.e., they don’t require a proof)

1a. $0 \bullet 0 = 0$
1b. $1 + 1 = 1$

2a. $1 \bullet 1 = 1$
2b. $0 + 0 = 0$

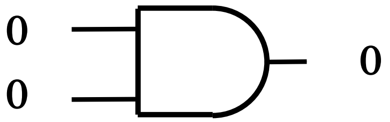
3a. $0 \bullet 1 = 1 \bullet 0 = 0$
3b. $1 + 0 = 0 + 1 = 1$

4a. If $x=0$, then $\overline{x} = 1$
4b. If $x=1$, then $\overline{x} = 0$

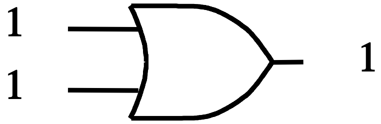
But here are some other ways to think about them

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$

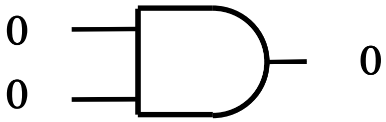


AND gate



OR gate

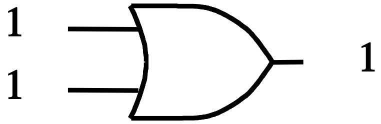
1a. $0 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

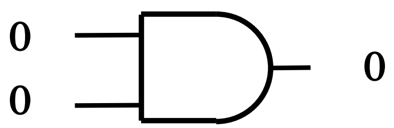
1b. $1 + 1 = 1$



OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

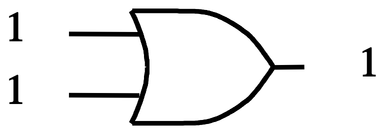
1a. $0 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

1b. $1 + 1 = 1$

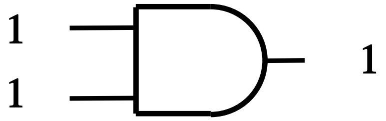


OR gate

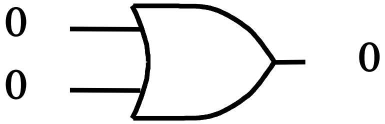
x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

2a. $1 \bullet 1 = 1$

2b. $0 + 0 = 0$

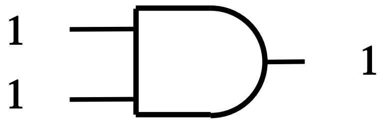


AND gate



OR gate

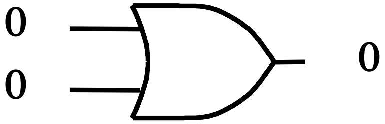
2a. $1 \bullet 1 = 1$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

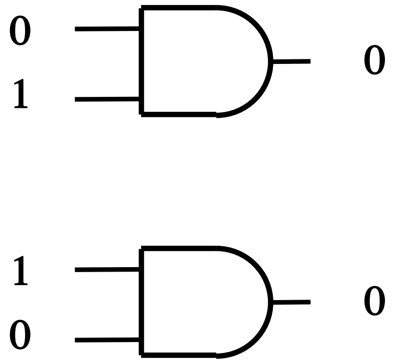
2b. $0 + 0 = 0$



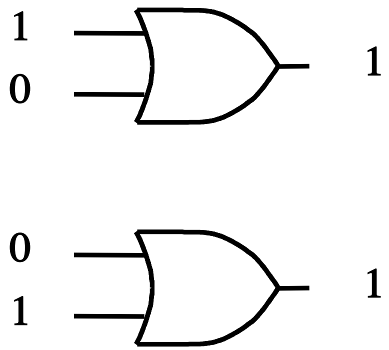
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

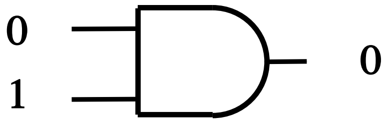
3a. $0 \cdot 1 = 1 \cdot 0 = 0$

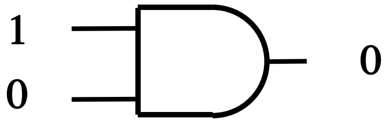


3b. $1 + 0 = 0 + 1 = 1$



3a. $0 \cdot 1 = 1 \cdot 0 = 0$

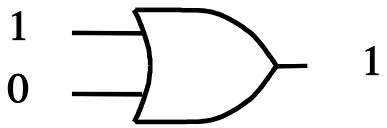


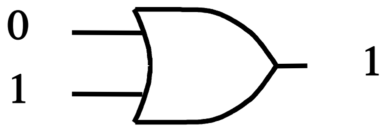


AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

3b. $1 + 0 = 0 + 1 = 1$

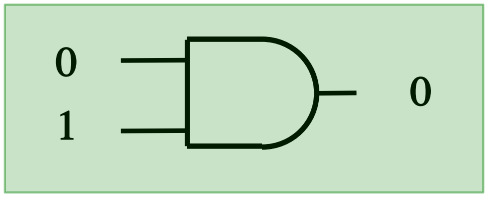


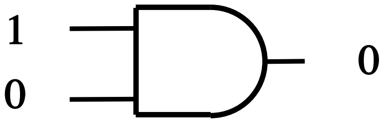


OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

3a. $0 \cdot 1 = 1 \cdot 0 = 0$

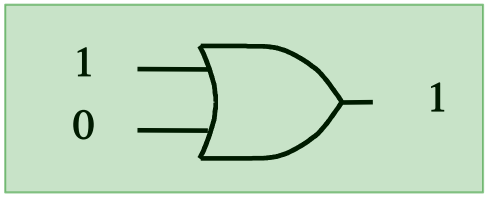


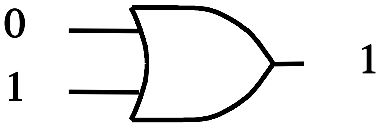


AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

3b. $1 + 0 = 0 + 1 = 1$



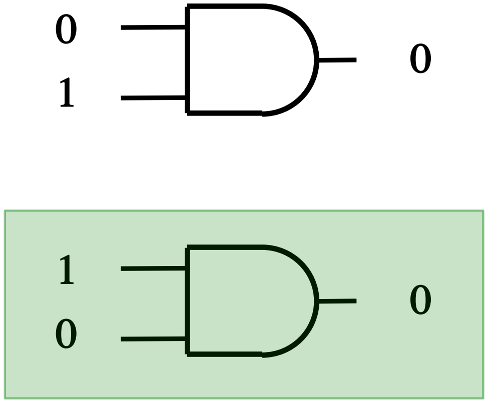


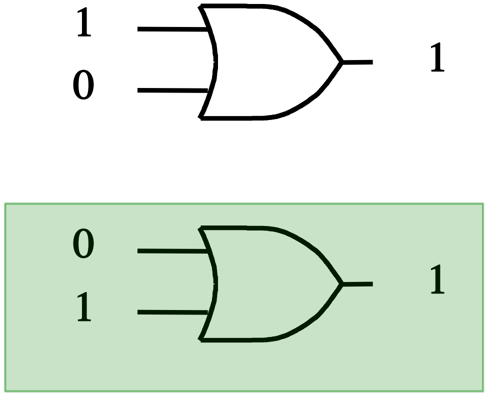
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

3a. $0 \cdot 1 = 1 \cdot 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$





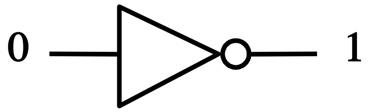
AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

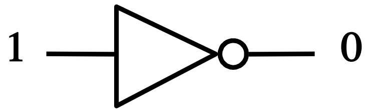
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

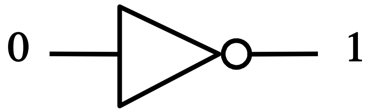
4a. If $x=0$, then $\overline{x} = 1$



4b. If $x=1$, then $\overline{x} = 0$



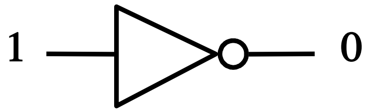
4a. If x=0, then $\overline{x}$ = 1



NOT gate

x	$\overline{x}$
0	1
1	0

4b. If x=1, then $\overline{x}$ = 0



NOT gate

x	$\overline{x}$
0	1
1	0

Single-Variable Theorems

5a. $x \bullet 0 = 0$

5b. $x + 1 = 1$

6a. $x \bullet 1 = x$

6b. $x + 0 = x$

7a. $x \bullet x = x$

7b. $x + x = x$

8a. $x \bullet \overline{x} = 0$

8b. $x + \overline{x} = 1$

9. $\overline{\overline{x}} = x$

5a. $x \bullet 0 = 0$

5a. $x \bullet 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let’s look at each case separately.

5a. $x \bullet 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let’s look at each case separately.

i) If x = 0, then we have

$0 \bullet 0 = 0$ // axiom 1a

5a. $x \bullet 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$0 \bullet 0 = 0 \qquad \text{// axiom 1a}$$

ii) If x = 1, then we have

$$1 \bullet 0 = 0 \qquad \text{// axiom 3a}$$

5b. $x + 1 = 1$

5b. $x + 1 = 1$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$0 + 1 = 1$ // axiom 3b

5b. $x + 1 = 1$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$0 + 1 = 1$ // axiom 3b

ii) If $x = 1$, then we have

$1 + 1 = 1$ // axiom 1b

6a. $x \bullet 1 = x$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$0 \bullet 1 = 0$ // axiom 3a

ii) If x = 1, then we have

$1 \bullet 1 = 1$ // axiom 2a

$$6a. \quad \boxed{x} \bullet 1 = \boxed{x}$$

The Boolean variable x can have only two possible values: 0 or 1. Let’s look at each case separately.

i) If x = 0, then we have

$$\boxed{0} \bullet 1 = \boxed{0} \qquad \text{// axiom 3a}$$

ii) If x = 1, then we have

$$\boxed{1} \bullet 1 = \boxed{1} \qquad \text{// axiom 2a}$$

6b. $x + 0 = x$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$0 + 0 = 0$ // axiom 2b

ii) If x = 1, then we have

$1 + 0 = 1$ // axiom 3b

6b. $\boxed{x} + 0 = \boxed{x}$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$\boxed{0} + 0 = \boxed{0} \qquad \text{// axiom 2b}$$

ii) If x = 1, then we have

$$\boxed{1} + 0 = \boxed{1} \qquad \text{// axiom 3b}$$

7a. $x \bullet x = x$

i) If $x = 0$, then we have

$$0 \bullet 0 = 0 \qquad \text{// axiom 1a}$$

ii) If $x = 1$, then we have

$$1 \bullet 1 = 1 \qquad \text{// axiom 2a}$$

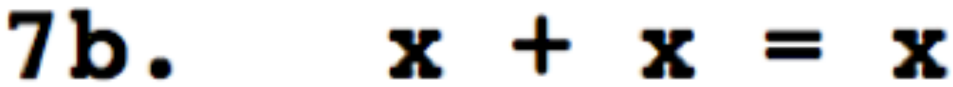
7a. $\boxed{x} \bullet \boxed{x} = \boxed{x}$

i) If x = 0, then we have

$\boxed{0} \bullet \boxed{0} = \boxed{0}$ // axiom 1a

ii) If x = 1, then we have

$\boxed{1} \bullet \boxed{1} = \boxed{1}$ // axiom 2a



7b. $x + x = x$

i) If $x = 0$, then we have

$0 + 0 = 0$ // axiom 2b

ii) If $x = 1$, then we have

$1 + 1 = 1$ // axiom 1b

7b. $\boxed{x} + \boxed{x} = \boxed{x}$

i) If x = 0, then we have

$$\boxed{0} + \boxed{0} = \boxed{0}$$ // axiom 2b

ii) If x = 1, then we have

$$\boxed{1} + \boxed{1} = \boxed{1}$$ // axiom 1b

8a. $x \bullet \overline{x} = 0$

i) If $x = 0$, then we have

$0 \bullet 1 = 0$ // axiom 3a

ii) If $x = 1$, then we have

$1 \bullet 0 = 0$ // axiom 3a

8a. $x \bullet \overline{x} = \boxed{0}$

i) If $x = 0$, then we have

$0 \bullet 1 = \boxed{0}$ // axiom 3a

ii) If $x = 1$, then we have

$1 \bullet 0 = \boxed{0}$ // axiom 3a

8b. $x + \overline{x} = 1$

i) If $x = 0$, then we have

$0 + 1 = 1$ // axiom 3b

ii) If $x = 1$, then we have

$1 + 0 = 1$ // axiom 3b

8b. $x + \overline{x} = \boxed{1}$

i) If x = 0, then we have

$0 + 1 = \boxed{1}$ // axiom 3b

ii) If x = 1, then we have

$1 + 0 = \boxed{1}$ // axiom 3b

$$9. \quad \overline{\overline{x}} = x$$

i) If x = 0, then we have

$$\overline{x} = 1$$ // axiom 4a

let $y = \overline{x} = 1$, then we have

$$\overline{y} = 0$$ // axiom 4b

Therefore,

$$\overline{\overline{x}} = x$$ (when x =0)

9. $\overline{\overline{x}} = x$

ii) If x = 1, then we have

$\overline{x} = 0$ // axiom 4b

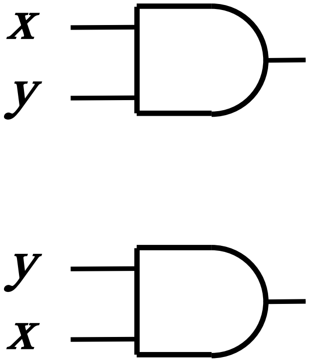
let $y = \overline{x} = 0$, then we have

$\overline{y} = 1$ // axiom 4a

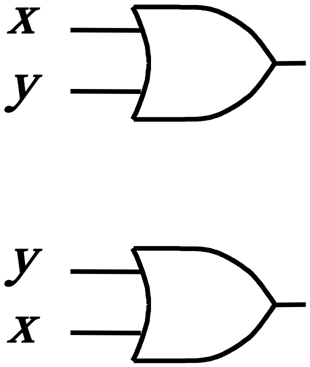
Therefore,

$\overline{\overline{x}} = x$ (when x =1)

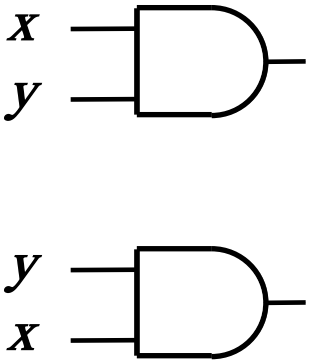
10a. $x \bullet y = y \bullet x$



10b. $x + y = y + x$



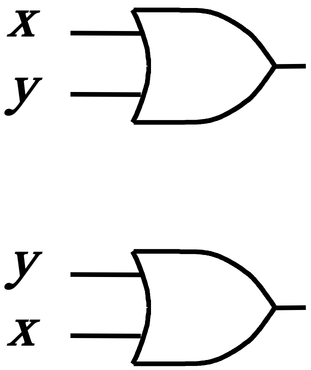
10a. $x \bullet y = y \bullet x$



AND gate

x	y	f
0	0	0
0	1	0
1	0	0
1	1	1

10b. $x + y = y + x$



OR gate

x	y	f
0	0	0
0	1	1
1	0	1
1	1	1

The order of the inputs does not matter.

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x	y • z	x•(y•z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x	$y \bullet z$	$x \bullet (y \bullet z)$
0	0	0	0	0	
0	0	1	0	0	
0	1	0	0	0	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	0	
1	1	0	1	0	
1	1	1	1	1	

Truth table for the left-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x	y • z	x•(y•z)
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	1	1	1

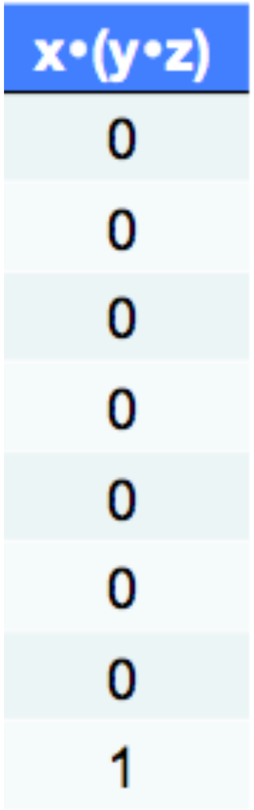
Truth table for the left-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

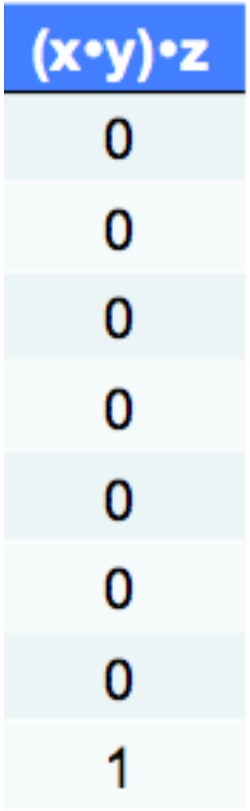
x	y	z	x • y	z	(x•y)•z
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	0	0	0
1	0	1	0	1	0
1	1	0	1	0	0
1	1	1	1	1	1

Truth table for the right-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$



x•(y•z)
0
0
0
0
0
0
0
1



(x•y)•z
0
0
0
0
0
0
0
1

These two are identical, which concludes the proof.

11b. $x + (y + z) = (x + y) + z$

x	y	z	x	y + z	x+(y+z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

11b. $x + (y + z) = (x + y) + z$

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	
0	0	1	0	1	
0	1	0	0	1	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	1	
1	1	0	1	1	
1	1	1	1	1	

Truth table for the left-hand side

11b. $x + (y + z) = (x + y) + z$

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Truth table for the left-hand side

11b. $x + (y + z) = (x + y) + z$

x	y	z	x + y	z	(x+y)+z
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	0	1
1	1	1	1	1	1

Truth table for the right-hand side

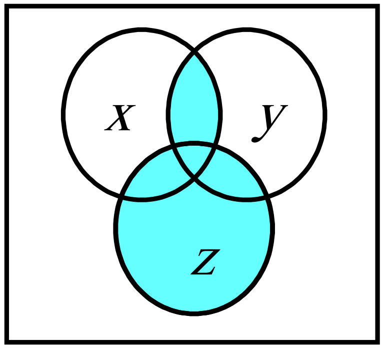
11b. $x + (y + z) = (x + y) + z$

x+(y+z)
0
1
1
1
1
1
1
1

(x+y)+z
0
1
1
1
1
1
1
1

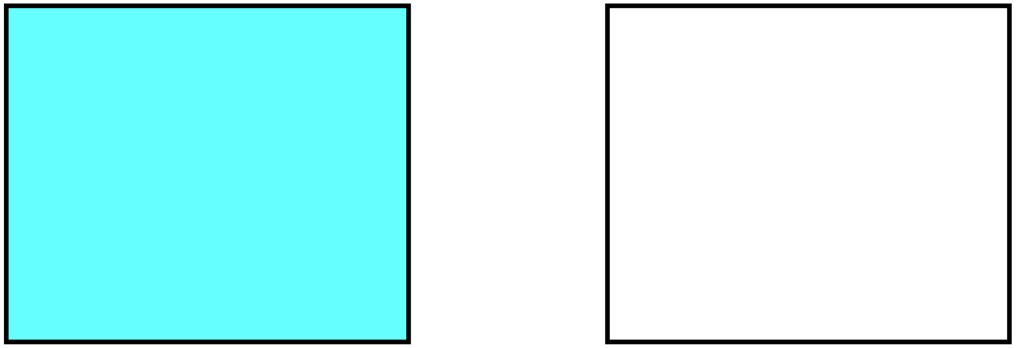
These two are identical, which concludes the proof.

The Venn Diagram Representation



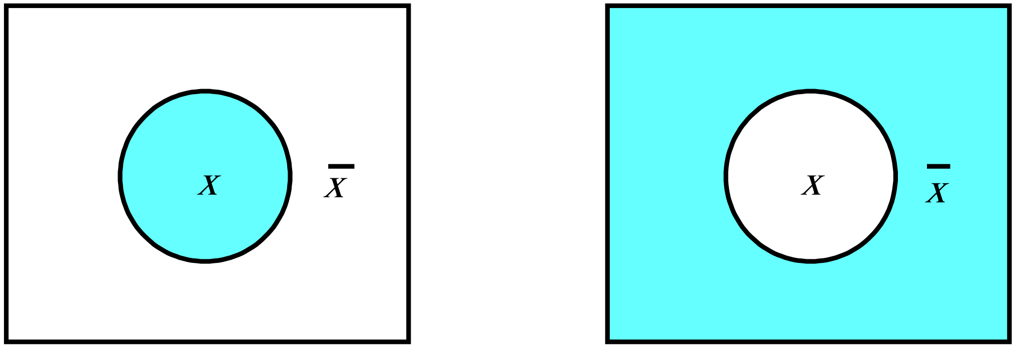
$xy + z$

Venn Diagram Basics



(a) Constant 1

(b) Constant 0

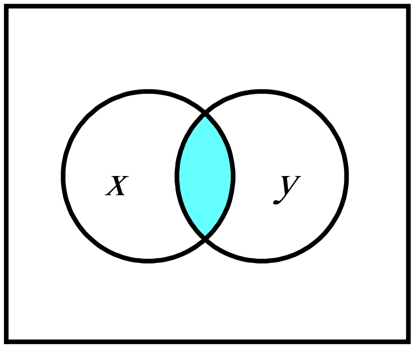


(c) Variable x

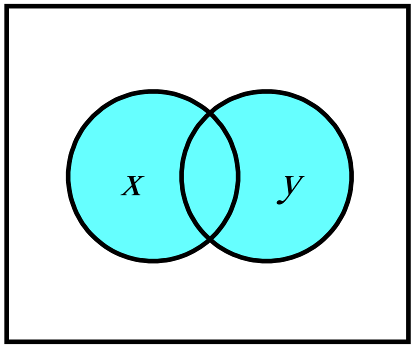
(d) $\overline{x}$

[Figure 2.14 from the textbook]

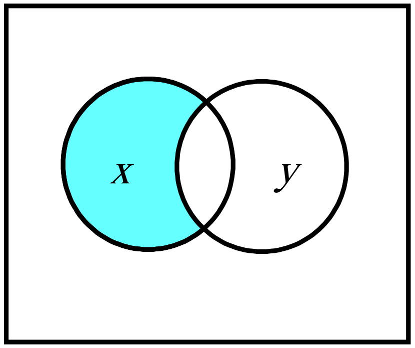
Venn Diagram Basics



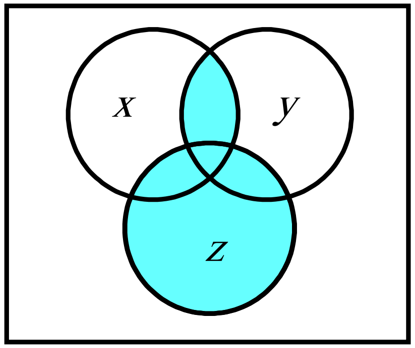
(e) $x \; y$



(f) $x + y$



(g) $x \; \bar{y}$



(h) $x y + z$

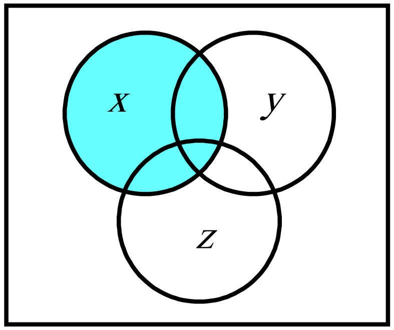
[Figure 2.14 from the textbook]

Let’s Prove the Distributive Properties

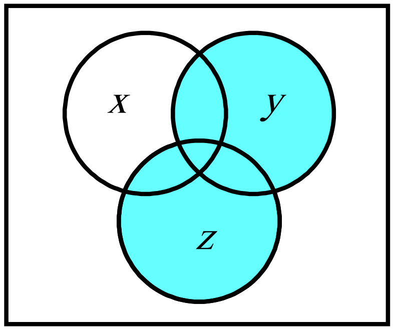
12a. $x \bullet (y + z) = x \bullet y + x \bullet z$

12b. $x + y \bullet z = (x + y) \bullet (x + z)$

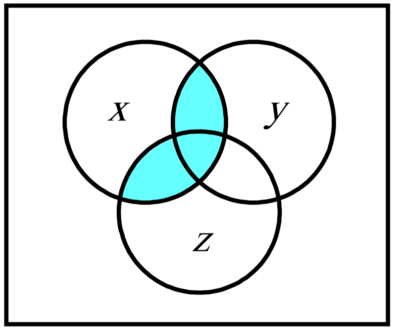
12a. $x \bullet (y + z) = x \bullet y + x \bullet z$



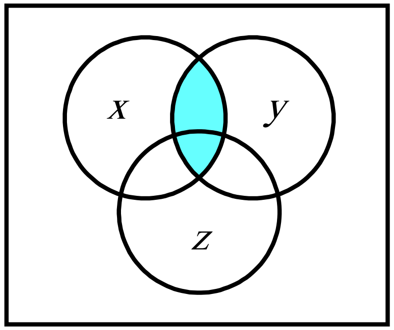
(a) x



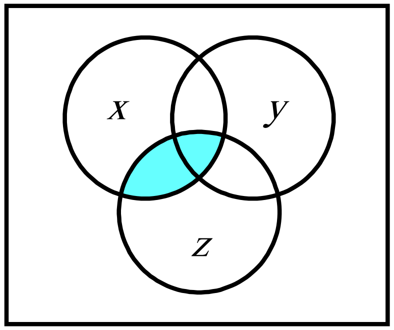
(b) $y + z$



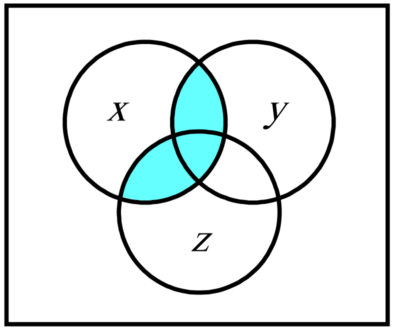
(c) $x \ (y + z)$



(d) $x \, y$



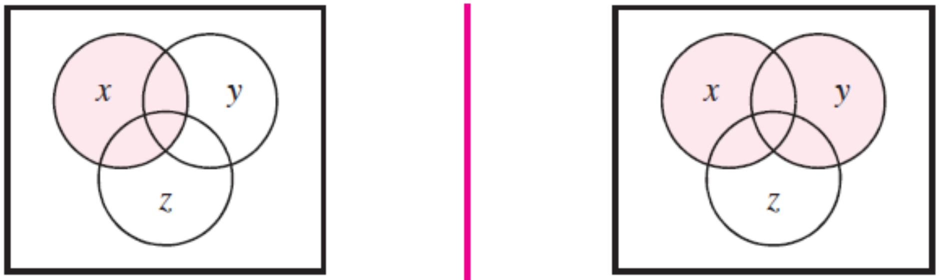
(e) $x \, z$



(f) $x y \ + x \, z$

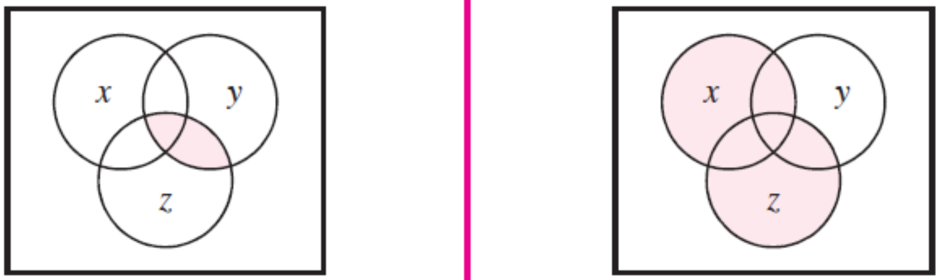
[Figure 2.15 from the textbook]

12b. $x + y \cdot z = (x + y) \cdot (x + z)$



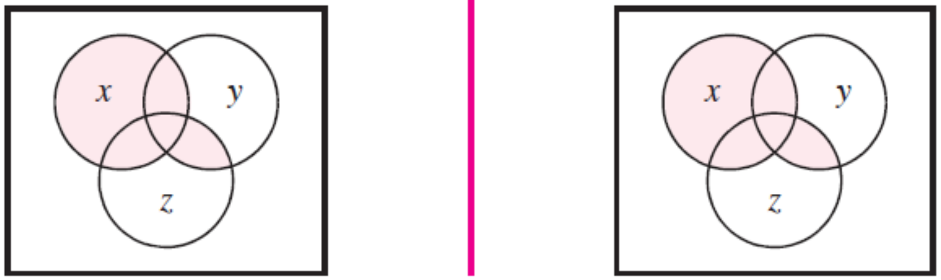
(a) x

(d) $x + y$



(b) $y \cdot z$

(e) $x + z$



(c) $x + y \cdot z$

(f) $(x + y) \cdot (x + z)$

[Figure 2.17 from the textbook]

Try to prove these ones at home

13a. $x + x \cdot y = x$

13b. $x \cdot (x + y) = x$

14a. $x \cdot y + x \cdot \overline{y} = x$

14b. $(x + y) \cdot (x + \overline{y}) = x$

DeMorgan’s Theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

Proof of DeMorgan’s theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$

[Figure 2.13 from the textbook]

Proof of DeMorgan’s theorem

15a. $\overline{\mathbf{x} \bullet \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0					
0	1					
1	0					
1	1					

LHS

RHS

Proof of DeMorgan’s theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0				
0	1	0				
1	0	0				
1	1	1				

LHS RHS

Proof of DeMorgan’s theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1			
0	1	0	1			
1	0	0	1			
1	1	1	0			

LHS

RHS

Proof of DeMorgan’s theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1	1		
0	1	0	1	1		
1	0	0	1	0		
1	1	1	0	0		

LHS

RHS

Proof of DeMorgan’s theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1	1	1	
0	1	0	1	1	0	
1	0	0	1	0	1	
1	1	1	0	0	0	

LHS RHS

Proof of DeMorgan’s theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$

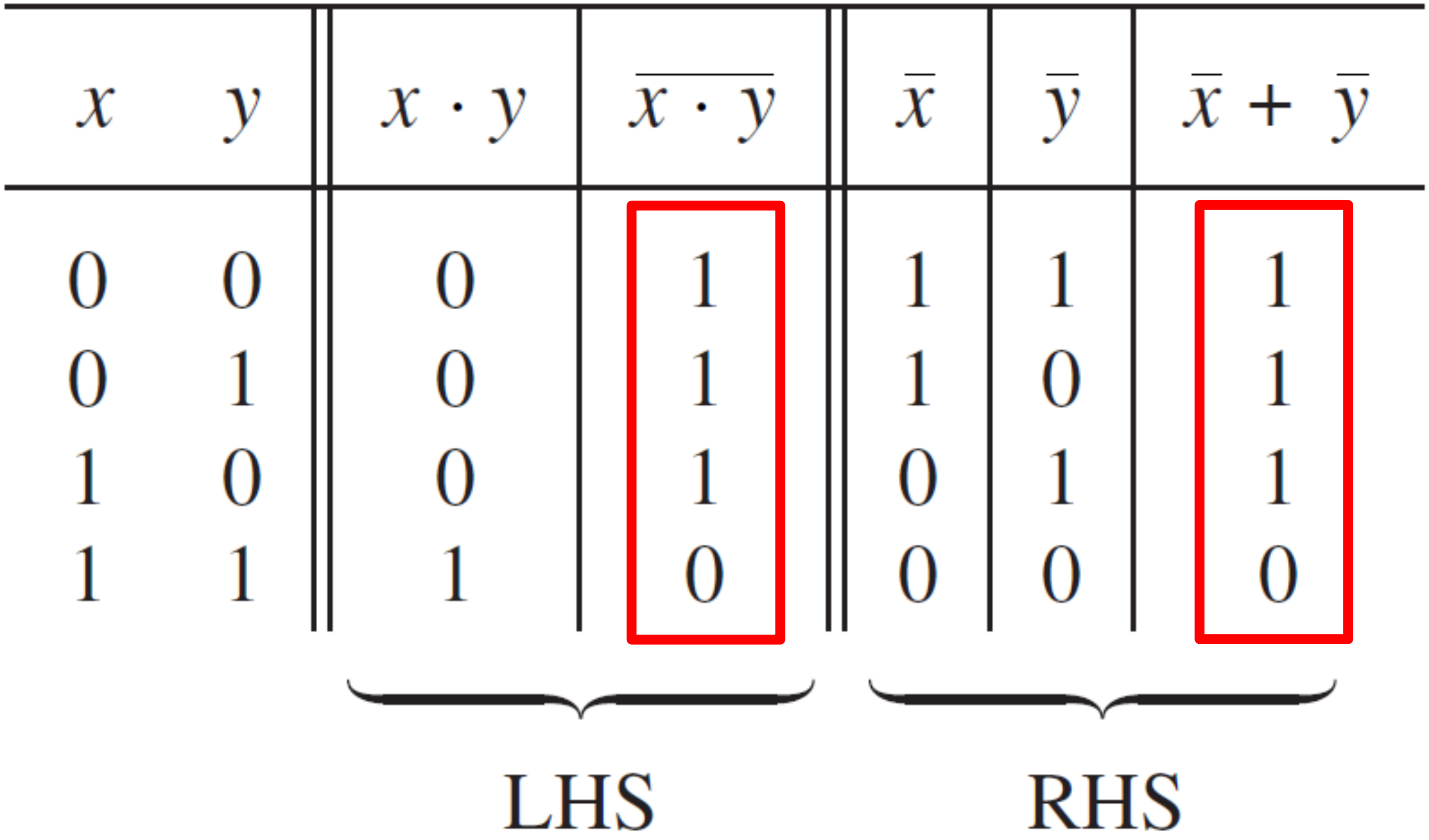
x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS

RHS

Proof of DeMorgan's theorem

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$



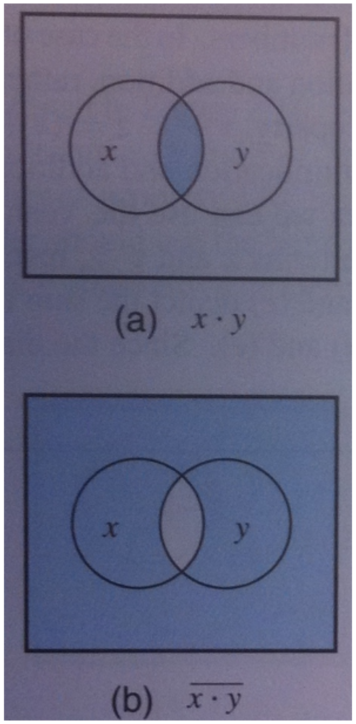
x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

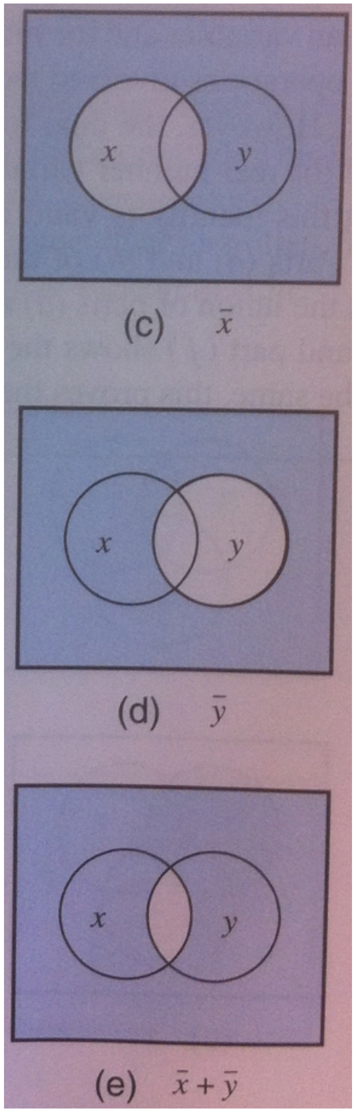
LHS RHS

These two columns are equal. Therefore, the theorem is true.

Alternative proof using Venn Diagrams

15a. $\overline{x \bullet y} = \overline{x} + \overline{y}$





[Figure 2.18 from the textbook]

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

Let's prove the other DeMorgan's theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

x	y	$x + y$	$\overline{x + y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0					
0	1					
1	0					
1	1					

LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \bar{x} \bullet \bar{y}$

x	y	$x + y$	$\overline{x + y}$	$\bar{x}$	$\bar{y}$	$\bar{x} \bullet \bar{y}$
0	0	0				
0	1	1				
1	0	1				
1	1	1				

LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \bar{x} \bullet \bar{y}$

x	y	$x + y$	$\overline{x + y}$	$\bar{x}$	$\bar{y}$	$\bar{x} \bullet \bar{y}$
0	0	0	1			
0	1	1	0			
1	0	1	0			
1	1	1	0			

LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \bar{x} \bullet \bar{y}$

x	y	$x + y$	$\overline{x + y}$	$\bar{x}$	$\bar{y}$	$\bar{x} \bullet \bar{y}$
0	0	0	1	1		
0	1	1	0	1		
1	0	1	0	0		
1	1	1	0	0		

LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

x	y	$x + y$	$\overline{x + y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0	0	1	1	1	
0	1	1	0	1	0	
1	0	1	0	0	1	
1	1	1	0	0	0	

LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

x	y	$x + y$	$\overline{x + y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

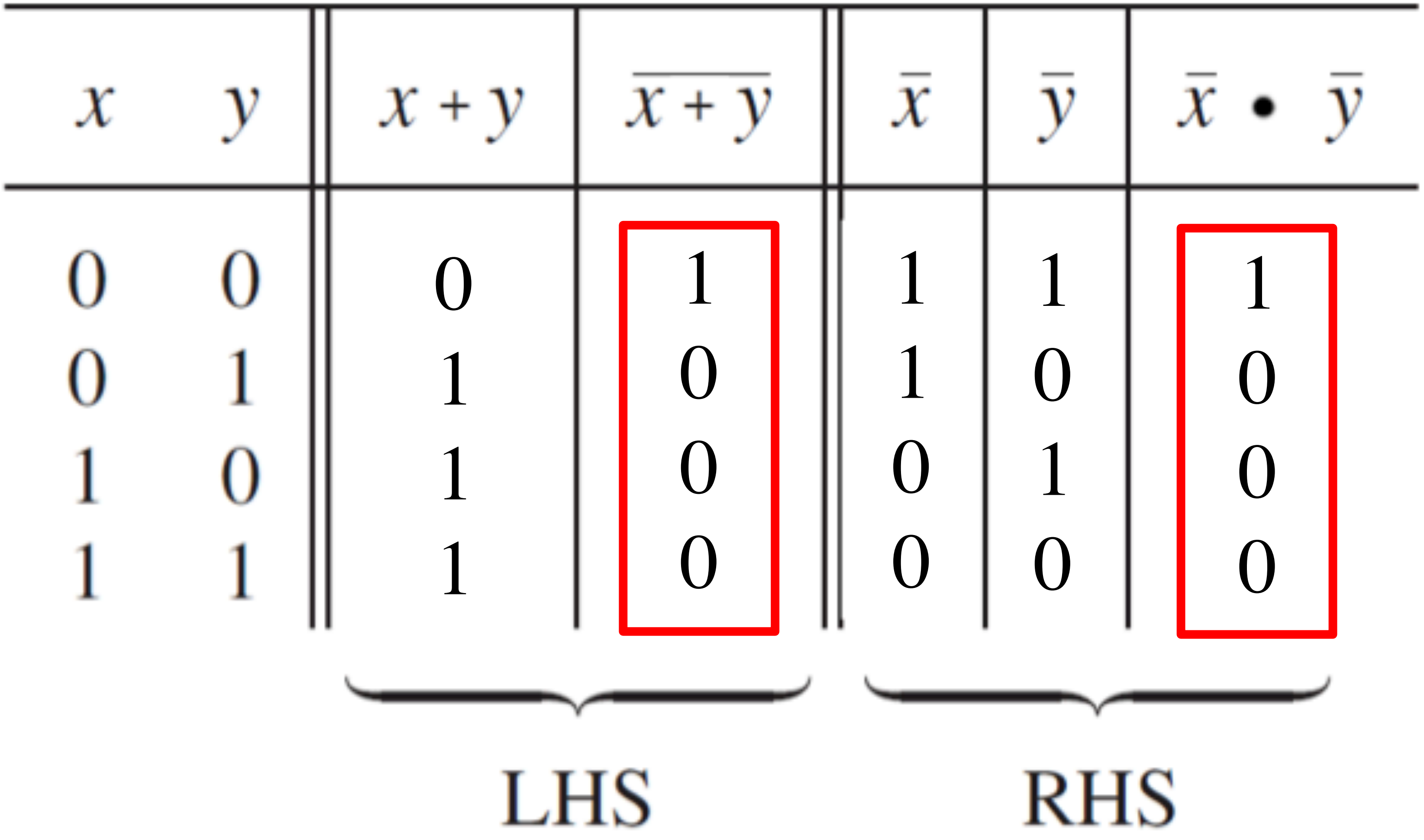
LHS

RHS

Let’s prove the other DeMorgan’s theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

These two columns are equal, so the theorem is true.



x	y	$x + y$	$\overline{x + y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

LHS

RHS

Try to prove these ones at home

16a. $x + \overline{x} \bullet y = x + y$

16b. $x \bullet (\overline{x} + y) = x \bullet y$

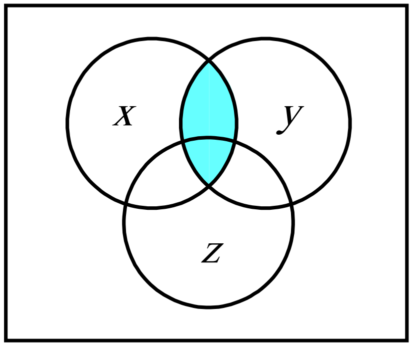
17a. $x \bullet y + y \bullet z + \overline{x} \bullet z = x \bullet y + \overline{x} \bullet z$

17b. $(x+y) \bullet (y+z) \bullet (\overline{x}+z) = (x+y) \bullet (\overline{x}+z)$

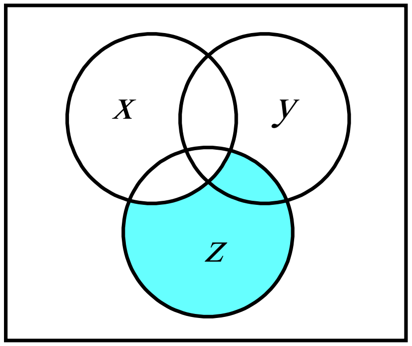
Venn Diagram Example
Proof of Property 17a

17a. $x \bullet y + y \bullet z + \overline{x} \bullet z = x \bullet y + \overline{x} \bullet z$

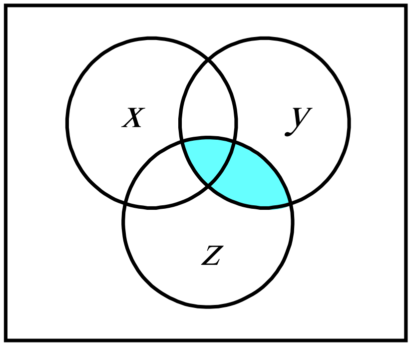
Left-Hand Side



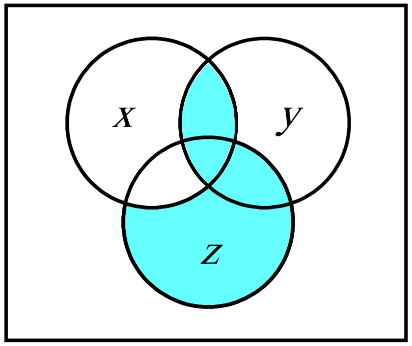
$x y$



$\overline{x}\, z$



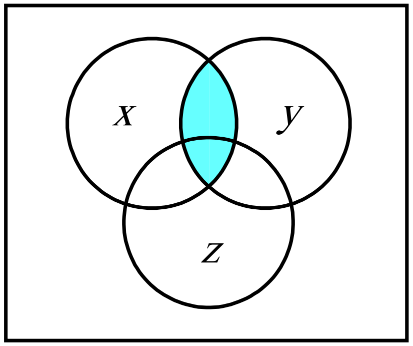
$y\, z$



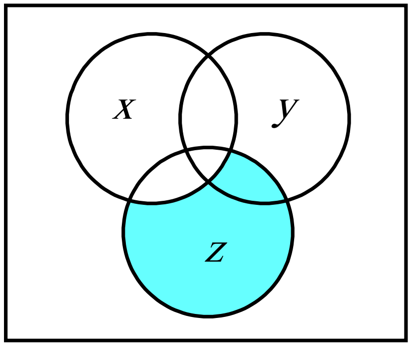
$x y + \overline{x} z + y z$

[Figure 2.16 from the textbook]

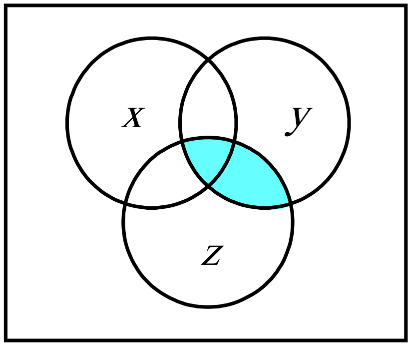
Left-Hand Side



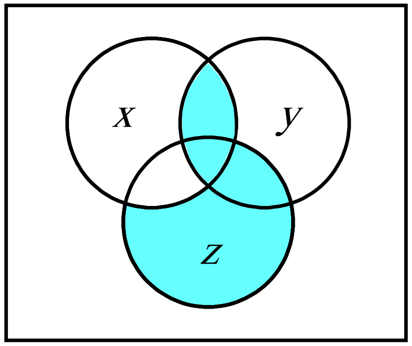
$x y$



$\overline{x} z$

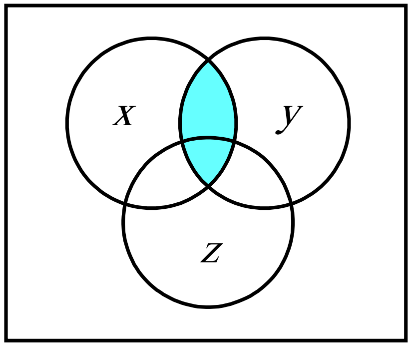


$y z$

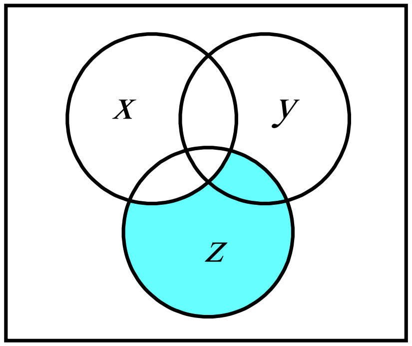


$x y + \overline{x} z + y z$

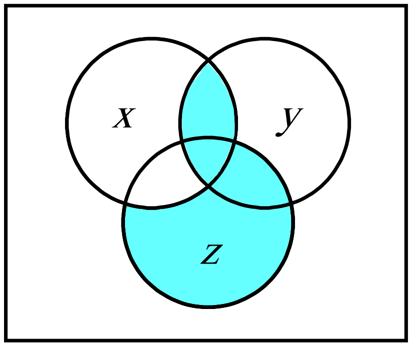
Right-Hand Side



$x \ y$



$\overline{x} \ z$

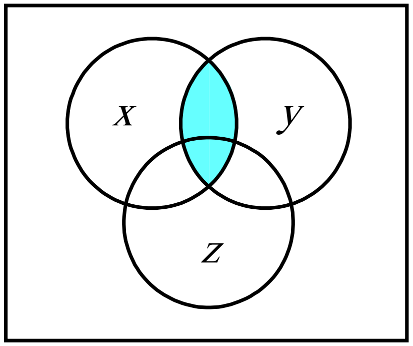


$x \ y + \overline{x} \ z$

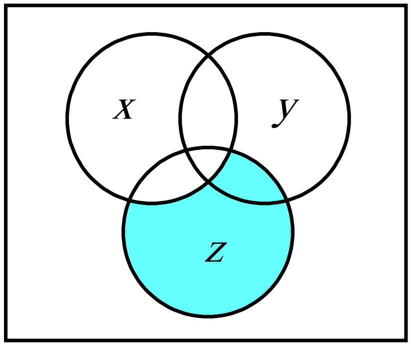
[Figure 2.16 from the textbook]

Left-Hand Side

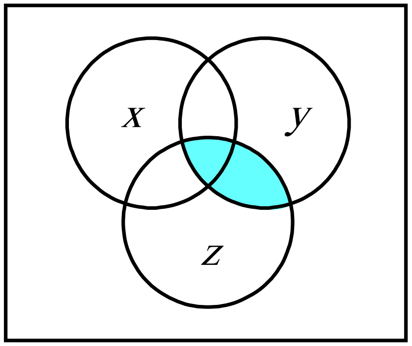
These two are equal



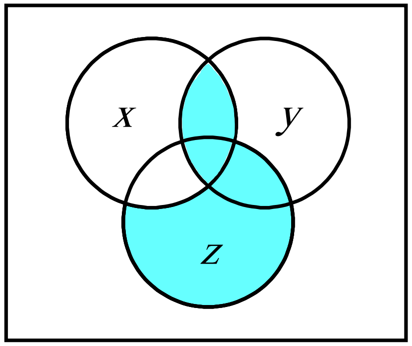
$x y$



$\overline{x} z$

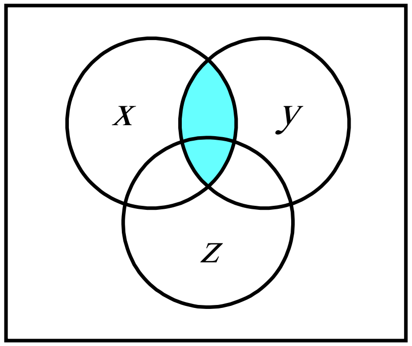


$y z$

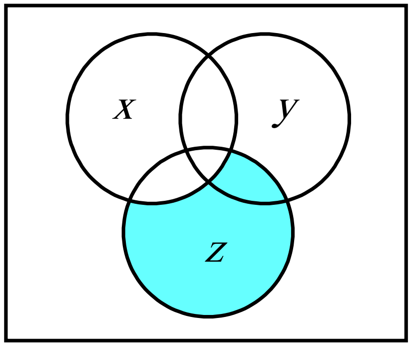


$x y + \overline{x} z + y z$

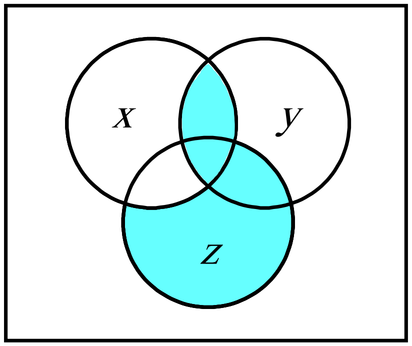
Right-Hand Side



$x \; y$



$\overline{x} \; z$



$x \; y + \overline{x} \; z$

[Figure 2.16 from the textbook]

Questions?

THE END