



CprE 281: Digital Logic

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<http://www.ece.iastate.edu/~alexs/classes/>

Addition of Unsigned Numbers

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Iowa State University, Ames, IA
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Administrative Stuff

- **HW5 is due today**

Administrative Stuff

- **No HW is due next Monday**
- **HW 6 will be due on Monday Oct 14.**

Administrative Stuff

- **Labs next week**
- **Mini-Project**
- **This is worth 3% of your grade (x2 labs)**
- **http://www.ece.iastate.edu/~alexs/classes/2018_Fall_281/labs/Project-Mini/**

Quick Review


Number Systems

$$N = d_n B^n + d_{n-1} B^{n-1} + \cdots + d_1 B^1 + d_0 B^0$$


Number Systems

$$N = d_n B^n + d_{n-1} B^{n-1} + \dots + d_1 B^1 + d_0 B^0$$

n-th digit
(most significant)



0-th digit
(least significant)



Number Systems

base

power

$$N = d_n B^n + d_{n-1} B^{n-1} + \dots + d_1 B^1 + d_0 B^0$$

n-th digit
(most significant)

0-th digit
(least significant)

The Decimal System

$$524_{10} = 5 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

The Decimal System

$$524_{10} = 5 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

$$= 5 \times 100 + 2 \times 10 + 4 \times 1$$

$$= 500 + 20 + 4$$

$$= 524_{10}$$

Another Way to Look at This

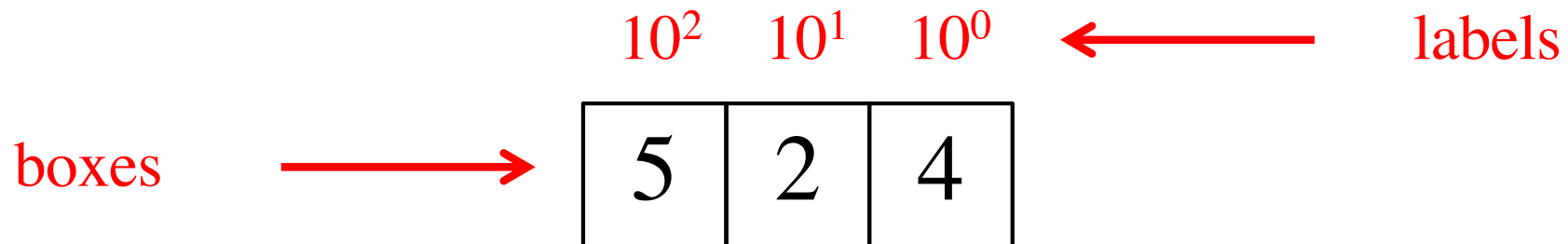
5	2	4
---	---	---

Another Way to Look at This

10^2 10^1 10^0

5	2	4
---	---	---

Another Way to Look at This



Each box can contain only one digit and has only one label. From right to left, the labels are increasing powers of the base, starting from 0.

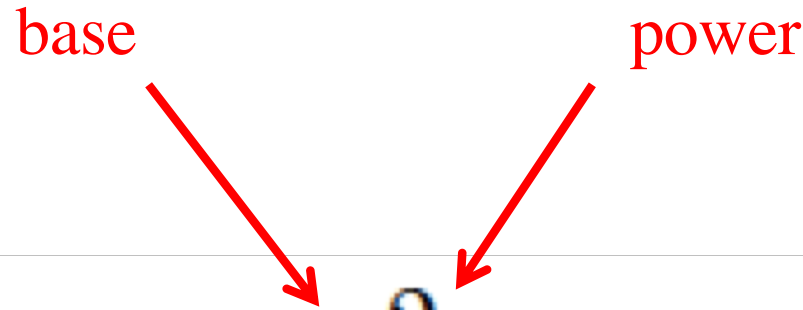
Base 7

$$524_7 = 5 \times 7^2 + 2 \times 7^1 + 4 \times 7^0$$

Base 7

base

power


$$524_7 = 5 \times 7^2 + 2 \times 7^1 + 4 \times 7^0$$

Base 7

base

power

$$524_7 = 5 \times 7^2 + 2 \times 7^1 + 4 \times 7^0$$

most significant
digit

least significant
digit

Base 7

$$524_7 = 5 \times 7^2 + 2 \times 7^1 + 4 \times 7^0$$

$$= 5 \times 49 + 2 \times 7 + 4 \times 1$$

$$= 245 + 14 + 4$$

$$= 263_{10}$$

Another Way to Look at This

$$\begin{array}{|c|c|c|} \hline 7^2 & 7^1 & 7^0 \\ \hline 5 & 2 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 10^2 & 10^1 & 10^0 \\ \hline 2 & 6 & 3 \\ \hline \end{array}$$

Binary Numbers (Base 2)

$$1001_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

Binary Numbers (Base 2)

$$\begin{aligned} 1001_2 &= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = \\ &= 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1 = \\ &= 8 + 0 + 0 + 1 = \\ &= 9_{10} \end{aligned}$$

Another Example

$$\begin{aligned} 11101_2 &= 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = \\ &= 1 \times 16 + 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 = \\ &= 16 + 8 + 4 + 0 + 1 = 29_{10} \end{aligned}$$

Powers of 2

$$2^{10} = 1024$$

$$2^9 = 512$$

$$2^8 = 256$$

$$2^7 = 128$$

$$2^6 = 64$$

$$2^5 = 32$$

$$2^4 = 16$$

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = 2$$

$$2^0 = 1$$

What is the value of this binary number?

- **0 0 1 0 1 1 0 0**

- **0 0 1 0 1 1 0 0**

- **$0*2^7 + 0*2^6 + 1*2^5 + 0*2^4 + 1*2^3 + 1*2^2 + 0*2^1 + 0*2^0$**

- **$0*128 + 0*64 + 1*32 + 0*16 + 1*8 + 1*4 + 0*2 + 0*1$**

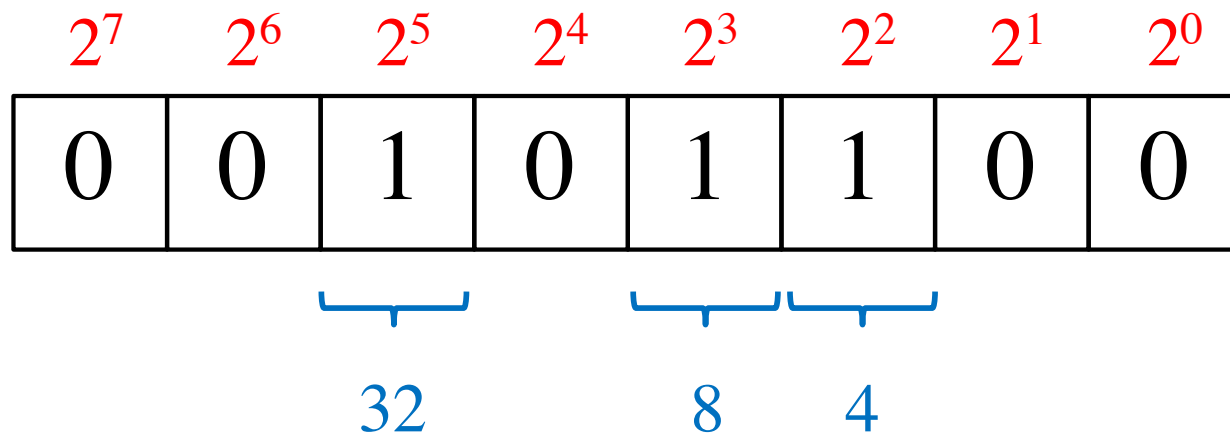
- **$0*128 + 0*64 + 1*32 + 0*16 + 1*8 + 1*4 + 0*2 + 0*1$**

- **$32 + 8 + 4 = 44$ (in decimal)**

Another Way to Look at This

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

Another Way to Look at This



Signed v.s. Unsigned Numbers

Two Different Types of Binary Numbers

Unsigned numbers

- All bits jointly represent a positive integer.
- Negative numbers cannot be represented this way.

Signed numbers

- The left-most bit represents the sign of the number.
- If that bit is 0, then the number is positive.
- If that bit is 1, then the number is negative.
- The magnitude of the largest number that can be represented in this way is twice smaller than the largest number in the unsigned representation.

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

This represents + 44.

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

This represents + 172.

Signed Representation (using the left-most bit as the sign)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

This represents + 44.

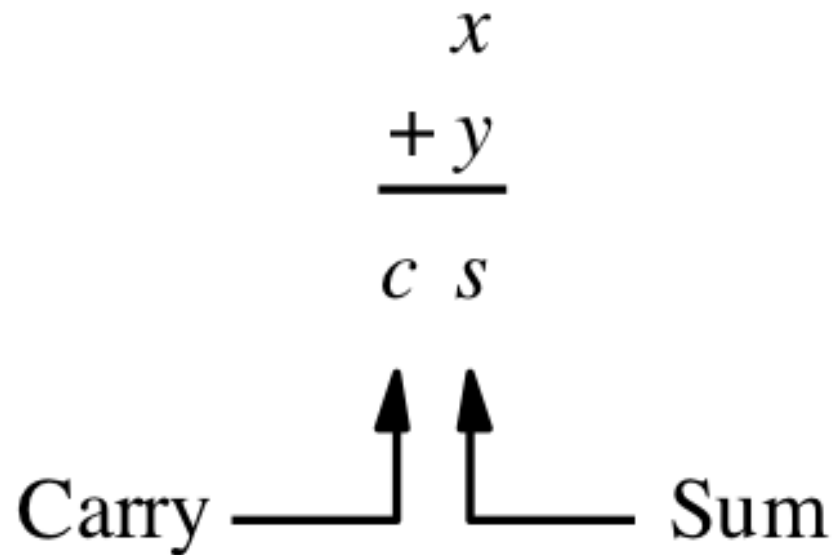
Signed Representation (using the left-most bit as the sign)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

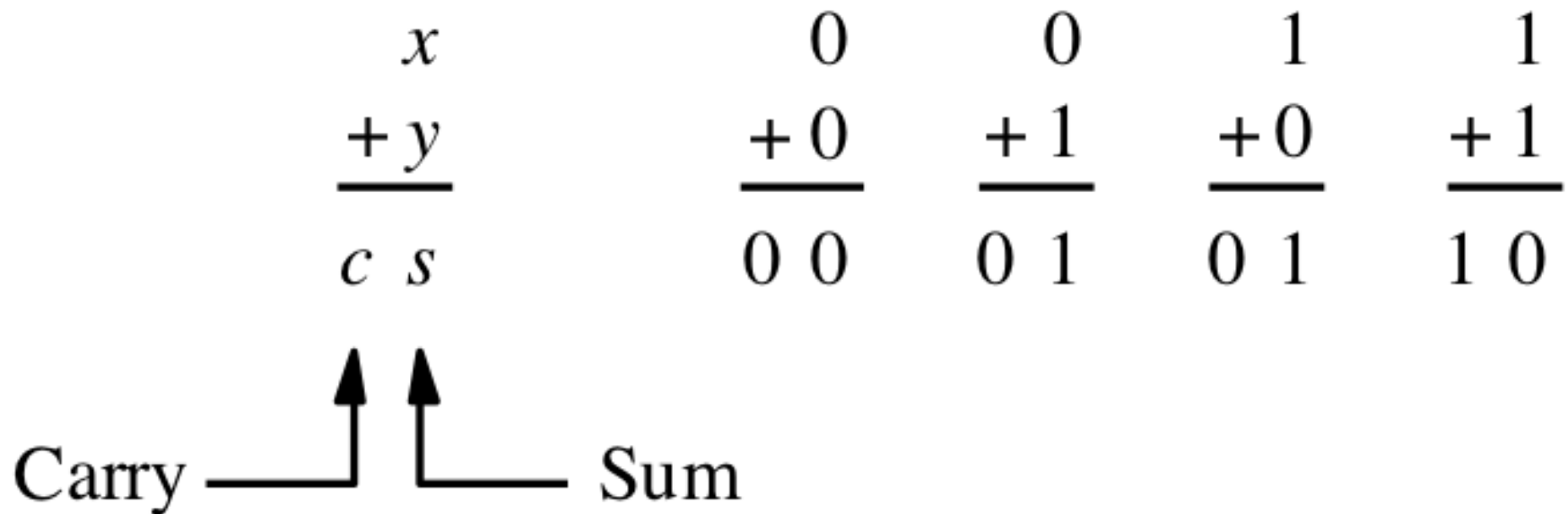
This represents -44 .

**Today's Lecture is About
Addition of **Unsigned** Numbers**

Addition of two 1-bit numbers



Addition of two 1-bit numbers (there are four possible cases)



Addition of two 1-bit numbers (the truth table)

x	y	Carry c	Sum s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	$+ 0$	$+ 1$	$+ 0$	$+ 1$
<u> </u>	<u> </u>	<u> </u>	<u> </u>	<u> </u>
$c\ s$	0 0	0 1	0 1	1 0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	<u>+ 0</u>	<u>+ 1</u>	<u>+ 0</u>	<u>+ 1</u>
$c \ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+y$	<u>+0</u>	<u>+1</u>	<u>+0</u>	<u>+1</u>
$c\ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+y$	$+0$	$+1$	$+0$	$+1$
$c\ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	<u>+ 0</u>	<u>+ 1</u>	<u>+ 0</u>	<u>+ 1</u>
$c \ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
<u>$+y$</u>	<u>+0</u>	<u>+1</u>	<u>+0</u>	<u>+1</u>
$c\ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
<u>$+y$</u>	<u>+0</u>	<u>+1</u>	<u>+0</u>	<u>+1</u>
$c\ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	<u>+ 0</u>	<u>+ 1</u>	<u>+ 0</u>	<u>+ 1</u>
$c \ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+y$	$+0$	$+1$	$+0$	$+1$
$\underline{\quad}$	$\underline{\quad}$	$\underline{\quad}$	$\underline{\quad}$	$\underline{\quad}$
c s	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+y$	<u>+0</u>	<u>+1</u>	<u>+0</u>	<u>+1</u>
c s	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	<u>+ 0</u>	<u>+ 1</u>	<u>+ 0</u>	<u>+ 1</u>
$c \ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	$+ 0$	$+ 1$	$+ 0$	$+ 1$
<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
c s	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+y$	$+0$	$+1$	$+0$	$+1$
<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
c s	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	0	0	1	1
$+ y$	<u>+ 0</u>	<u>+ 1</u>	<u>+ 0</u>	<u>+ 1</u>
$c \ s$	0 0	0 1	0 1	1 0

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

?

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

AND

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

?

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

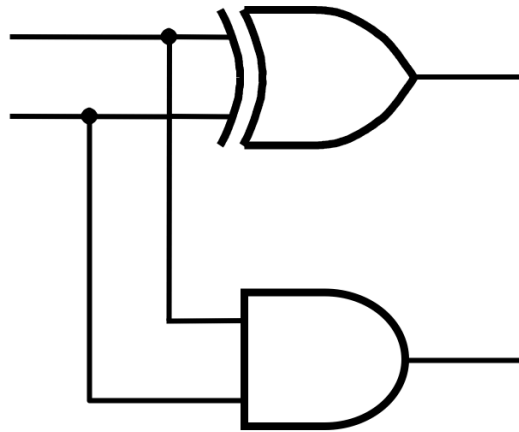
XOR

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

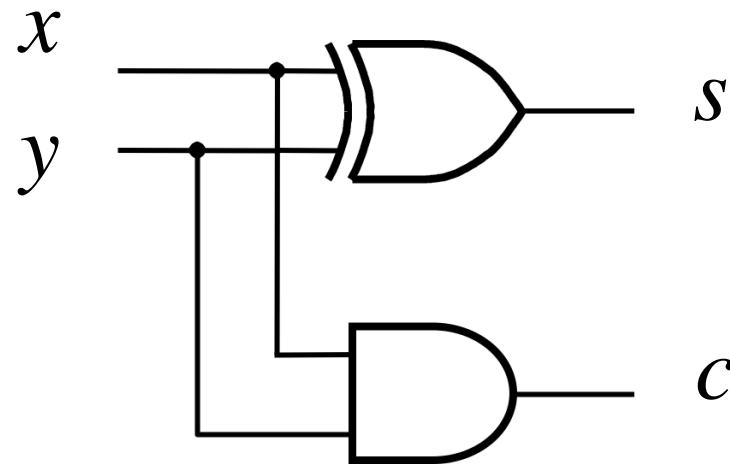
x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers



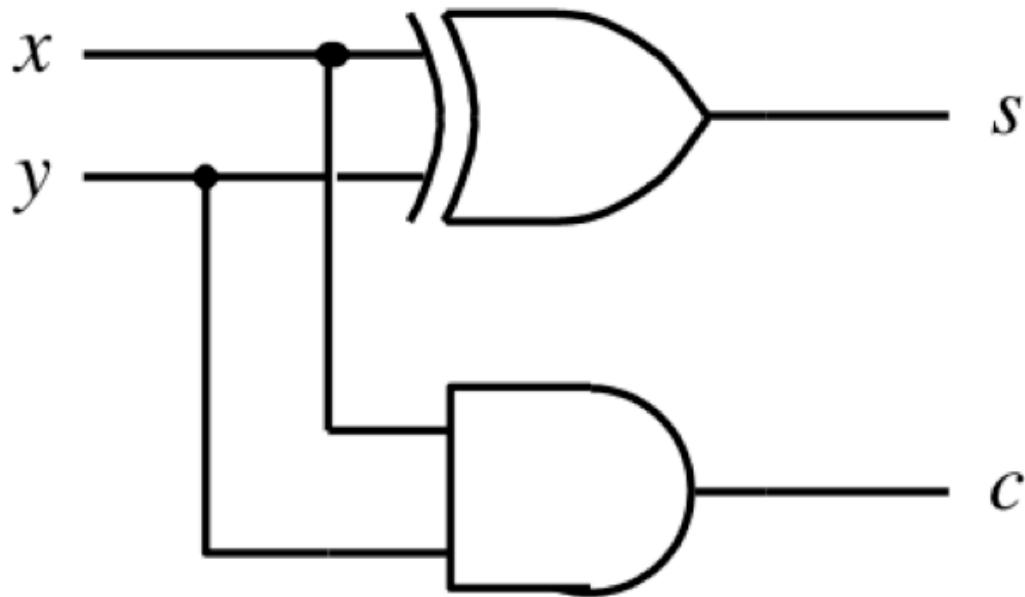
x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers

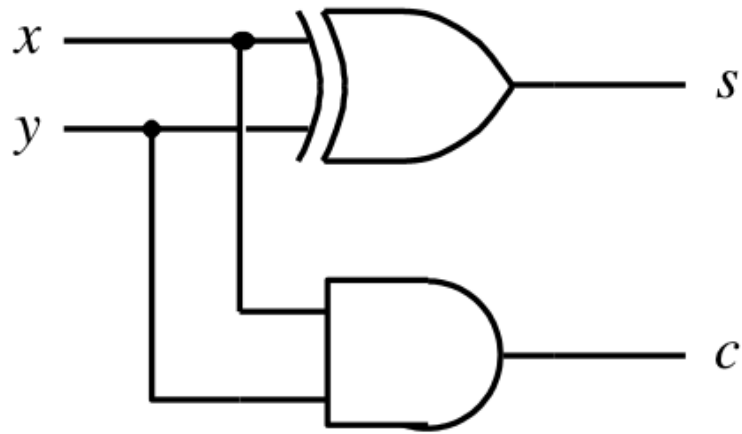


x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Addition of two 1-bit numbers (the logic circuit)



The Half-Adder



(c) Circuit



(d) Graphical symbol

Addition of Multibit Unsigned Numbers

Analogy with addition in base 10

$$\begin{array}{r} + \quad \quad \quad X_2 \quad X_1 \quad X_0 \\ \quad \quad \quad Y_2 \quad Y_1 \quad Y_0 \\ \hline \quad \quad \quad S_2 \quad S_1 \quad S_0 \end{array}$$

Analogy with addition in base 10

$$\begin{array}{r} + \quad 3 \quad 8 \quad 9 \\ \quad 1 \quad 5 \quad 7 \\ \hline \quad 5 \quad 4 \quad 6 \end{array}$$

Analogy with addition in base 10

carry	0	1	1	0
		3	8	9
+		1	5	7
		<hr/>		
		5	4	6

Analogy with addition in base 10

$$\begin{array}{rcccc} & C_3 & C_2 & C_1 & C_0 \\ + & & X_2 & X_1 & X_0 \\ & & Y_2 & Y_1 & Y_0 \\ \hline & & S_2 & S_1 & S_0 \end{array}$$

Another example in base 10

$$\begin{array}{r} + \quad \quad 9 \quad 3 \quad 8 \\ \quad \quad 2 \quad 1 \quad 4 \\ \hline \quad \quad 1 \quad 1 \quad 5 \quad 2 \end{array}$$

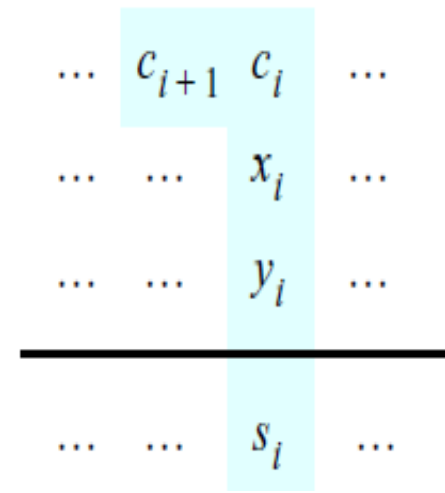
Another example in base 10

carry	1	0	1	0
		9	3	8
+		2	1	4
		<hr/>		
		1	5	2

Addition of multibit numbers

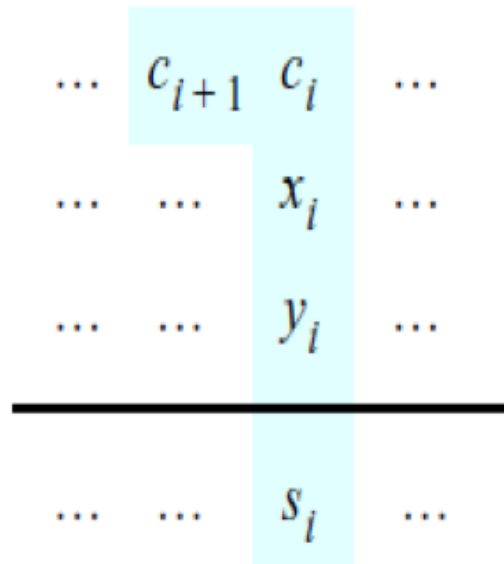
Generated carries \longrightarrow 1 1 1 0

$$\begin{array}{r}
 X = x_4x_3x_2x_1x_0 \quad 01111 \quad (15)_{10} \\
 + Y = y_4y_3y_2y_1y_0 \quad +01010 \quad + (10)_{10} \\
 \hline
 S = s_4s_3s_2s_1s_0 \quad 11001 \quad (25)_{10}
 \end{array}$$



Bit position i

Problem Statement and Truth Table



c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

[Figure 3.2b from the textbook]

[Figure 3.3a from the textbook]

Let's fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

		$x_i y_i$			
		00	01	11	10
c_i	0				
	1				

$s_i =$

		$x_i y_i$			
		00	01	11	10
c_i	0				
	1				

$c_{i+1} =$

Let's fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

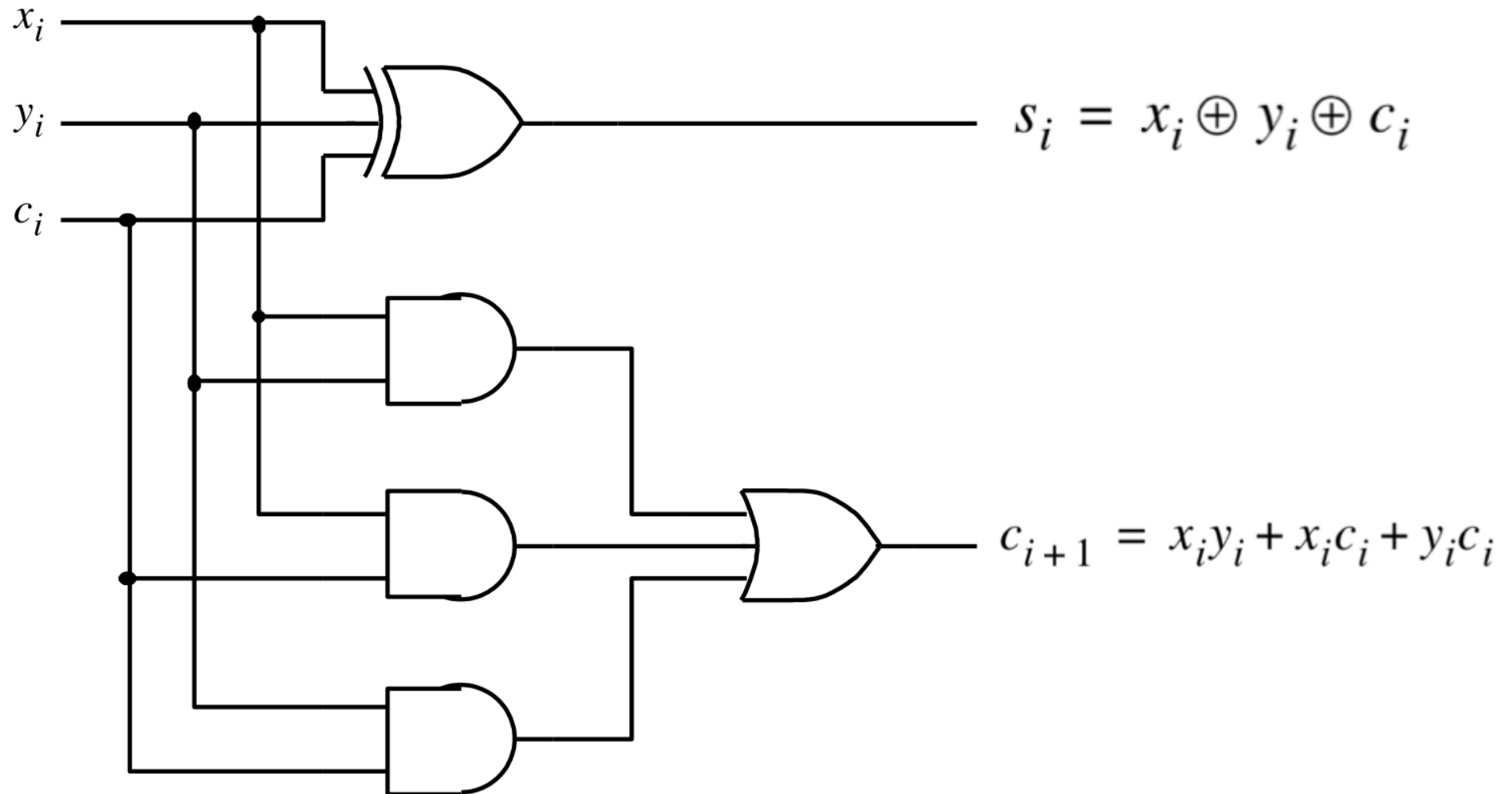
	$x_i y_i$				
c_i		00	01	11	10
0			1		1
1		1		1	

$$s_i = x_i \oplus y_i \oplus c_i$$

	$x_i y_i$				
c_i		00	01	11	10
0				1	
1			1	1	1

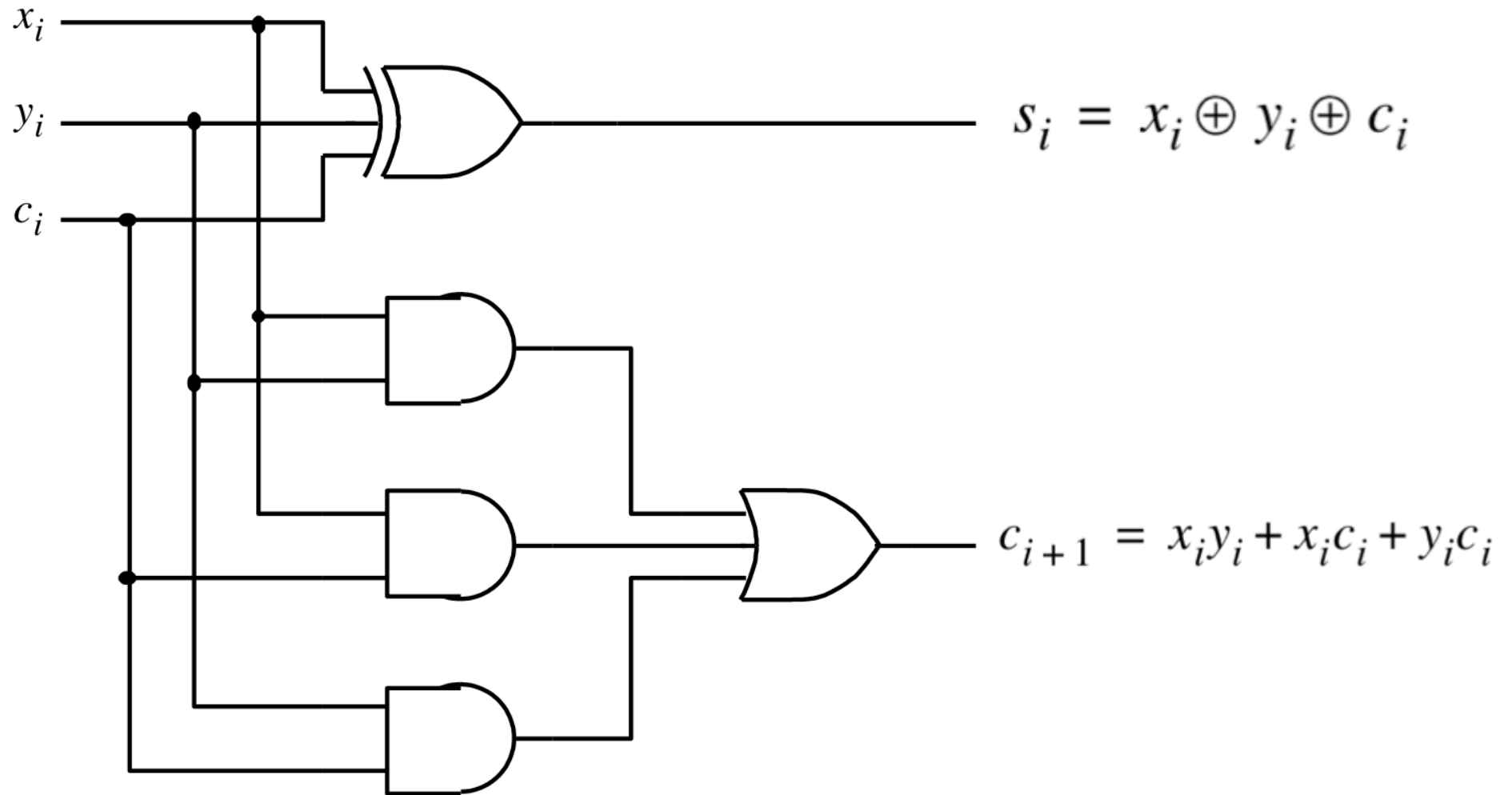
$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

The circuit for the two expressions



[Figure 3.3c from the textbook]

This is called the Full-Adder



[Figure 3.3c from the textbook]

XOR Magic

$$s_i = \bar{x}_i y_i \bar{c}_i + x_i \bar{y}_i \bar{c}_i + \bar{x}_i \bar{y}_i c_i + x_i y_i c_i$$

XOR Magic

$$s_i = \bar{x}_i y_i \bar{c}_i + x_i \bar{y}_i \bar{c}_i + \bar{x}_i \bar{y}_i c_i + x_i y_i c_i$$

$$s_i = (\bar{x}_i y_i + x_i \bar{y}_i) \bar{c}_i + (\bar{x}_i \bar{y}_i + x_i y_i) c_i$$

$$= (x_i \oplus y_i) \bar{c}_i + \overline{(x_i \oplus y_i)} c_i$$

$$= (x_i \oplus y_i) \oplus c_i$$

XOR Magic

$$s_i = \bar{x}_i y_i \bar{c}_i + x_i \bar{y}_i \bar{c}_i + \bar{x}_i \bar{y}_i c_i + x_i y_i c_i$$

Can you prove this?

$$s_i = (\bar{x}_i y_i + x_i \bar{y}_i) \bar{c}_i + (\bar{x}_i \bar{y}_i + x_i y_i) c_i$$

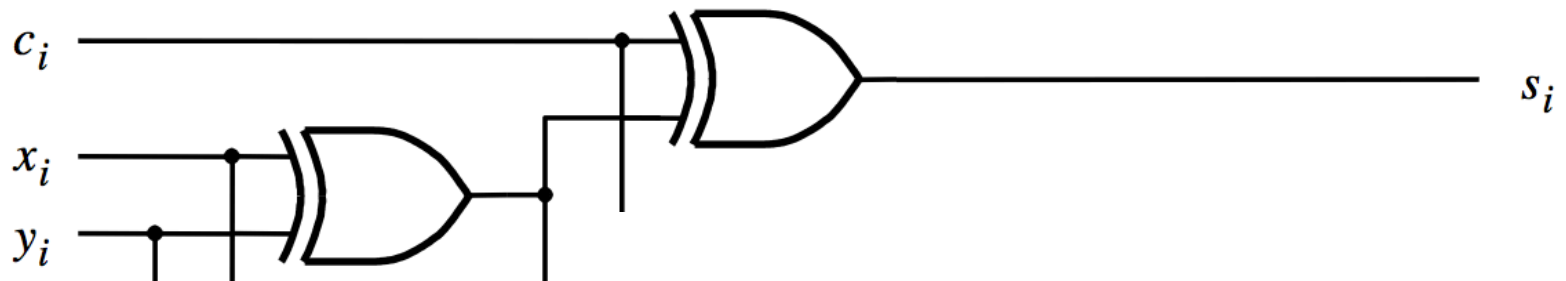
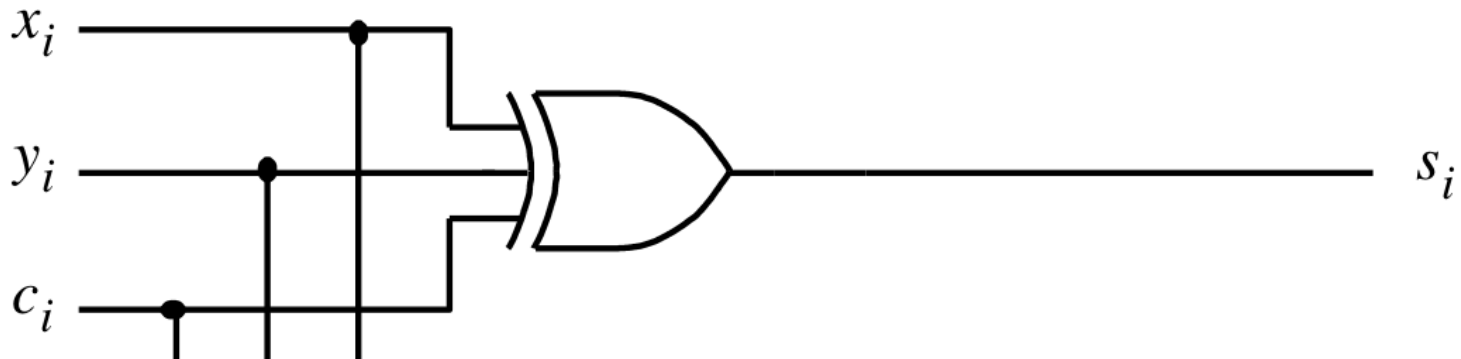
$$= (x_i \oplus y_i) \bar{c}_i + \overline{(x_i \oplus y_i)} c_i$$

$$= (x_i \oplus y_i) \oplus c_i$$

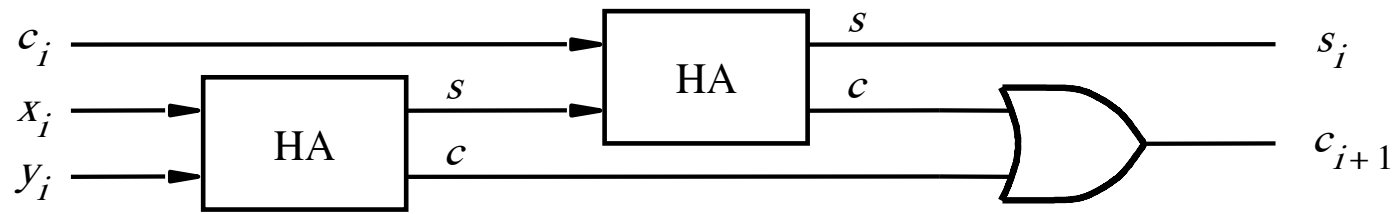
XOR Magic

(s_i can be implemented in two different ways)

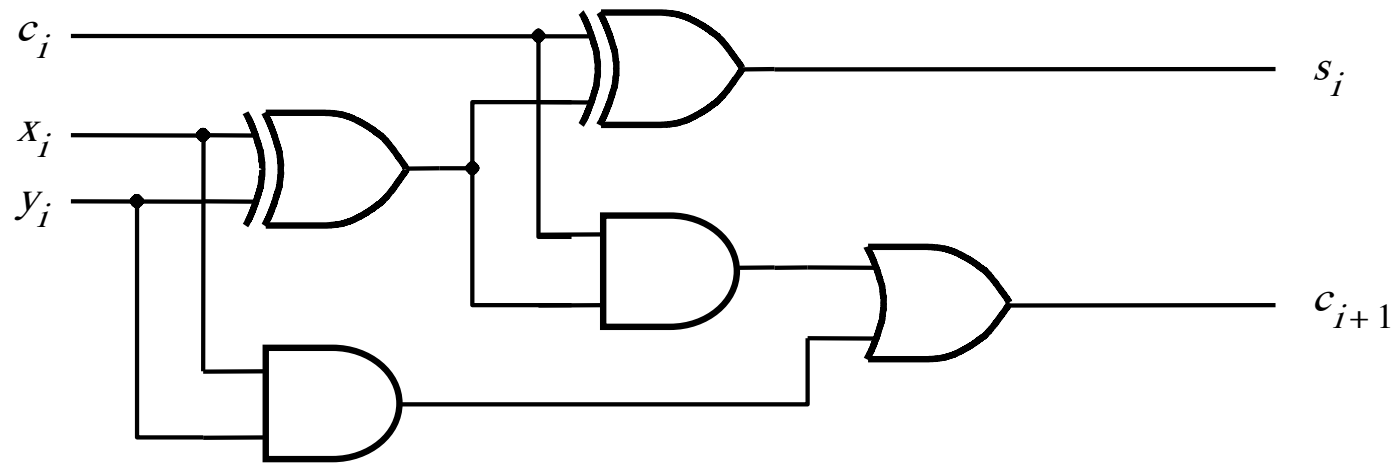
$$s_i = x_i \oplus y_i \oplus c_i$$



A decomposed implementation of the full-adder circuit

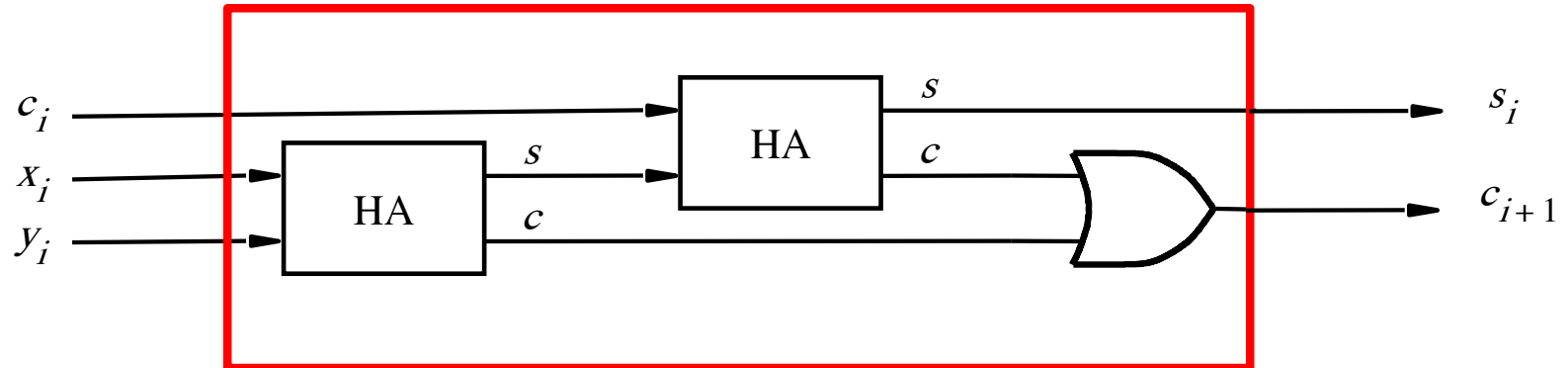


(a) Block diagram

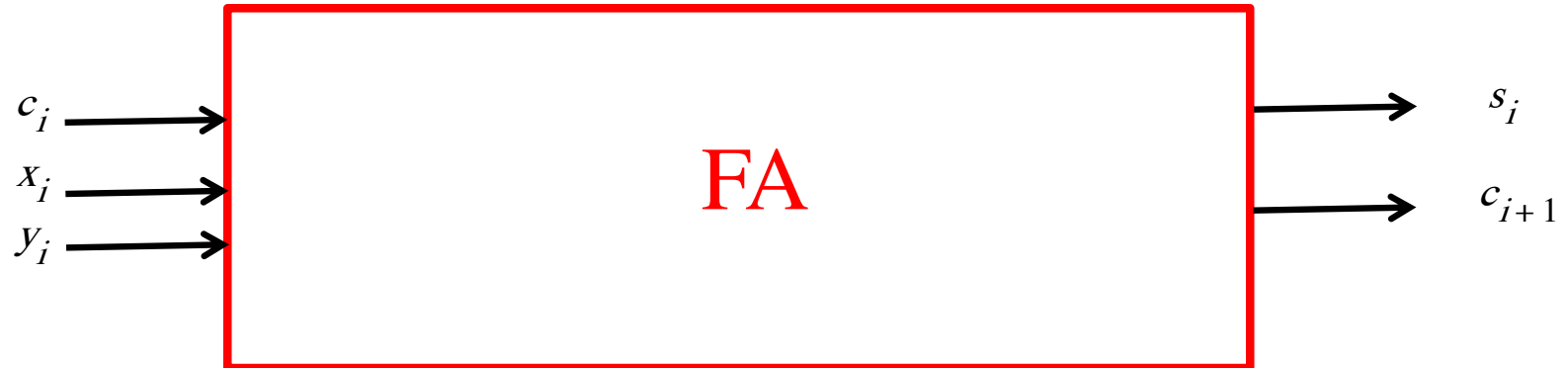


(b) Detailed diagram

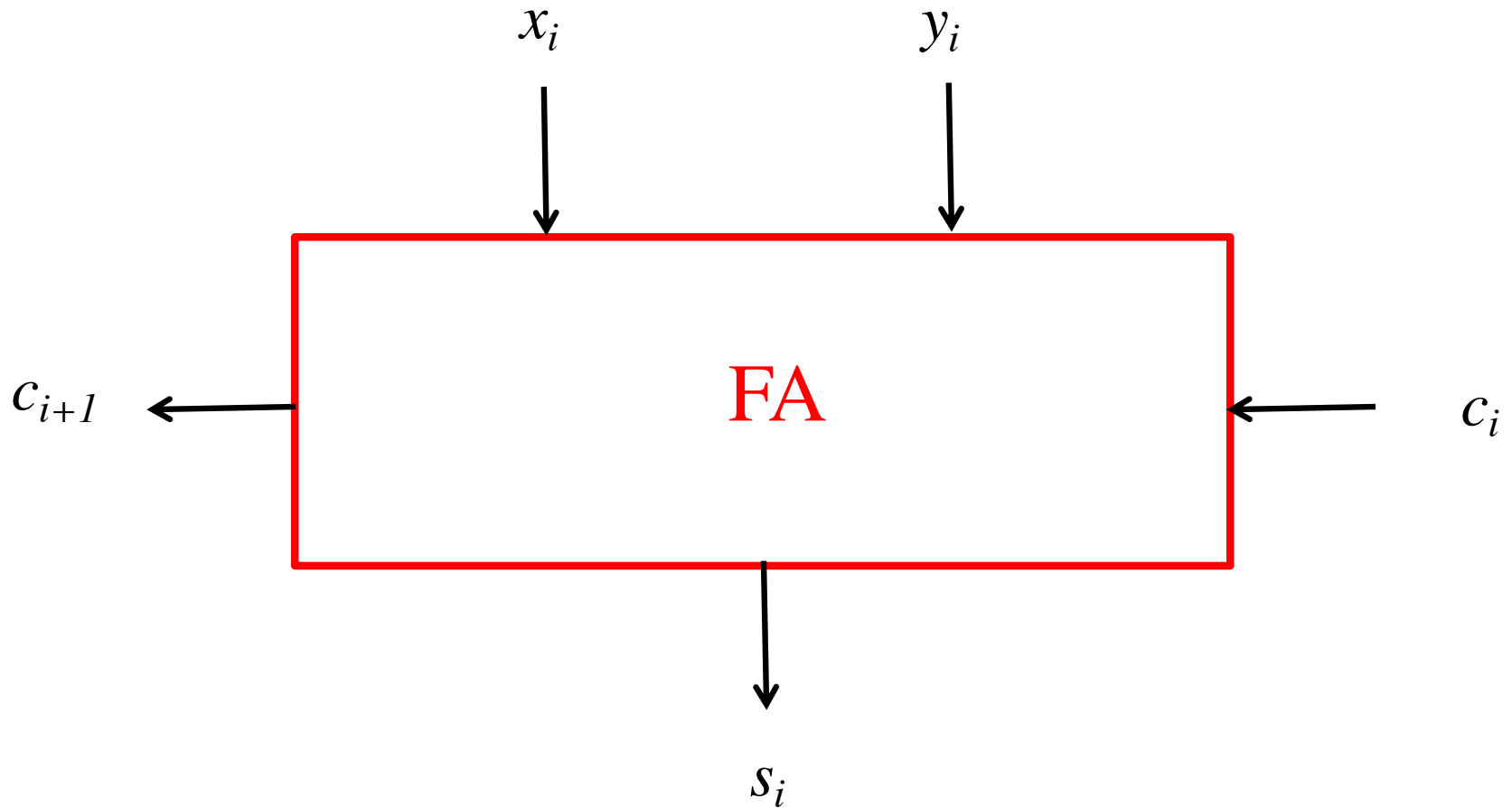
The Full-Adder Abstraction



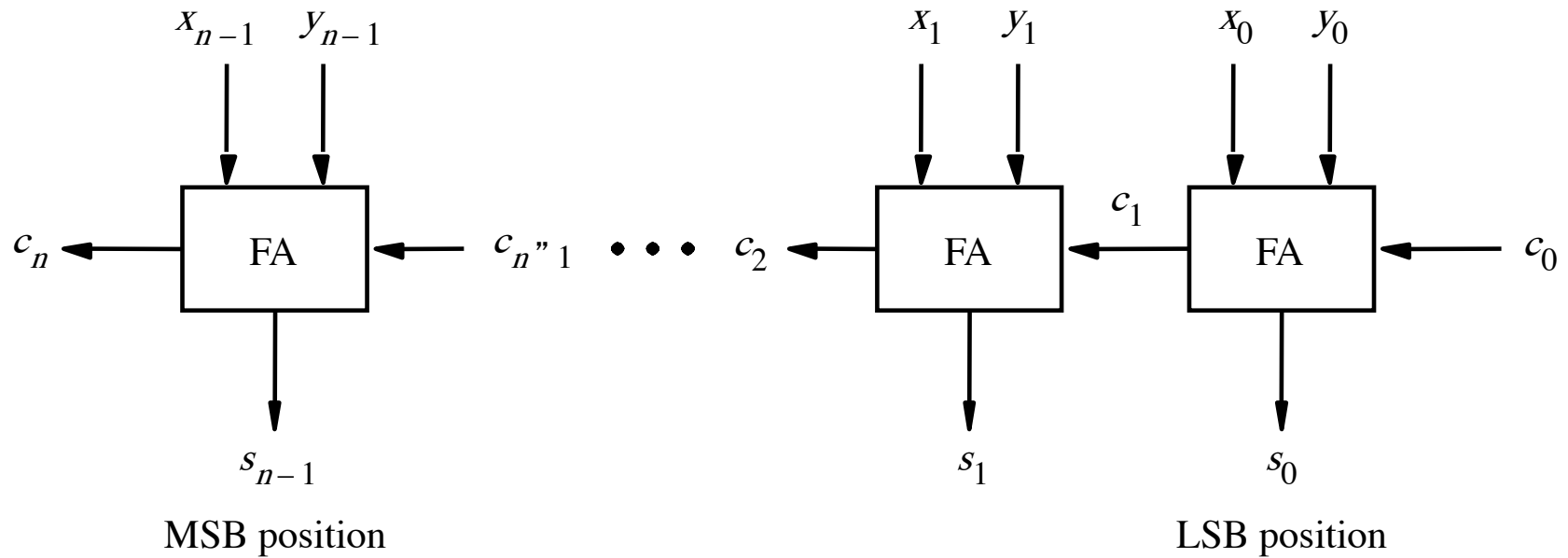
The Full-Adder Abstraction



We can place the arrows anywhere

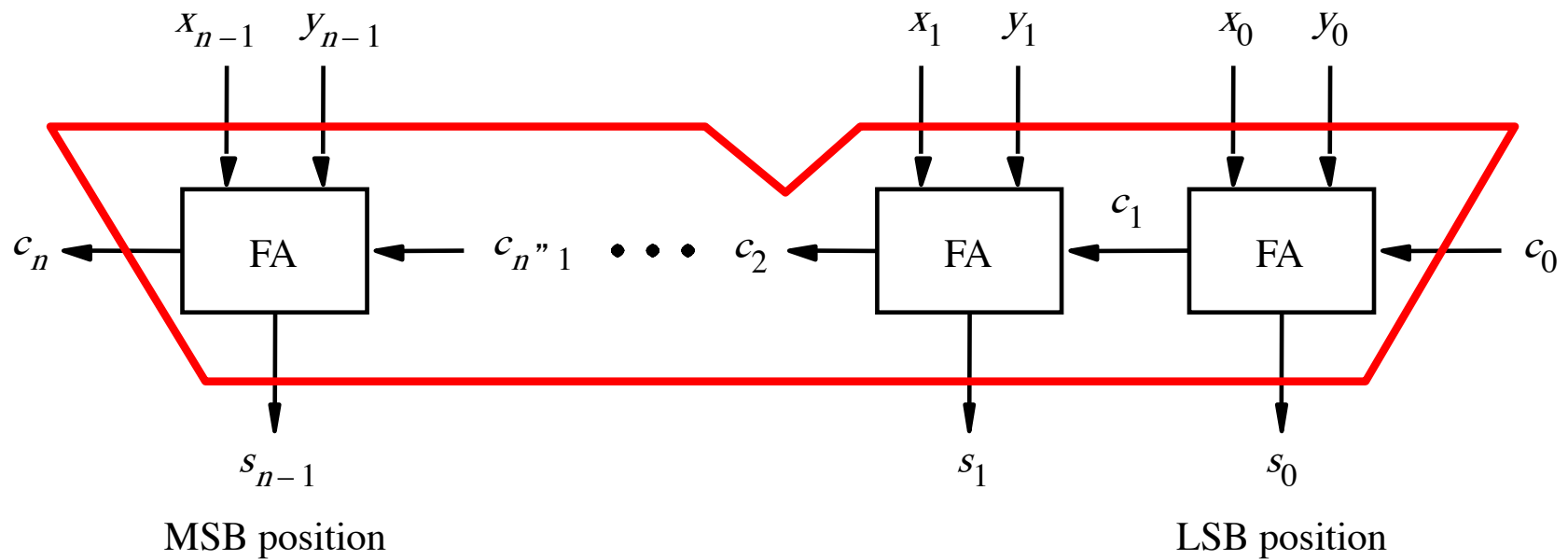


n-bit ripple-carry adder

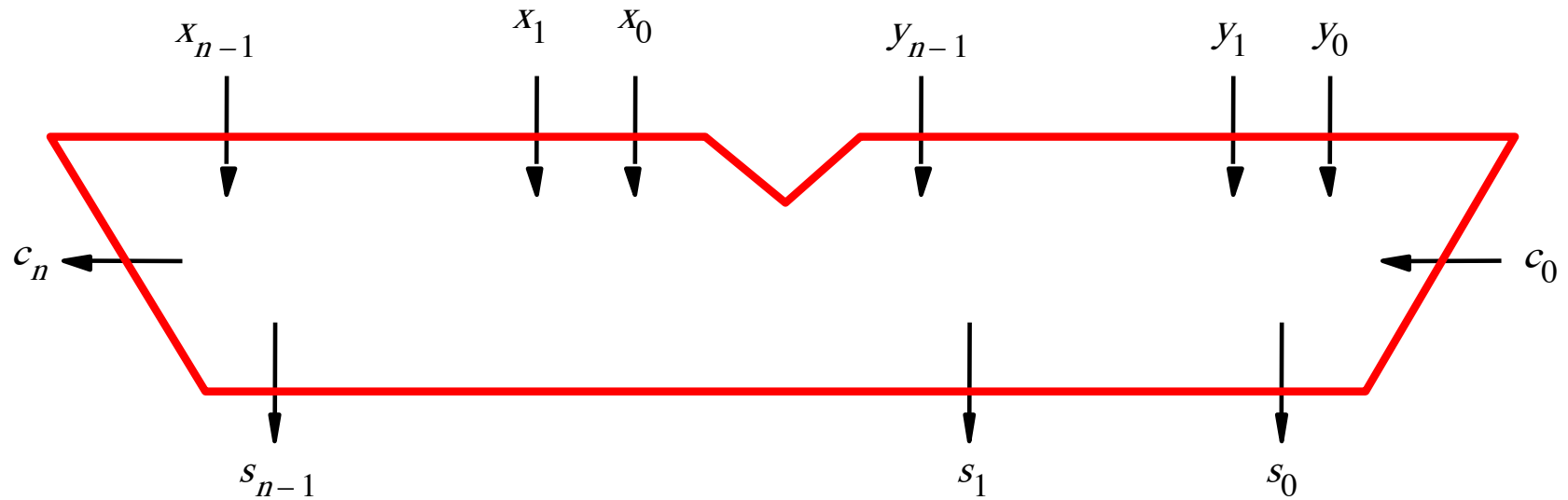


[Figure 3.5 from the textbook]

n-bit ripple-carry adder abstraction



The x and y lines are typically grouped together for better visualization, but the underlying logic remains the same

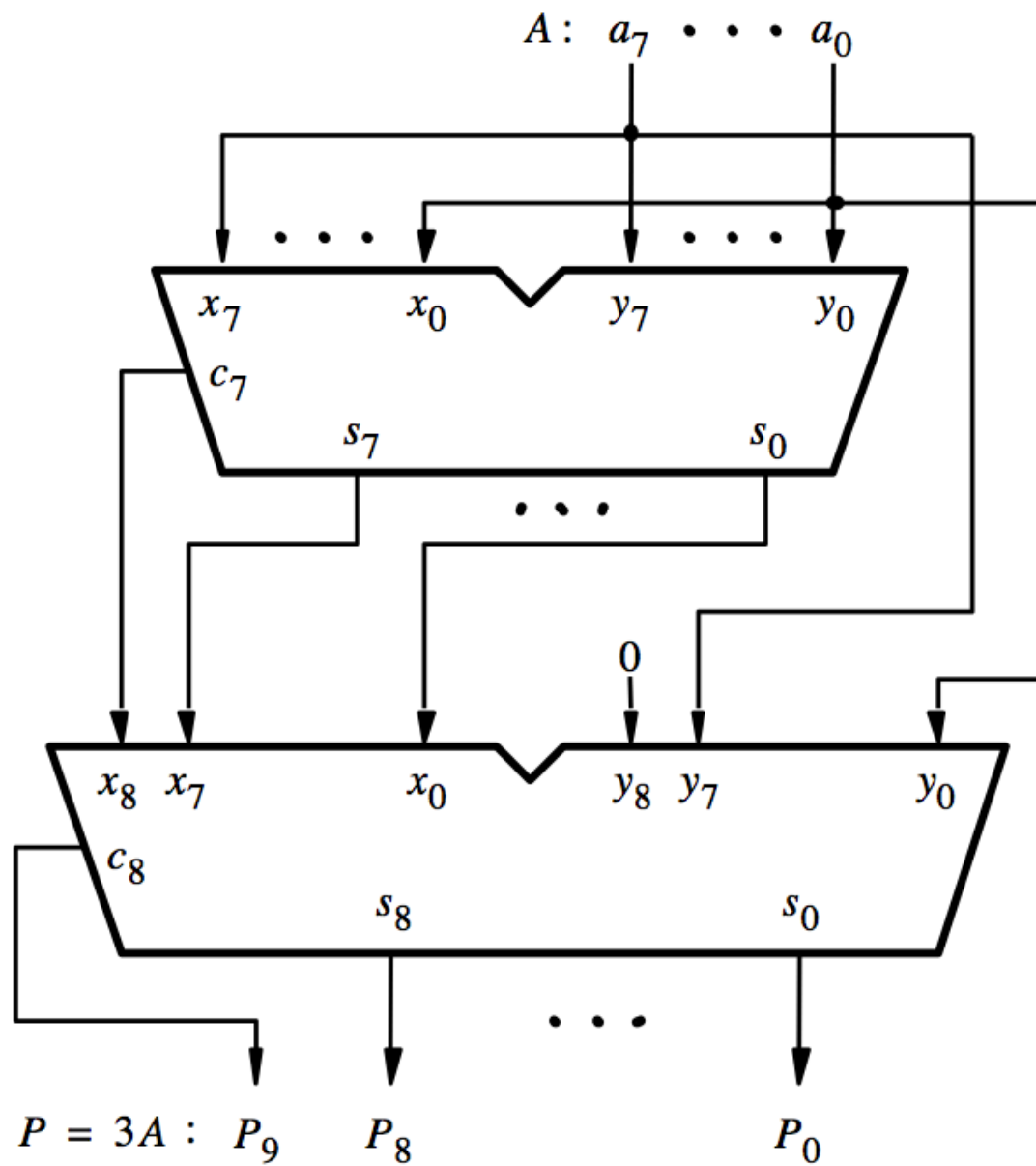


Design Example:

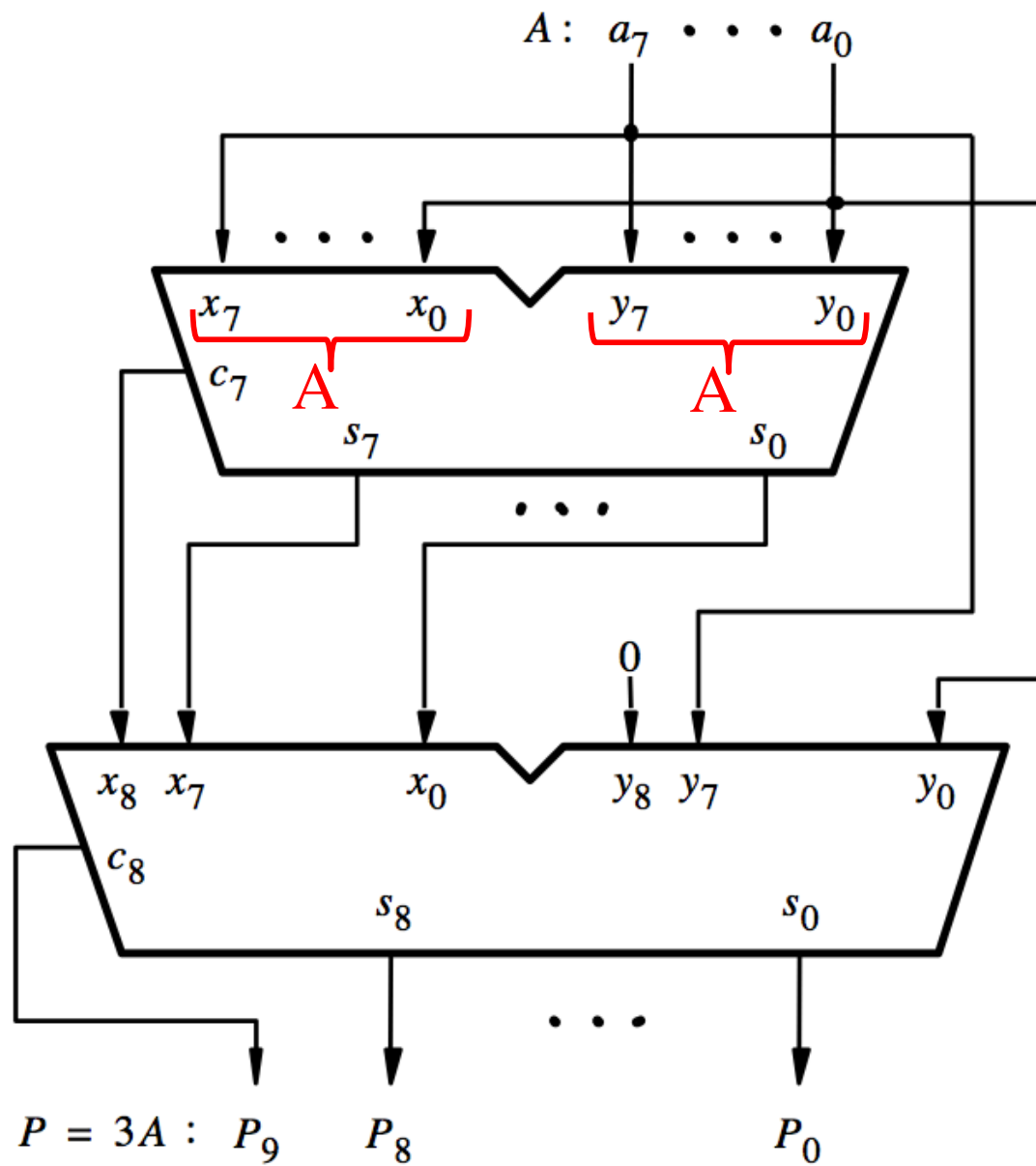
Create a circuit that multiplies a number by 3

How to Get 3A from A?

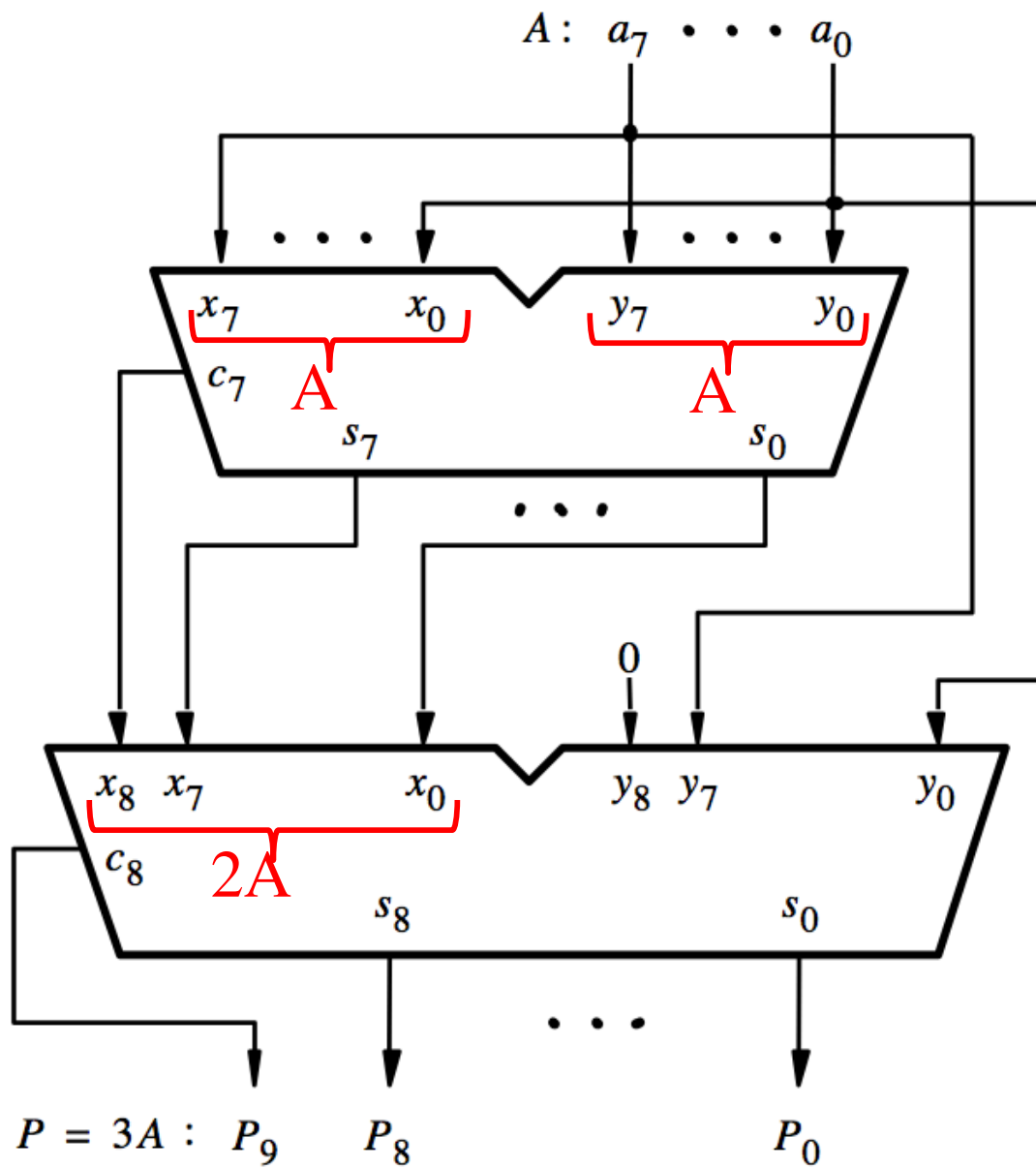
- $3A = A + A + A$
- $3A = (A+A) + A$
- $3A = 2A + A$



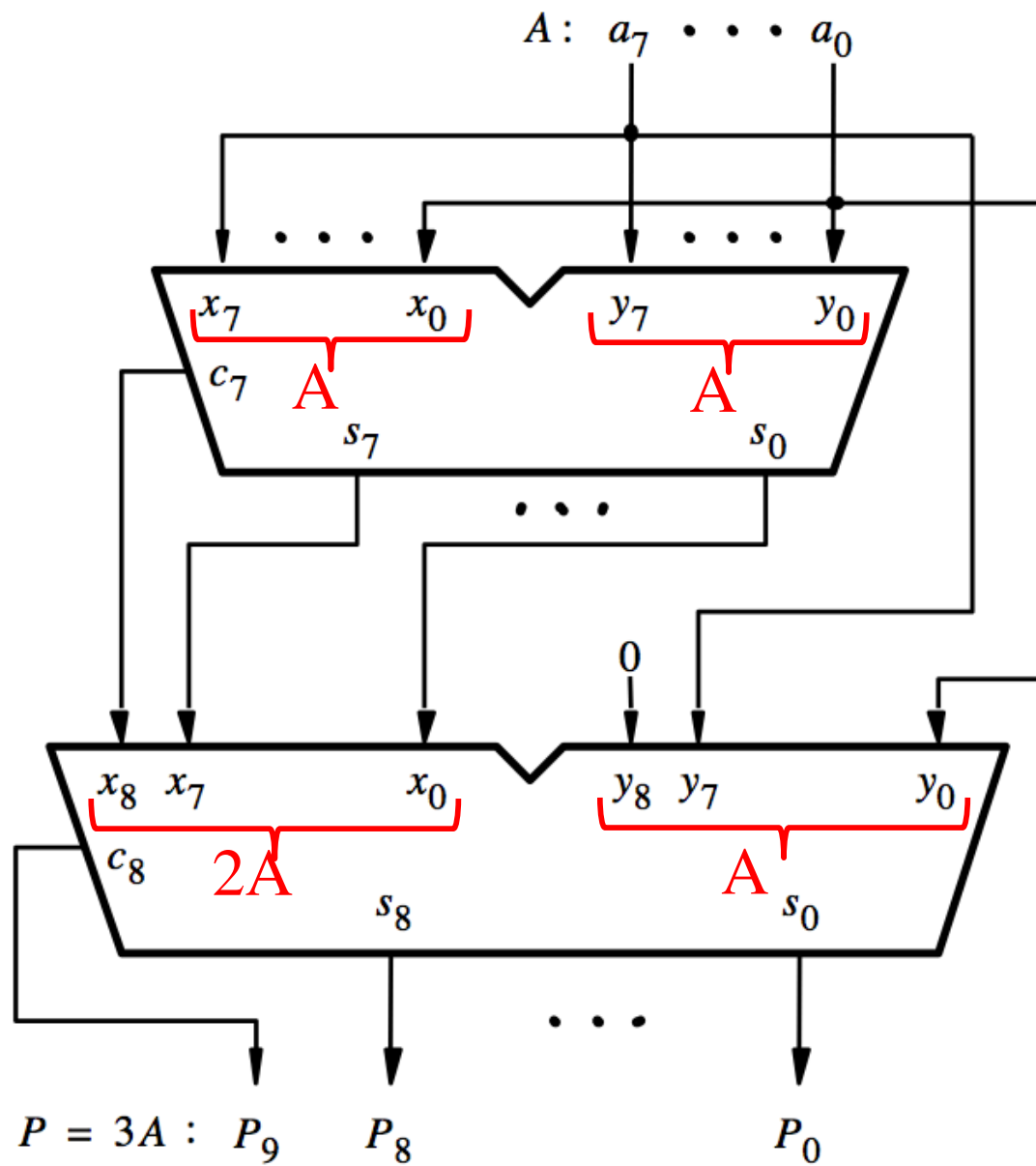
[Figure 3.6a from the textbook]



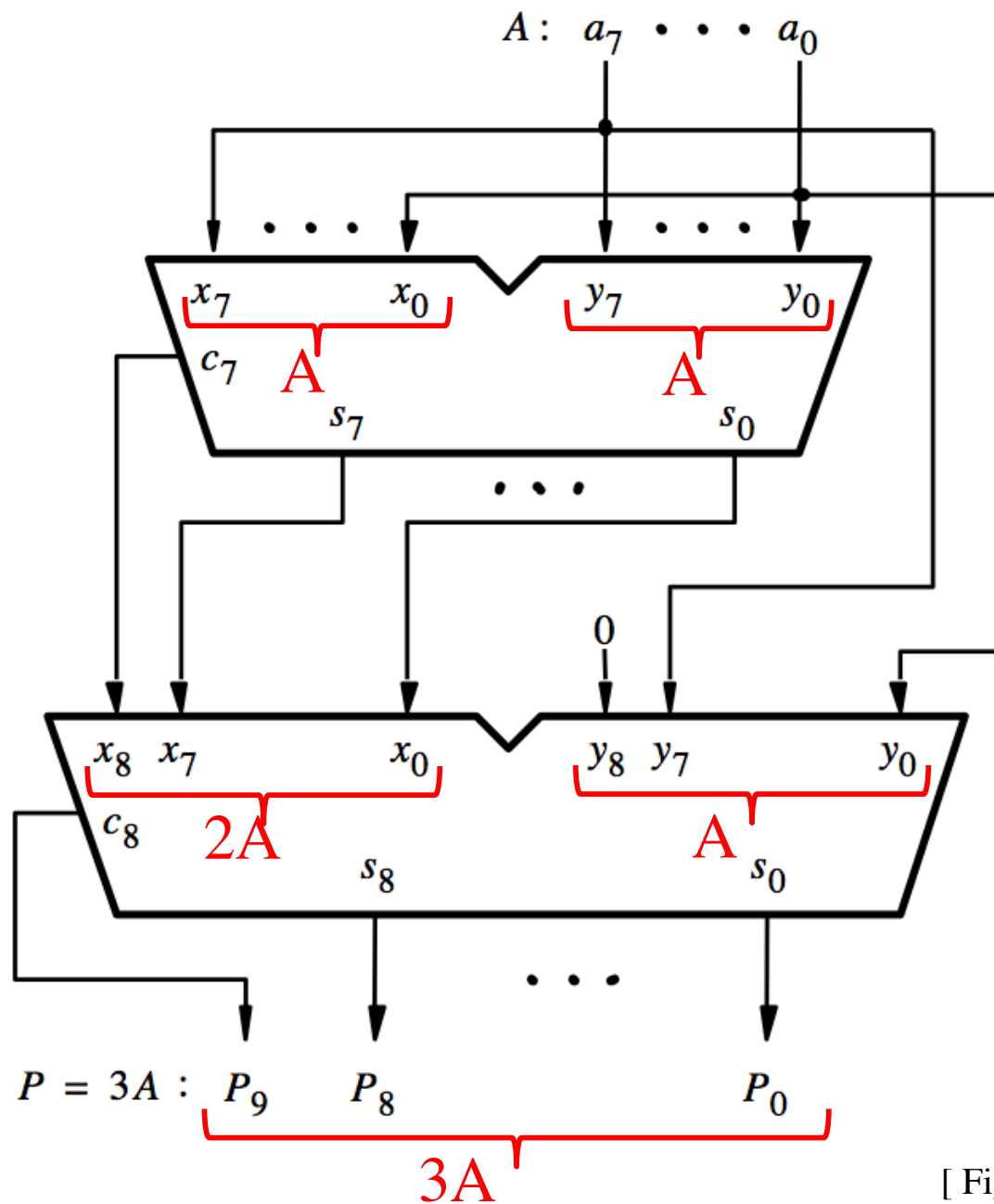
[Figure 3.6a from the textbook]



[Figure 3.6a from the textbook]



[Figure 3.6a from the textbook]



[Figure 3.6a from the textbook]

Decimal Multiplication by 10

What happens when we multiply a number by 10?

$$4 \times 10 = ?$$

$$542 \times 10 = ?$$

$$1245 \times 10 = ?$$

Decimal Multiplication by 10

What happens when we multiply a number by 10?

$$4 \times 10 = 40$$

$$542 \times 10 = 5420$$

$$1245 \times 10 = 12450$$

Decimal Multiplication by 10

What happens when we multiply a number by 10?

$$4 \times 10 = 40$$

$$542 \times 10 = 5420$$

$$1245 \times 10 = 12450$$

You simply add a zero as the rightmost number

Binary Multiplication by 2

What happens when we multiply a number by 2?

011 times 2 = ?

101 times 2 = ?

110011 times 2 = ?

Binary Multiplication by 2

What happens when we multiply a number by 2?

$$011 \text{ times } 2 = 0110$$

$$101 \text{ times } 2 = 1010$$

$$110011 \text{ times } 2 = 1100110$$

Binary Multiplication by 2

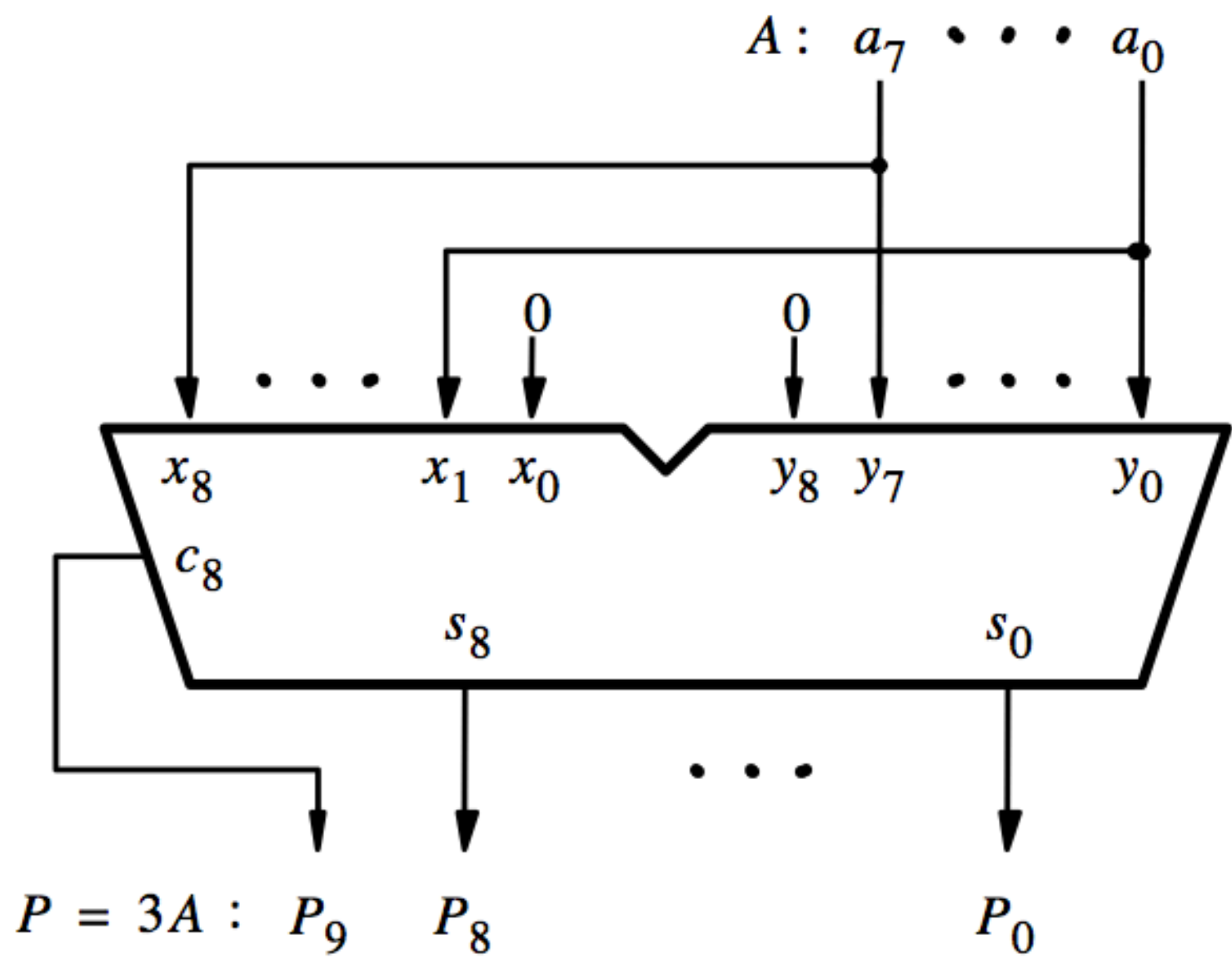
What happens when we multiply a number by 2?

$$011 \text{ times } 2 = 0110$$

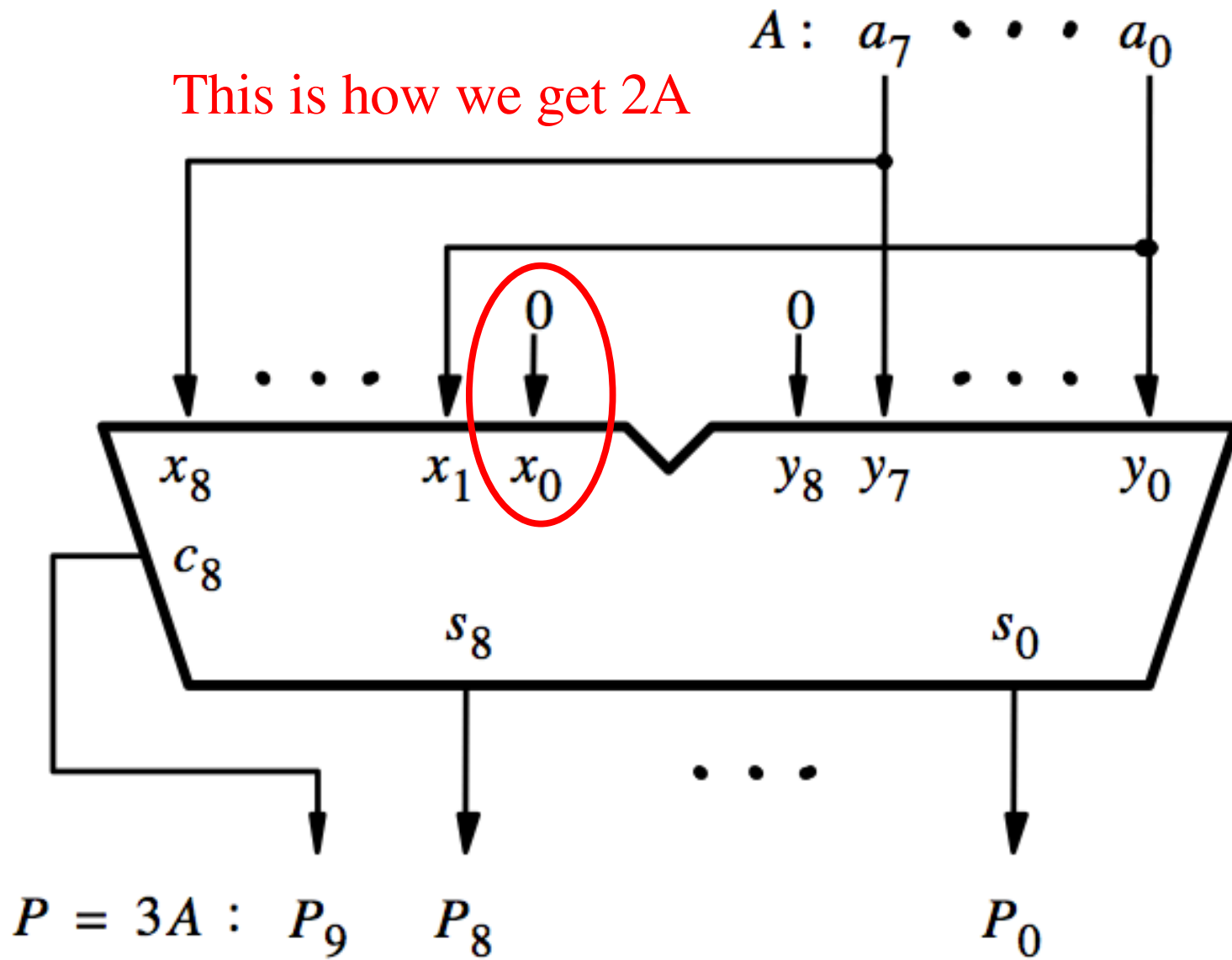
$$101 \text{ times } 2 = 1010$$

$$110011 \text{ times } 2 = 1100110$$

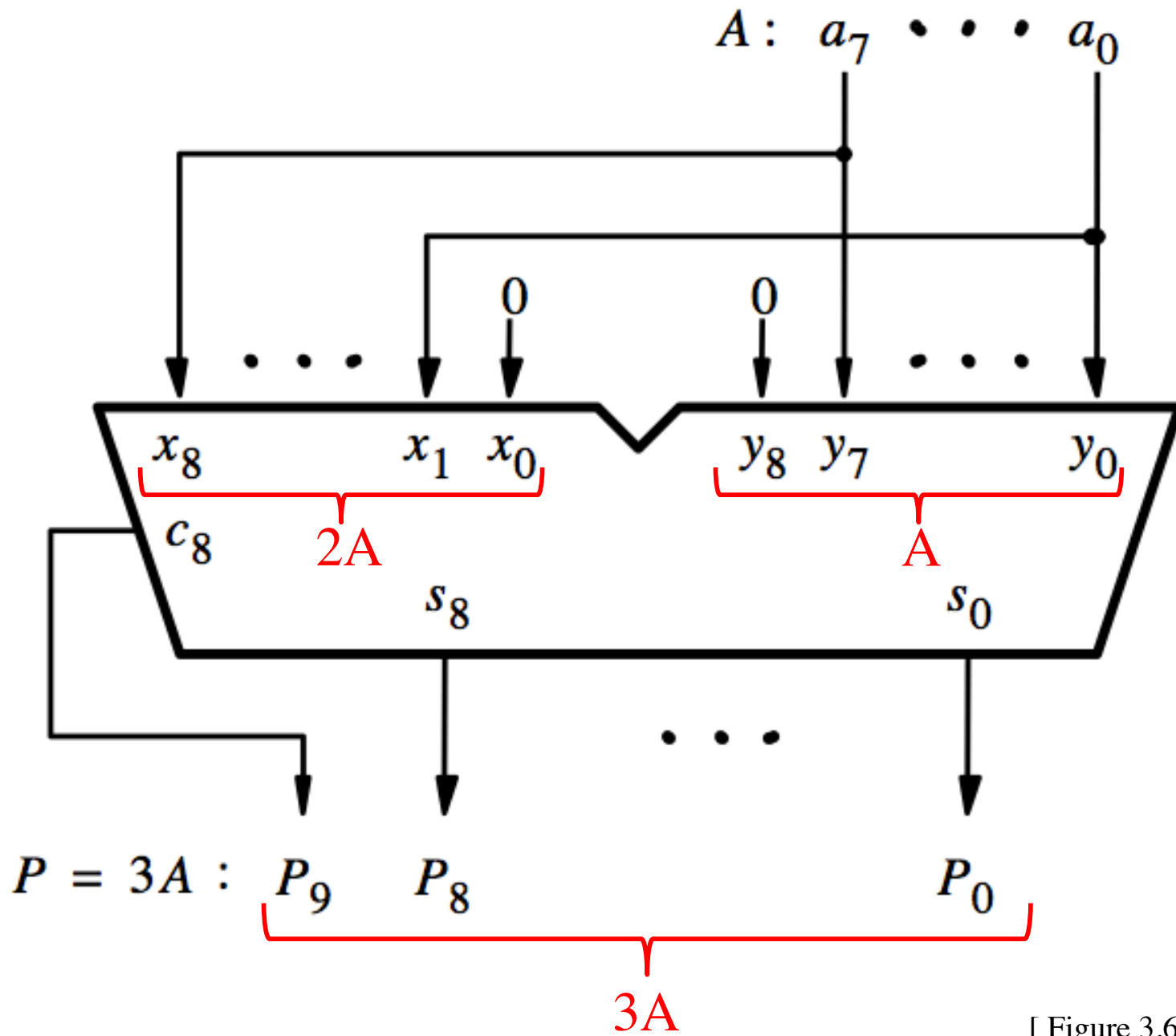
You simply add a zero as the rightmost number



[Figure 3.6b from the textbook]



[Figure 3.6b from the textbook]



[Figure 3.6b from the textbook]

Questions?

THE END