



CprE 281: Digital Logic

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<http://www.ece.iastate.edu/~alexs/classes/>

Boolean Algebra

CprE 281: Digital Logic
Iowa State University, Ames, IA
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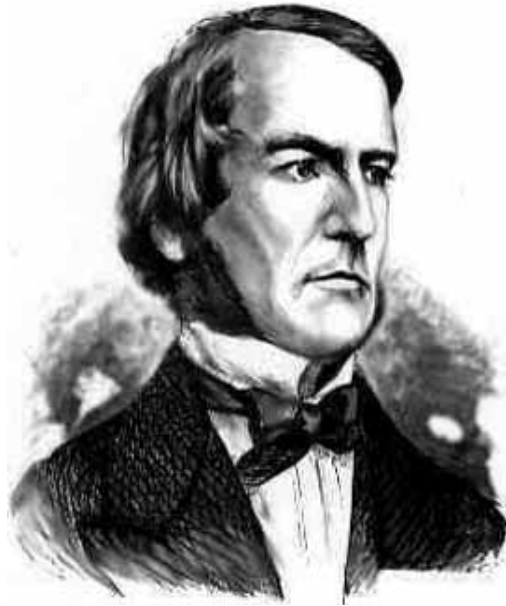
Administrative Stuff

- **HW1 is due today**

Administrative Stuff

- **HW2 is out**
- **It is due on Monday Sep 9 @ 4pm.**
- **Submit it on paper before the start of the lecture**

Boolean Algebra



George Boole
1815-1864

- **An algebraic structure consists of**
 - a set of elements $\{0, 1\}$
 - binary operators $\{+, \cdot\}$
 - and a unary operator $\{ ' \}$ or $\{ \bar{\ } \}$
- **Introduced by George Boole in 1854**
- **An effective means of describing circuits built with switches**
- **A powerful tool that can be used for designing and analyzing logic circuits**

Axioms of Boolean Algebra

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \cdot 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \cdot 1 = 1 \cdot 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

4a. If $x=0$, then $\bar{x} = 1$

4b. If $x=1$, then $\bar{x} = 0$

Single-Variable Theorems

$$5a. \quad \mathbf{x \cdot 0 = 0}$$

$$5b. \quad \mathbf{x + 1 = 1}$$

$$6a. \quad \mathbf{x \cdot 1 = x}$$

$$6b. \quad \mathbf{x + 0 = x}$$

$$7a. \quad \mathbf{x \cdot x = x}$$

$$7b. \quad \mathbf{x + x = x}$$

$$8a. \quad \mathbf{x \cdot \overline{x} = 0}$$

$$8b. \quad \mathbf{x + \overline{x} = 1}$$

$$9. \quad \mathbf{\overline{\overline{x}} = x}$$

Two- and Three-Variable Properties

10a. $x \cdot y = y \cdot x$

Commutative

10b. $x + y = y + x$

11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Associative

11b. $x + (y + z) = (x + y) + z$

12a. $x \cdot (y + z) = x \cdot y + x \cdot z$

Distributive

12b. $x + y \cdot z = (x + y) \cdot (x + z)$

13a. $x + x \cdot y = x$

Absorption

13b. $x \cdot (x + y) = x$

Two- and Three-Variable Properties

$$14a. \quad x \cdot y + x \cdot \bar{y} = x$$

Combining

$$14b. \quad (x + y) \cdot (x + \bar{y}) = x$$

$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

DeMorgan's
theorem

$$15b. \quad \overline{x + y} = \bar{x} \cdot \bar{y}$$

$$16a. \quad x + \bar{x} \cdot y = x + y$$

$$16b. \quad x \cdot (\bar{x} + y) = x \cdot y$$

$$17a. \quad x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$$

Consensus

$$17b. \quad (x+y) \cdot (y+z) \cdot (\bar{x}+z) = (x+y) \cdot (\bar{x}+z)$$

Now, let's prove all of these

The First Four are Axioms (i.e., they don't require a proof)

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \cdot 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \cdot 1 = 1 \cdot 0 = 0$

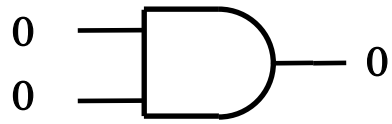
3b. $1 + 0 = 0 + 1 = 1$

4a. If $x=0$, then $\bar{x} = 1$

4b. If $x=1$, then $\bar{x} = 0$

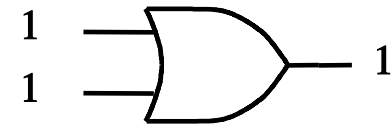
**But here are some other ways
to think about them**

1a. $0 \cdot 0 = 0$



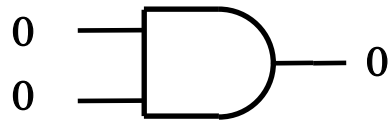
AND gate

1b. $1 + 1 = 1$



OR gate

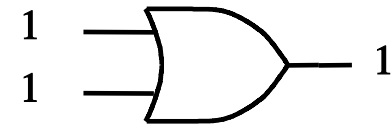
1a. 0 • 0 = 0



AND gate

x ₁	x ₂	f
0	0	0
0	1	0
1	0	0
1	1	1

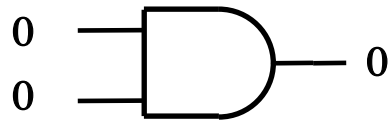
1b. 1 + 1 = 1



OR gate

x ₁	x ₂	f
0	0	0
0	1	1
1	0	1
1	1	1

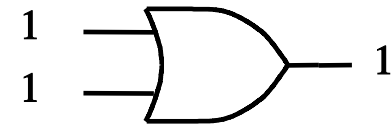
1a. 0 • 0 = 0



AND gate

x ₁	x ₂	f
0	0	0
0	1	0
1	0	0
1	1	1

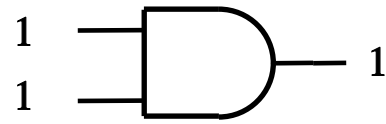
1b. 1 + 1 = 1



OR gate

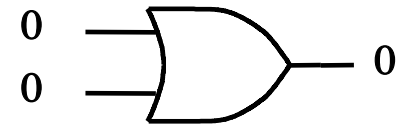
x ₁	x ₂	f
0	0	0
0	1	1
1	0	1
1	1	1

2a. $1 \cdot 1 = 1$



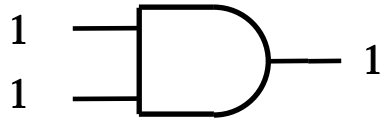
AND gate

2b. $0 + 0 = 0$



OR gate

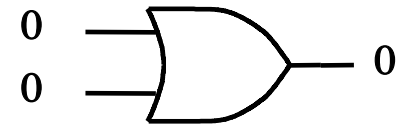
2a. $1 \cdot 1 = 1$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

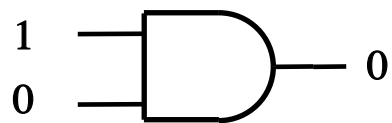
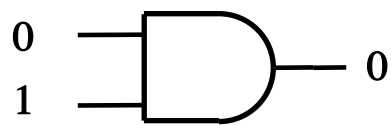
2b. $0 + 0 = 0$



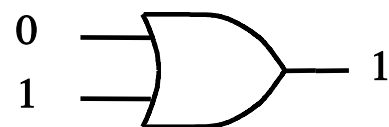
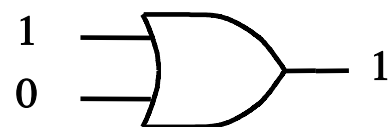
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

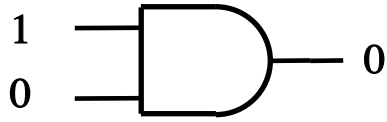
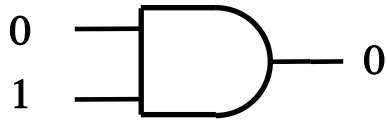
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



3b. $1 + 0 = 0 + 1 = 1$



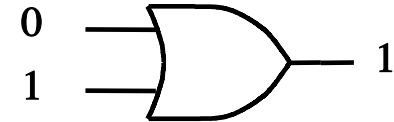
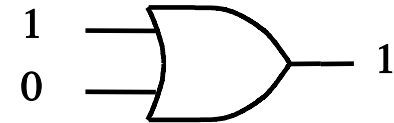
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

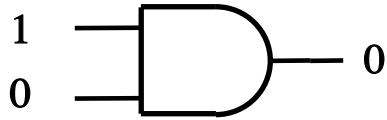
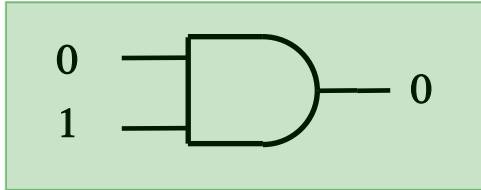
3b. $1 + 0 = 0 + 1 = 1$



OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

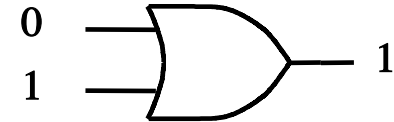
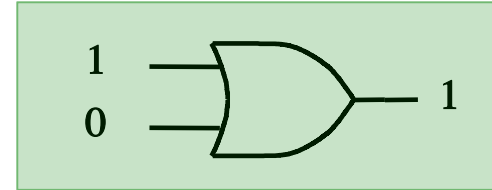
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

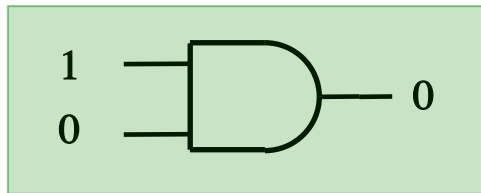
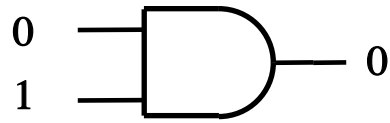
3b. $1 + 0 = 0 + 1 = 1$



OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

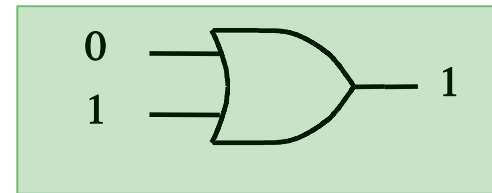
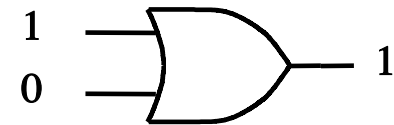
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

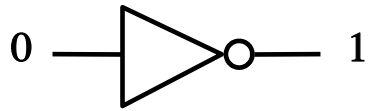
3b. $1 + 0 = 0 + 1 = 1$



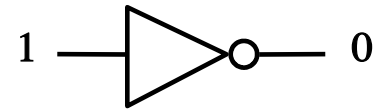
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

4a. If $x=0$, then $\bar{x} = 1$

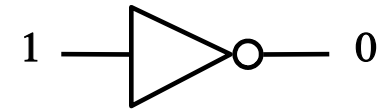
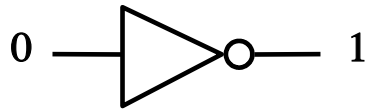


4b. If $x=1$, then $\bar{x} = 0$



4a. If $x=0$, then $\bar{x} = 1$

4b. If $x=1$, then $\bar{x} = 0$



NOT gate

x	\bar{x}
0	1
1	0

NOT gate

x	\bar{x}
0	1
1	0

Single-Variable Theorems

$$5a. \quad \mathbf{x \cdot 0 = 0}$$

$$5b. \quad \mathbf{x + 1 = 1}$$

$$6a. \quad \mathbf{x \cdot 1 = x}$$

$$6b. \quad \mathbf{x + 0 = x}$$

$$7a. \quad \mathbf{x \cdot x = x}$$

$$7b. \quad \mathbf{x + x = x}$$

$$8a. \quad \mathbf{x \cdot \overline{x} = 0}$$

$$8b. \quad \mathbf{x + \overline{x} = 1}$$

$$9. \quad \mathbf{\overline{\overline{x}} = x}$$

$$5a. \quad \mathbf{x} \cdot \mathbf{0} = \mathbf{0}$$

$$5a. \quad x \cdot 0 = 0$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

$$5a. \quad x \cdot 0 = 0$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 \cdot 0 = 0$$

// axiom 1a

$$5a. \quad x \cdot 0 = 0$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 \cdot 0 = 0 \quad // \text{ axiom 1a}$$

ii) If $x = 1$, then we have

$$1 \cdot 0 = 0 \quad // \text{ axiom 3a}$$

5b. **$x + 1 = 1$**

$$\mathbf{5b. \quad x + 1 = 1}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 + 1 = 1$$

// axiom 3b

$$\mathbf{5b. \quad x + 1 = 1}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 + 1 = 1 \quad // \text{ axiom 3b}$$

ii) If $x = 1$, then we have

$$1 + 1 = 1 \quad // \text{ axiom 1b}$$

$$\mathbf{6a. \quad x \cdot 1 = x}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 \cdot 1 = 0 \quad // \text{ axiom 3a}$$

ii) If $x = 1$, then we have

$$1 \cdot 1 = 1 \quad // \text{ axiom 2a}$$

$$6a. \quad \boxed{x} \cdot 1 = \boxed{x}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$\boxed{0} \cdot 1 = \boxed{0}$$

// axiom 3a

ii) If $x = 1$, then we have

$$\boxed{1} \cdot 1 = \boxed{1}$$

// axiom 2a

$$\mathbf{6b. \quad x + 0 = x}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$0 + 0 = 0 \quad // \text{ axiom 2b}$$

ii) If $x = 1$, then we have

$$1 + 0 = 1 \quad // \text{ axiom 3b}$$

$$6b. \quad \boxed{x} + 0 = \boxed{x}$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$$\boxed{0} + 0 = \boxed{0} \quad // \text{ axiom 2b}$$

ii) If $x = 1$, then we have

$$\boxed{1} + 0 = \boxed{1} \quad // \text{ axiom 3b}$$

$$\mathbf{7a. \quad x \cdot x = x}$$

i) If $x = 0$, then we have

$$0 \cdot 0 = 0 \quad // \text{ axiom 1a}$$

ii) If $x = 1$, then we have

$$1 \cdot 1 = 1 \quad // \text{ axiom 2a}$$

$$7a. \quad \boxed{x} \cdot \boxed{x} = \boxed{x}$$

i) If $x = 0$, then we have

$$\boxed{0} \cdot \boxed{0} = \boxed{0}$$

// axiom 1a

ii) If $x = 1$, then we have

$$\boxed{1} \cdot \boxed{1} = \boxed{1}$$

// axiom 2a

$$7b. \quad \mathbf{x + x = x}$$

i) If $x = 0$, then we have

$$0 + 0 = 0 \quad // \text{ axiom 2b}$$

ii) If $x = 1$, then we have

$$1 + 1 = 1 \quad // \text{ axiom 1b}$$

$$7b. \quad \boxed{x} + \boxed{x} = \boxed{x}$$

i) If $x = 0$, then we have

$$\boxed{0} + \boxed{0} = \boxed{0}$$

// axiom 2b

ii) If $x = 1$, then we have

$$\boxed{1} + \boxed{1} = \boxed{1}$$

// axiom 1b

$$8a. \quad \mathbf{x} \cdot \overline{\mathbf{x}} = \mathbf{0}$$

i) If $\mathbf{x} = \mathbf{0}$, then we have

$$\mathbf{0} \cdot \mathbf{1} = \mathbf{0} \quad // \text{ axiom 3a}$$

ii) If $\mathbf{x} = \mathbf{1}$, then we have

$$\mathbf{1} \cdot \mathbf{0} = \mathbf{0} \quad // \text{ axiom 3a}$$

$$8a. \quad x \cdot \overline{x} = 0$$

i) If $x = 0$, then we have

$$0 \cdot 1 = 0$$

// axiom 3a

ii) If $x = 1$, then we have

$$1 \cdot 0 = 0$$

// axiom 3a

$$\mathbf{8b.} \quad \mathbf{x + \bar{x} = 1}$$

i) If $x = 0$, then we have

$$\mathbf{0 + 1 = 1} \quad // \text{ axiom 3b}$$

ii) If $x = 1$, then we have

$$\mathbf{1 + 0 = 1} \quad // \text{ axiom 3b}$$

$$8b. \quad x + \bar{x} = 1$$

i) If $x = 0$, then we have

$$0 + 1 = 1 \quad // \text{ axiom 3b}$$

ii) If $x = 1$, then we have

$$1 + 0 = 1 \quad // \text{ axiom 3b}$$

$$9. \quad \overline{\overline{x}} = x$$

i) If $x = 0$, then we have

$$\overline{x} = 1 \quad // \text{ axiom 4a}$$

let $y = \overline{x} = 1$, then we have

$$\overline{y} = 0 \quad // \text{ axiom 4b}$$

Therefore,

$$\overline{\overline{x}} = x \quad (\text{when } x = 0)$$

$$9. \quad \overline{\overline{x}} = x$$

ii) If $x = 1$, then we have

$$\overline{x} = 0 \quad // \text{ axiom 4b}$$

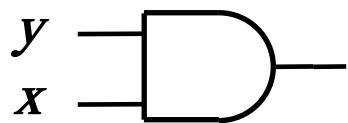
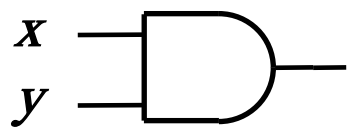
let $y = \overline{x} = 0$, then we have

$$\overline{y} = 1 \quad // \text{ axiom 4a}$$

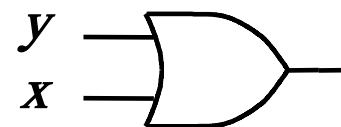
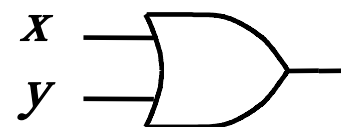
Therefore,

$$\overline{\overline{x}} = x \quad (\text{when } x = 1)$$

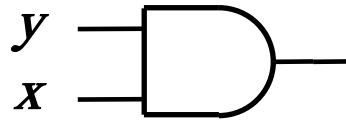
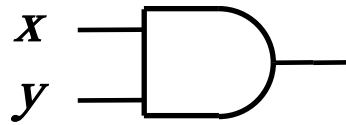
10a. $\mathbf{x \cdot y = y \cdot x}$



10b. $\mathbf{x + y = y + x}$



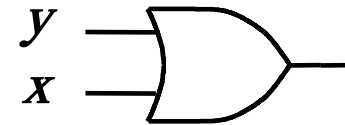
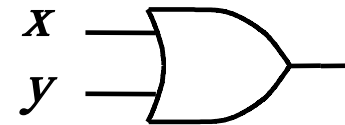
10a. $x \cdot y = y \cdot x$



AND gate

x	y	f
0	0	0
0	1	0
1	0	0
1	1	1

10b. $x + y = y + x$



OR gate

x	y	f
0	0	0
0	1	1
1	0	1
1	1	1

The order of the inputs does not matter.

$$11a. \quad \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$$

x	y	z	x	y · z	x·(y·z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

$$11a. \quad \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$$

x	y	z	x	y · z	x·(y·z)
0	0	0	0	0	
0	0	1	0	0	
0	1	0	0	0	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	0	
1	1	0	1	0	
1	1	1	1	1	

Truth table for the left-hand side

$$11a. \quad \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$$

x	y	z	x	y · z	x·(y·z)
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	1	1	1

Truth table for the left-hand side

11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

x	y	z	$x \cdot y$	z	$(x \cdot y) \cdot z$
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	0	0	0
1	0	1	0	1	0
1	1	0	1	0	0
1	1	1	1	1	1

Truth table for the right-hand side

$$11a. \quad \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$$

$\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$
0
0
0
0
0
0
0
0
1

$(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$
0
0
0
0
0
0
0
0
1

These two are identical, which concludes the proof.

11b. **$x + (y + z) = (x + y) + z$**

x	y	z	x	y + z	x+(y+z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

11b. $\mathbf{x + (y + z) = (x + y) + z}$

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	
0	0	1	0	1	
0	1	0	0	1	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	1	
1	1	0	1	1	
1	1	1	1	1	

Truth table for the left-hand side

11b. **$x + (y + z) = (x + y) + z$**

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Truth table for the left-hand side

11b. $\mathbf{x + (y + z) = (x + y) + z}$

x	y	z	x + y	z	(x+y)+z
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	0	1
1	1	1	1	1	1

Truth table for the right-hand side

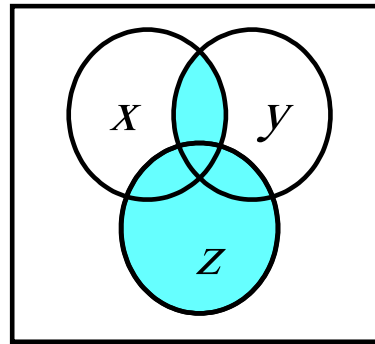
11b. **$x + (y + z) = (x + y) + z$**

$x+(y+z)$
0
1
1
1
1
1
1
1
1

$(x+y)+z$
0
1
1
1
1
1
1
1
1

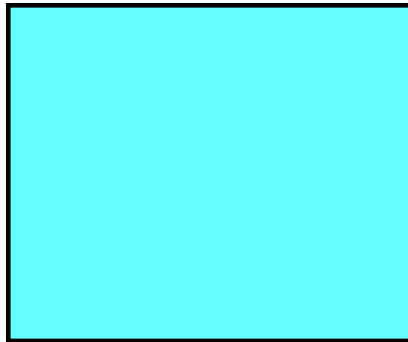
These two are identical, which concludes the proof.

The Venn Diagram Representation



$$xy + z$$

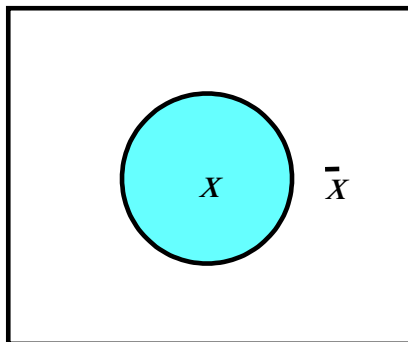
Venn Diagram Basics



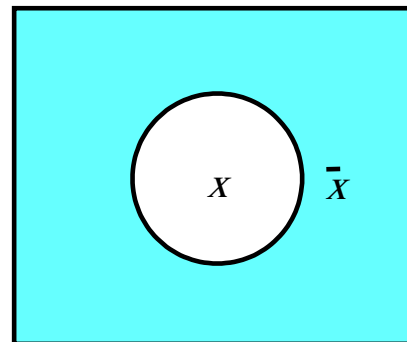
(a) Constant 1



(b) Constant 0

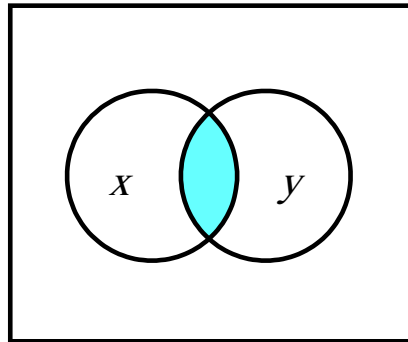


(c) Variable x

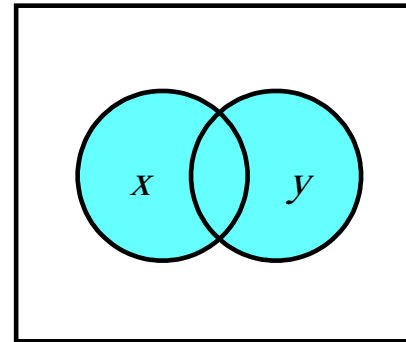


(d) \bar{x}

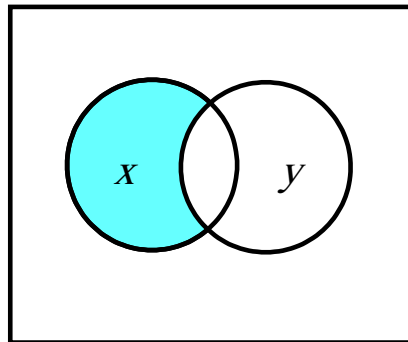
Venn Diagram Basics



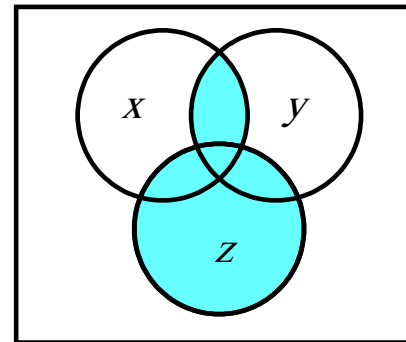
(e) $x \cap y$



(f) $x \cup y$



(g) $x \cap \bar{y}$



(h) $(x \cap y) \cup z$

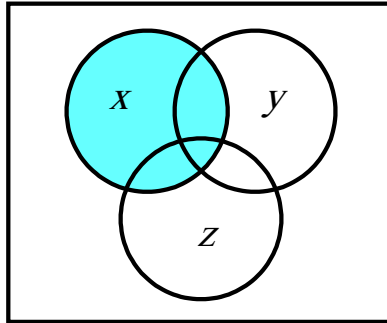
Let's Prove the Distributive Properties

$$12a. \quad \mathbf{x \cdot (y + z) = x \cdot y + x \cdot z}$$

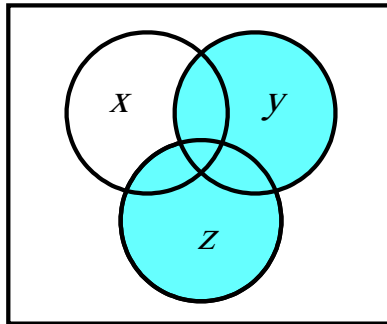
$$12b. \quad \mathbf{x + y \cdot z = (x + y) \cdot (x + z)}$$

12a.

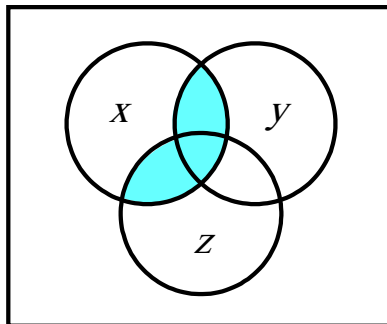
$$\mathbf{x \cdot (y + z) = x \cdot y + x \cdot z}$$



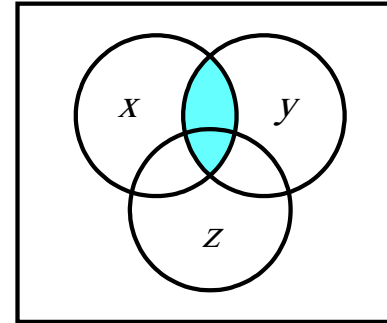
(a) x



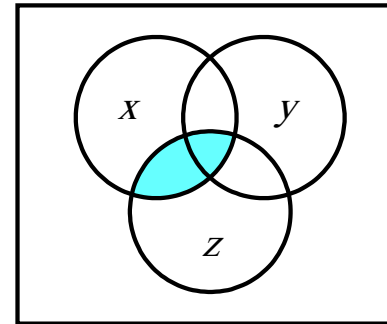
(b) $y + z$



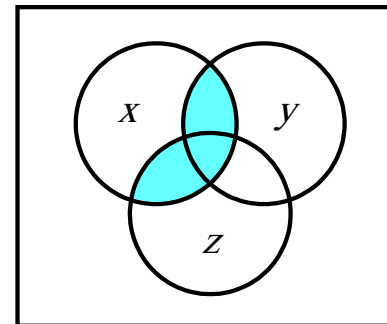
(c) $x \cdot (y + z)$



(d) $x \cdot y$

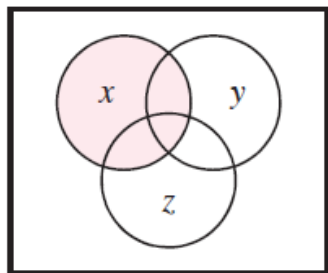


(e) $x \cdot z$

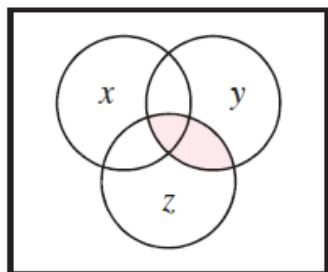


(f) $x \cdot y + x \cdot z$

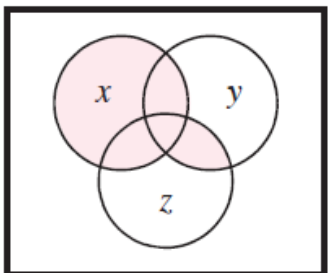
12b. $x + y \cdot z = (x + y) \cdot (x + z)$



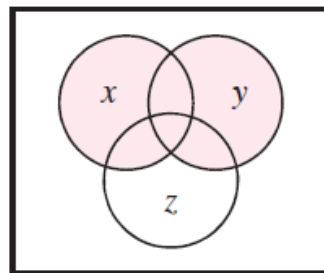
(a) x



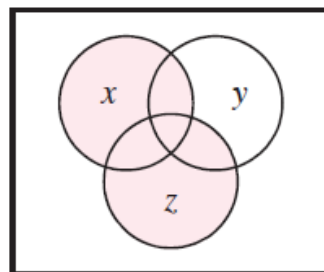
(b) $y \cdot z$



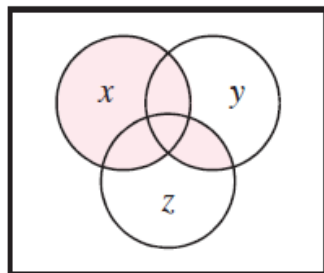
(c) $x + y \cdot z$



(d) $x + y$



(e) $x + z$



(f) $(x + y) \cdot (x + z)$

Try to prove these ones at home

$$13a. \quad \mathbf{x + x \cdot y = x}$$

$$13b. \quad \mathbf{x \cdot (x + y) = x}$$

$$14a. \quad \mathbf{x \cdot y + x \cdot \bar{y} = x}$$

$$14b. \quad \mathbf{(x + y) \cdot (x + \bar{y}) = x}$$

DeMorgan's Theorem

$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

$$15b. \quad \overline{x + y} = \bar{x} \cdot \bar{y}$$

Proof of DeMorgan's theorem

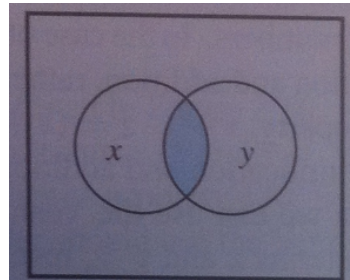
$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$

x	y	$x \cdot y$	$\overline{x \cdot y}$	\bar{x}	\bar{y}	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

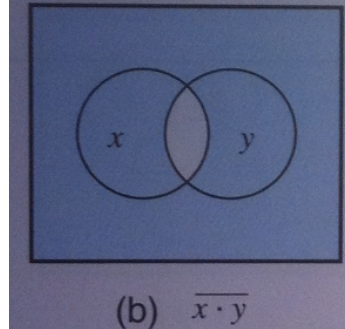
$\underbrace{\hspace{15em}}_{\text{LHS}} \qquad \underbrace{\hspace{15em}}_{\text{RHS}}$

Alternative Proof of DeMorgan's theorem

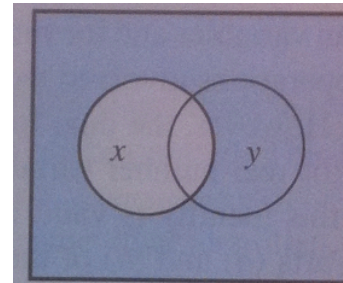
$$15a. \quad \overline{x \cdot y} = \bar{x} + \bar{y}$$



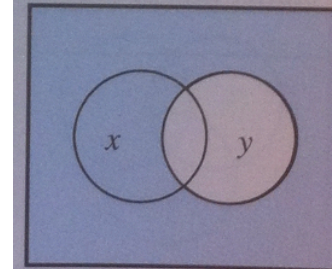
(a) $x \cdot y$



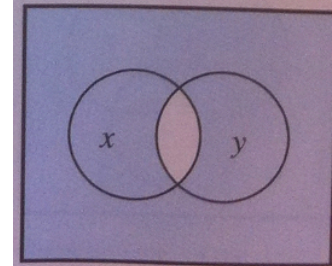
(b) $\overline{x \cdot y}$



(c) \bar{x}



(d) \bar{y}



(e) $\bar{x} + \bar{y}$

Try to prove these ones at home

16a. $x + \bar{x} \cdot y = x + y$

16b. $x \cdot (\bar{x} + y) = x \cdot y$

17a. $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

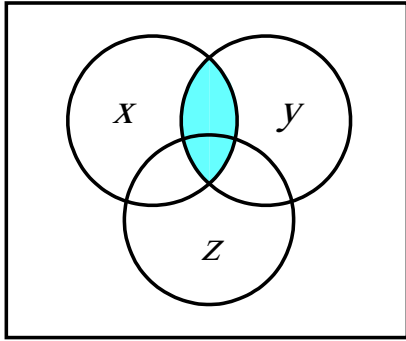
17b. $(x+y) \cdot (y+z) \cdot (\bar{x}+z) = (x+y) \cdot (\bar{x}+z)$

Venn Diagram Example

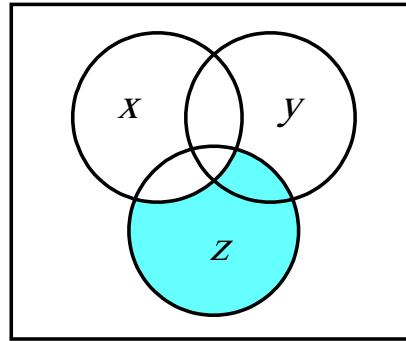
Proof of Property 17a

$$17a. \quad x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$$

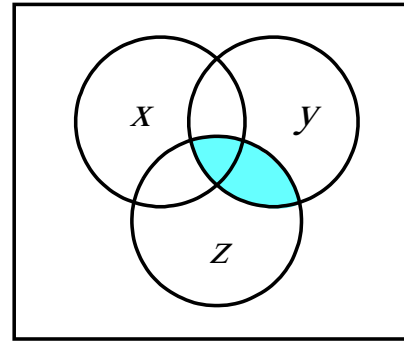
Left-Hand Side



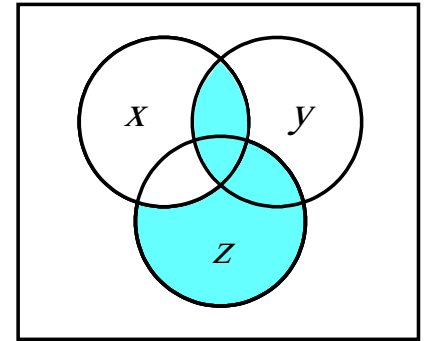
$$x y$$



$$\bar{x} z$$

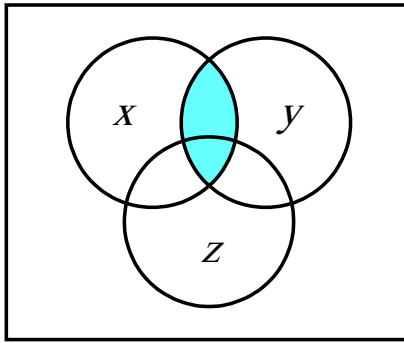


$$y z$$

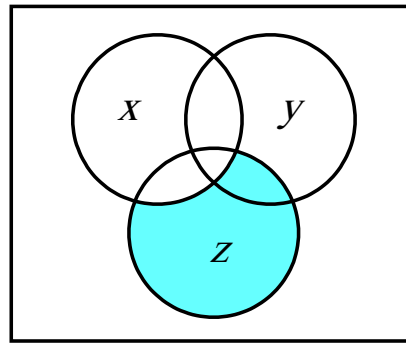


$$x y + \bar{x} z + y z$$

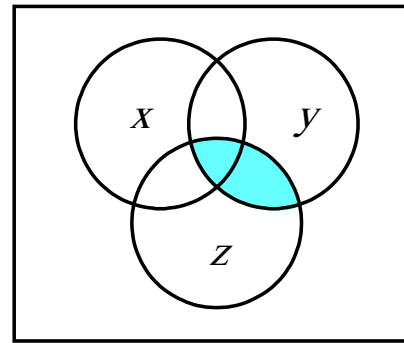
Left-Hand Side



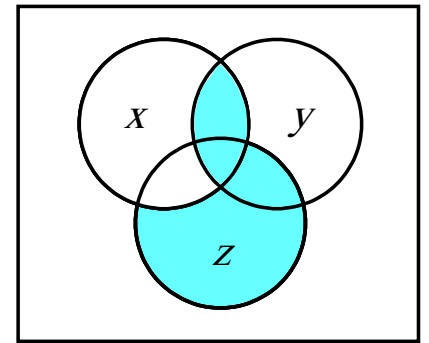
$$x y$$



$$\bar{x} z$$

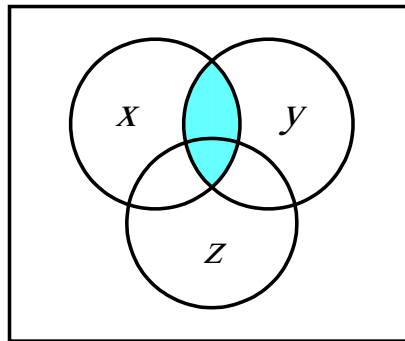


$$y z$$

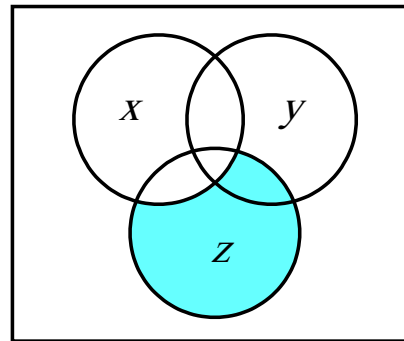


$$x y + \bar{x} z + y z$$

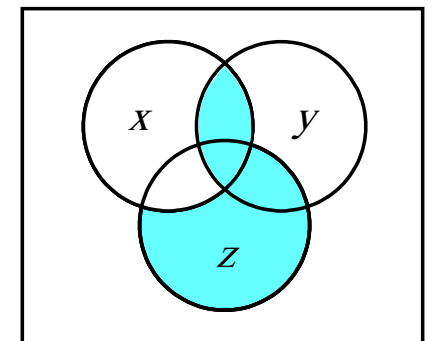
Right-Hand Side



$$x y$$

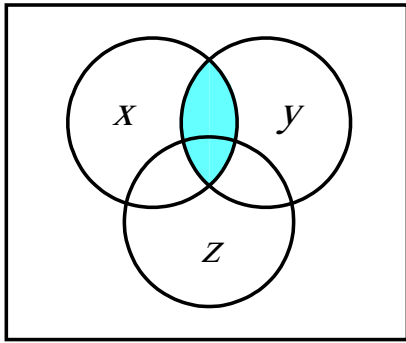


$$\bar{x} z$$

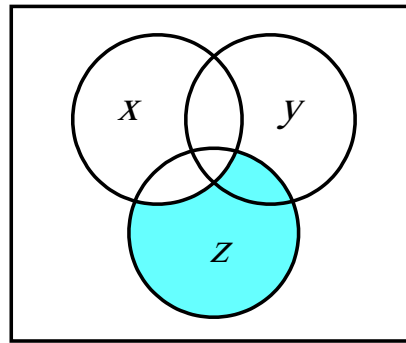


$$x y + \bar{x} z$$

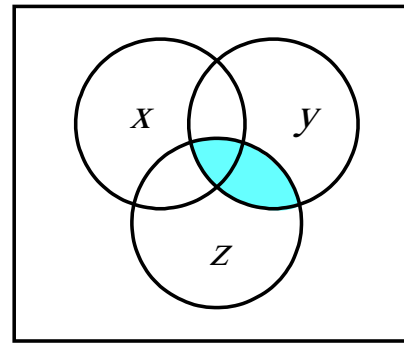
Left-Hand Side



$$x y$$

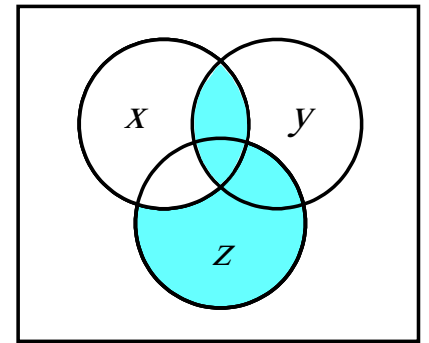


$$\bar{x} z$$



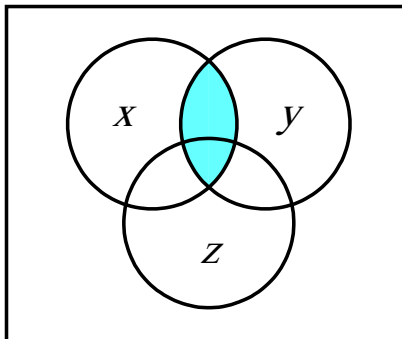
$$y z$$

These two are equal

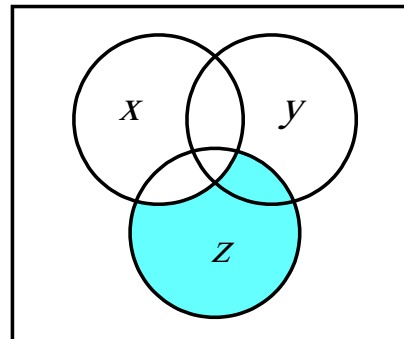


$$x y + \bar{x} z + y z$$

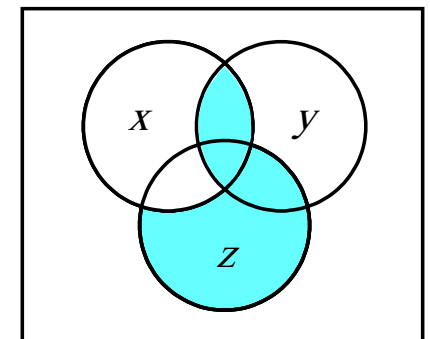
Right-Hand Side



$$x y$$



$$\bar{x} z$$



$$x y + \bar{x} z$$

Questions?

THE END