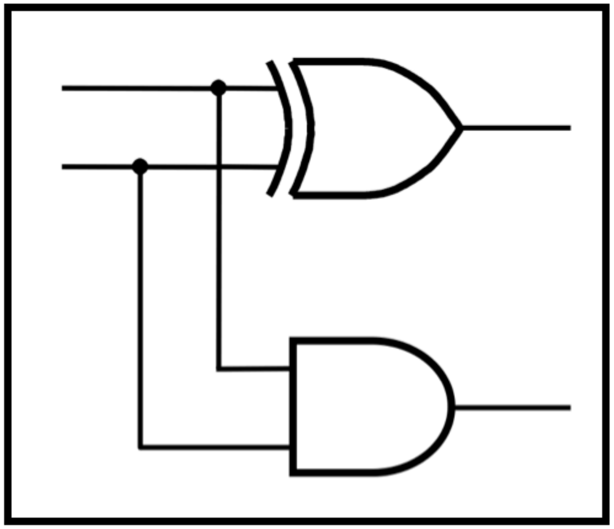


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Multiplication

CprE 281: Digital Logic
Iowa State University, Ames, IA

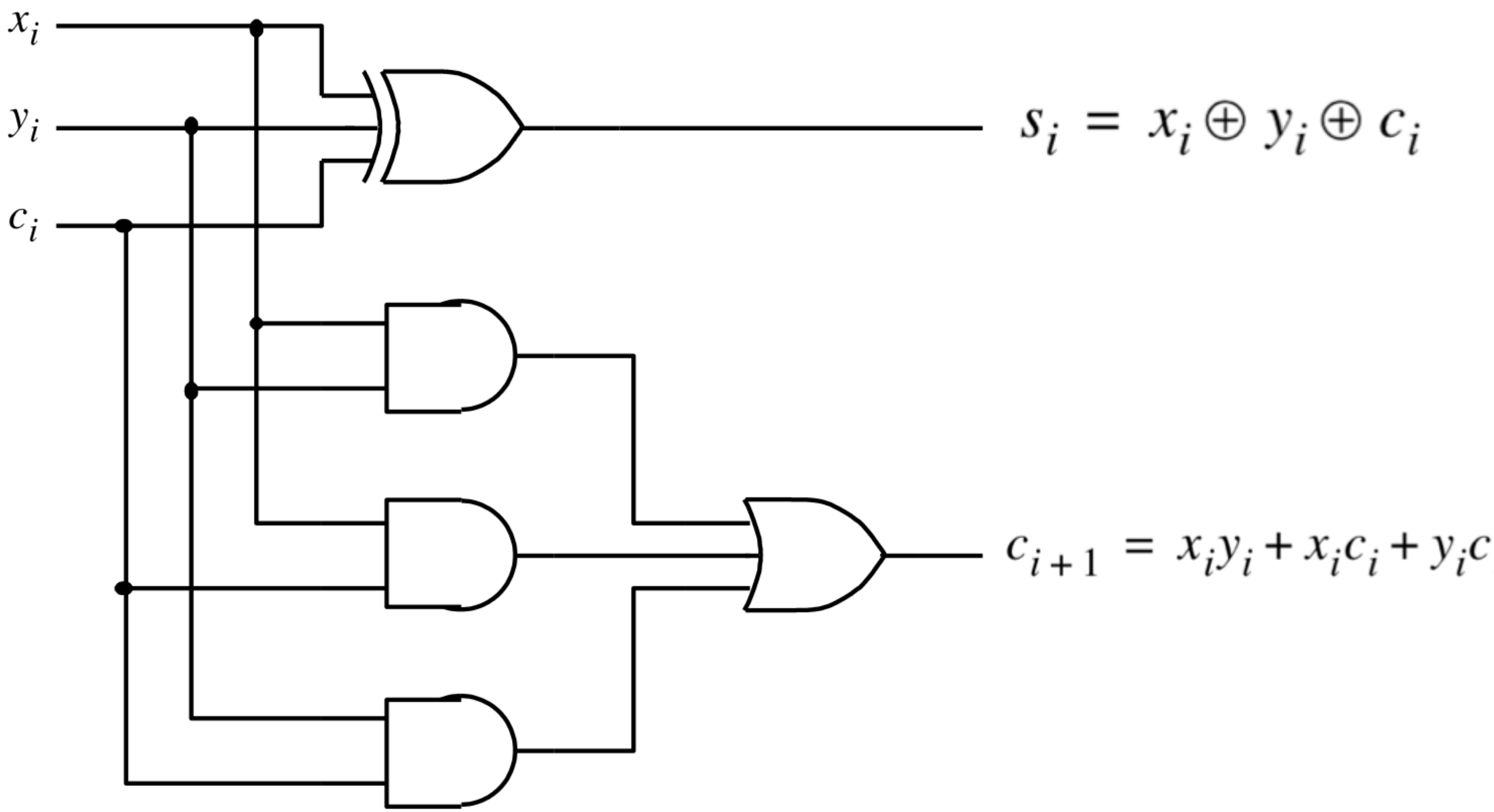
Administrative Stuff

- HW 6 is out

- It is due on Monday Oct 8 @ 4pm

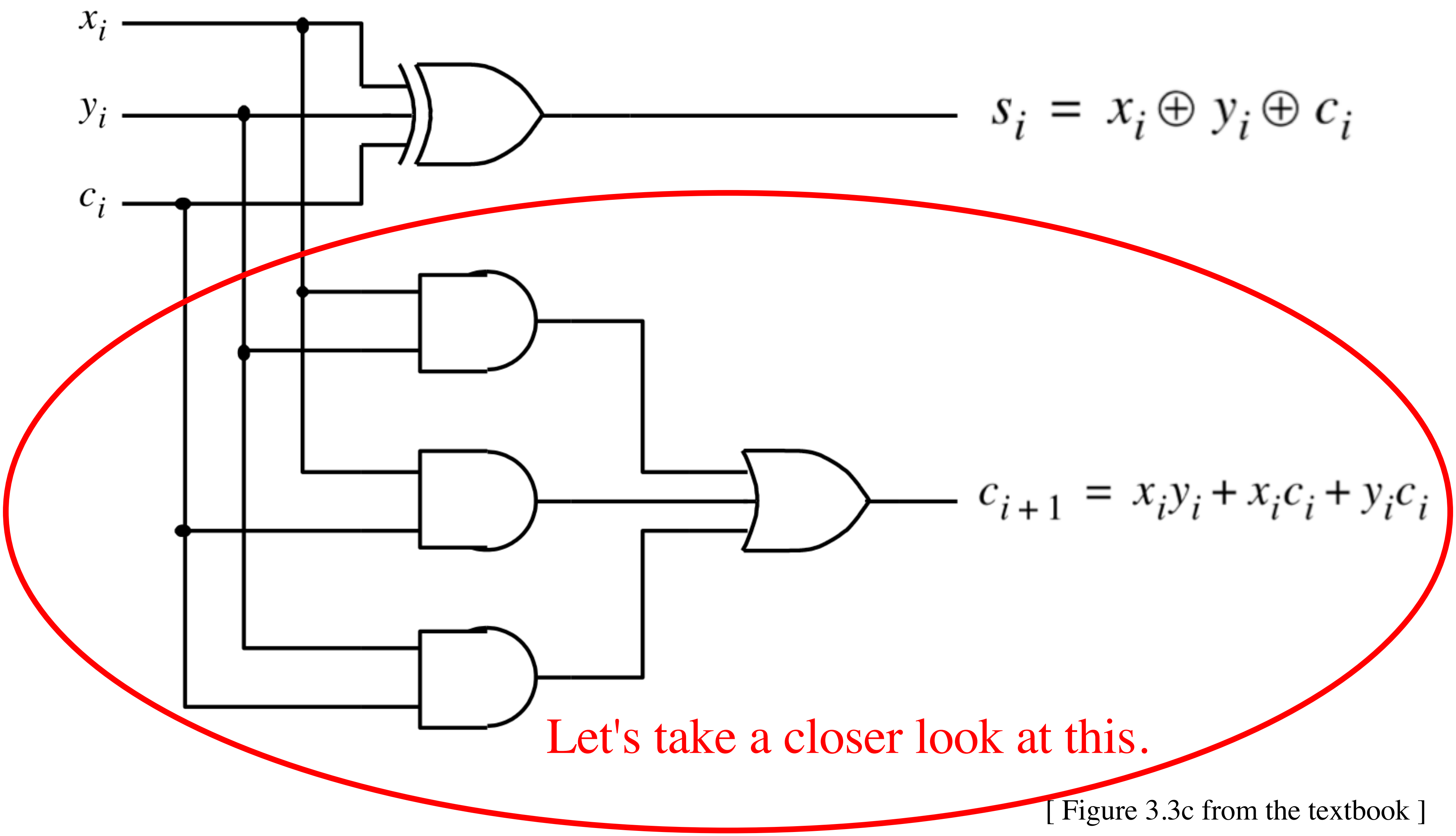
Quick Review

The Full-Adder Circuit



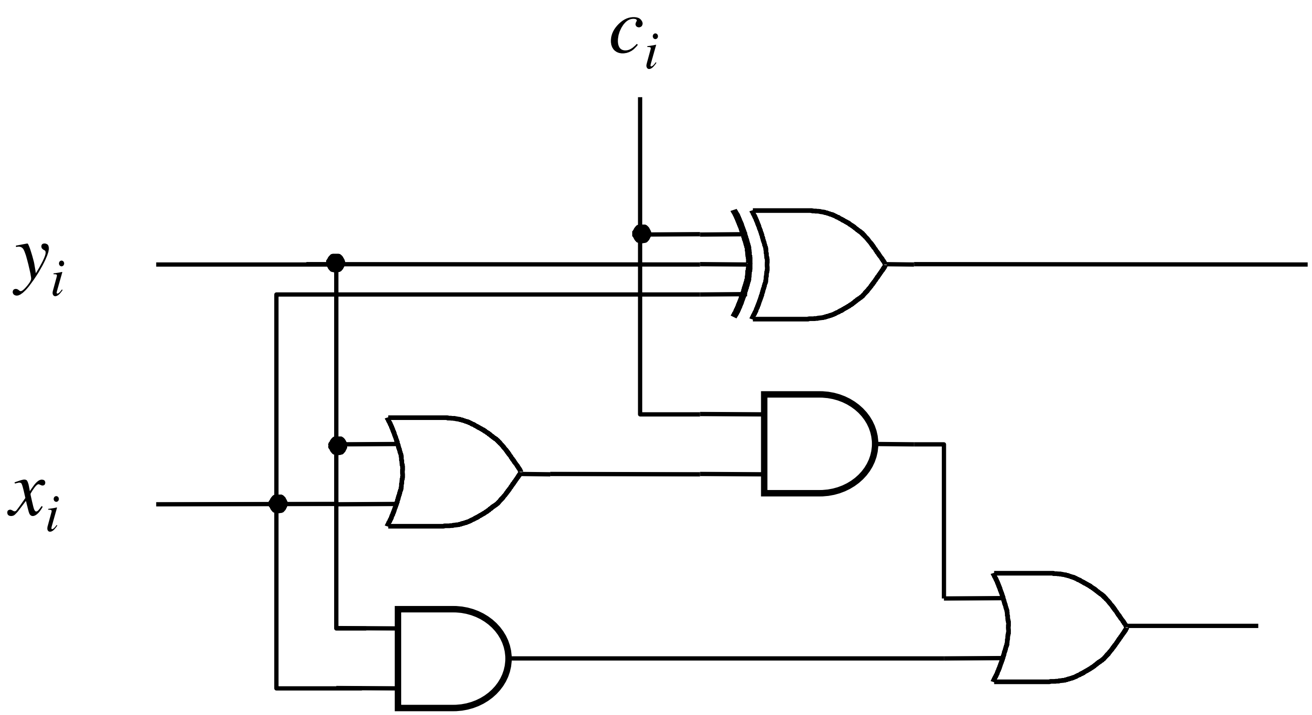
[Figure 3.3c from the textbook]

The Full-Adder Circuit



[Figure 3.3c from the textbook]

Another Way to Draw the Full-Adder Circuit



$$s_i = x_i \oplus y_i \oplus c_i$$

$$c_{i+1} = x_i\, y_i + (x_i + y_i\,)c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

Decomposing the Carry Expression

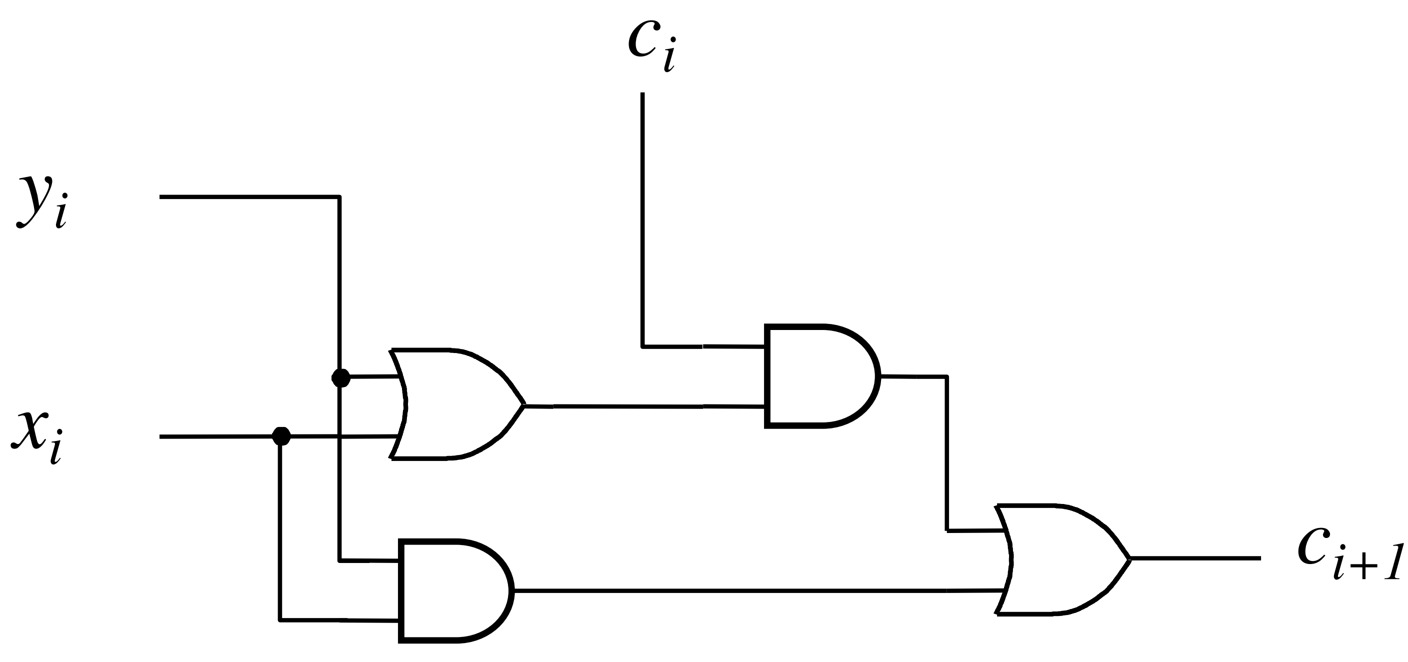
$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

$$c_{i+1} = x_i y_i + (x_i + y_i) c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

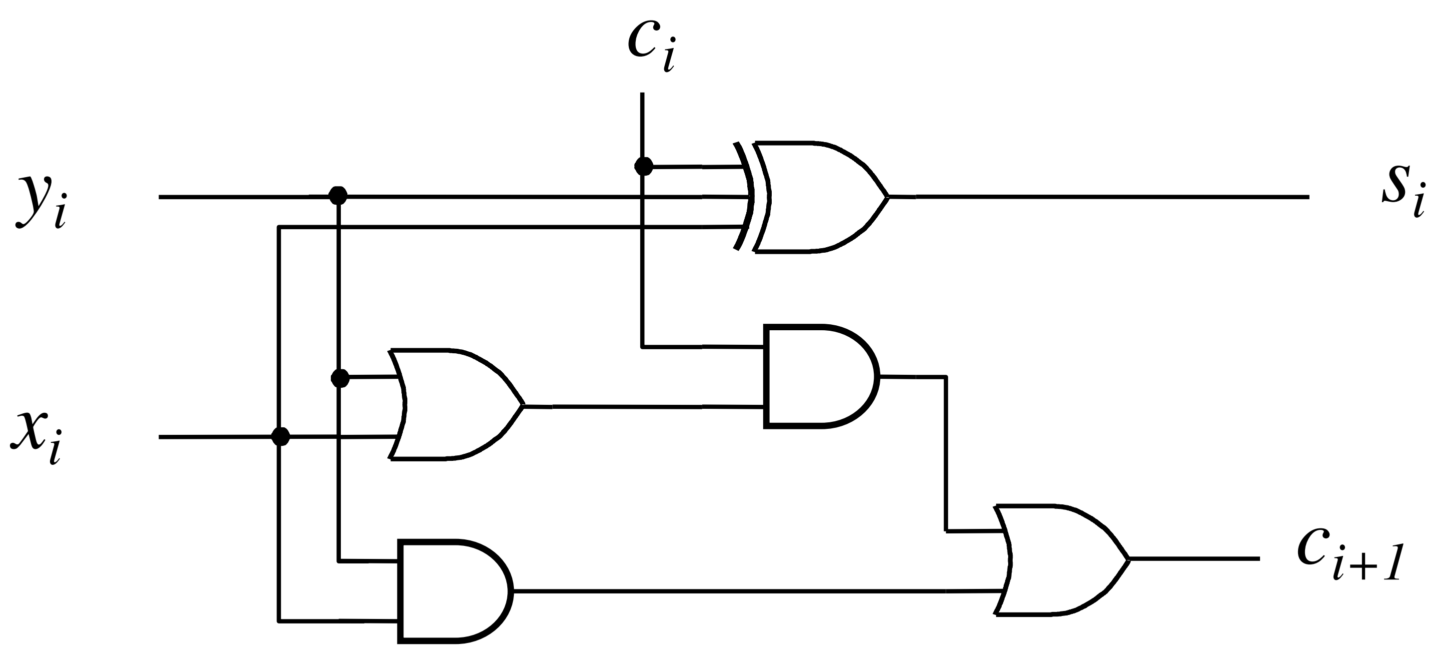
$$c_{i+1} = x_i\, y_i + (x_i + y_i)c_i$$



Another Way to Draw the Full-Adder Circuit

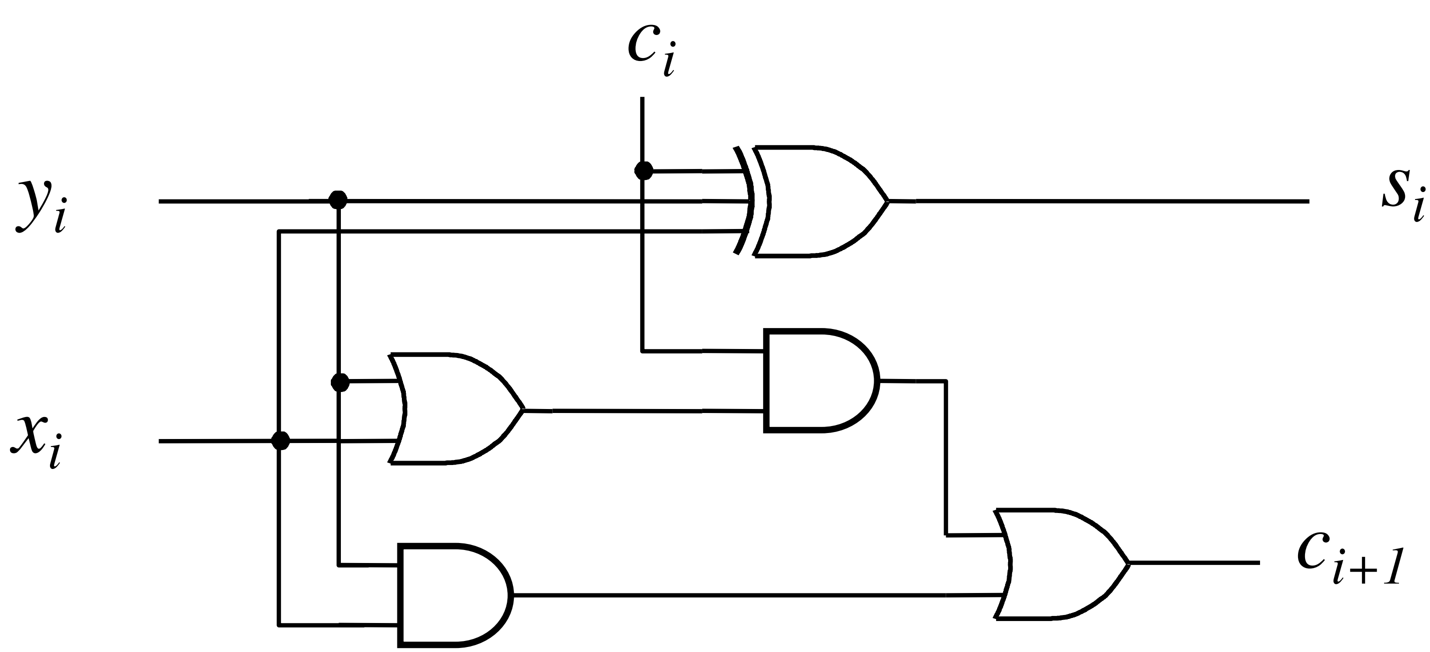
$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

$$c_{i+1} = x_i\, y_i + (x_i + y_i) c_i$$



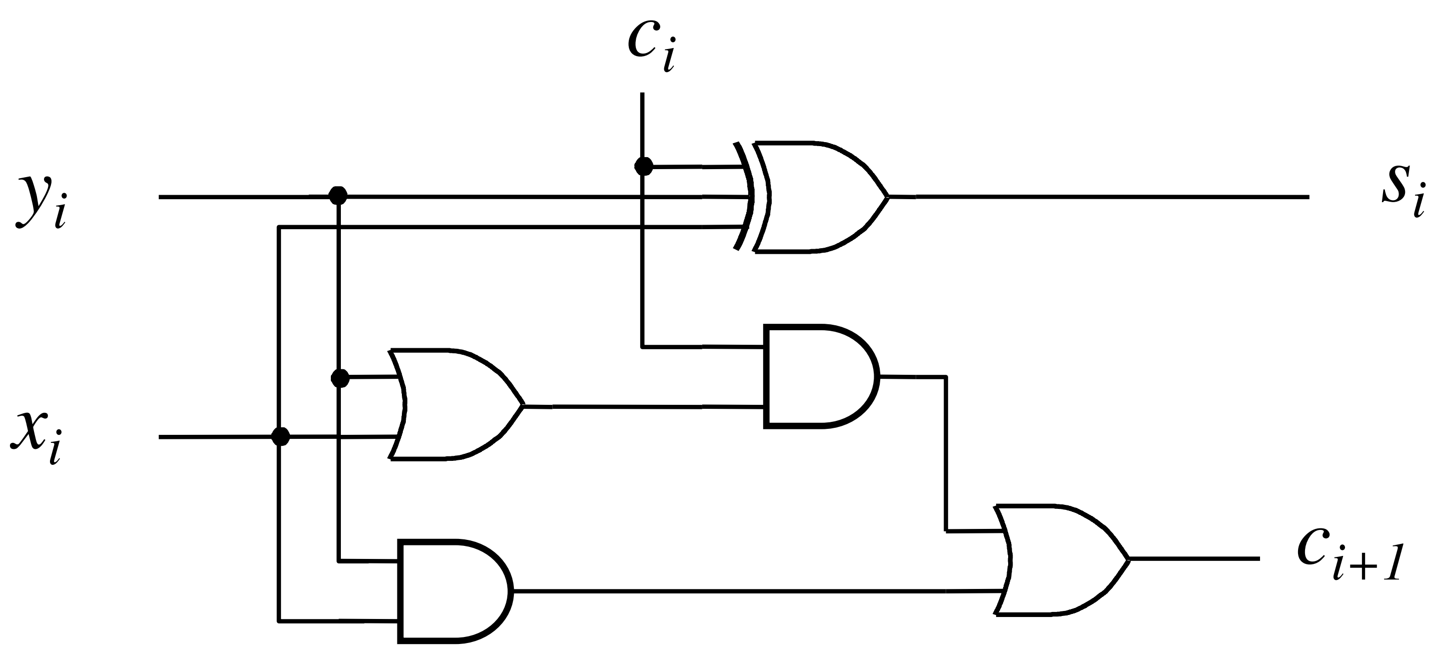
Another Way to Draw the Full-Adder Circuit

$$c_{i+1} = x_i\, y_i + (x_i + y_i)c_i$$



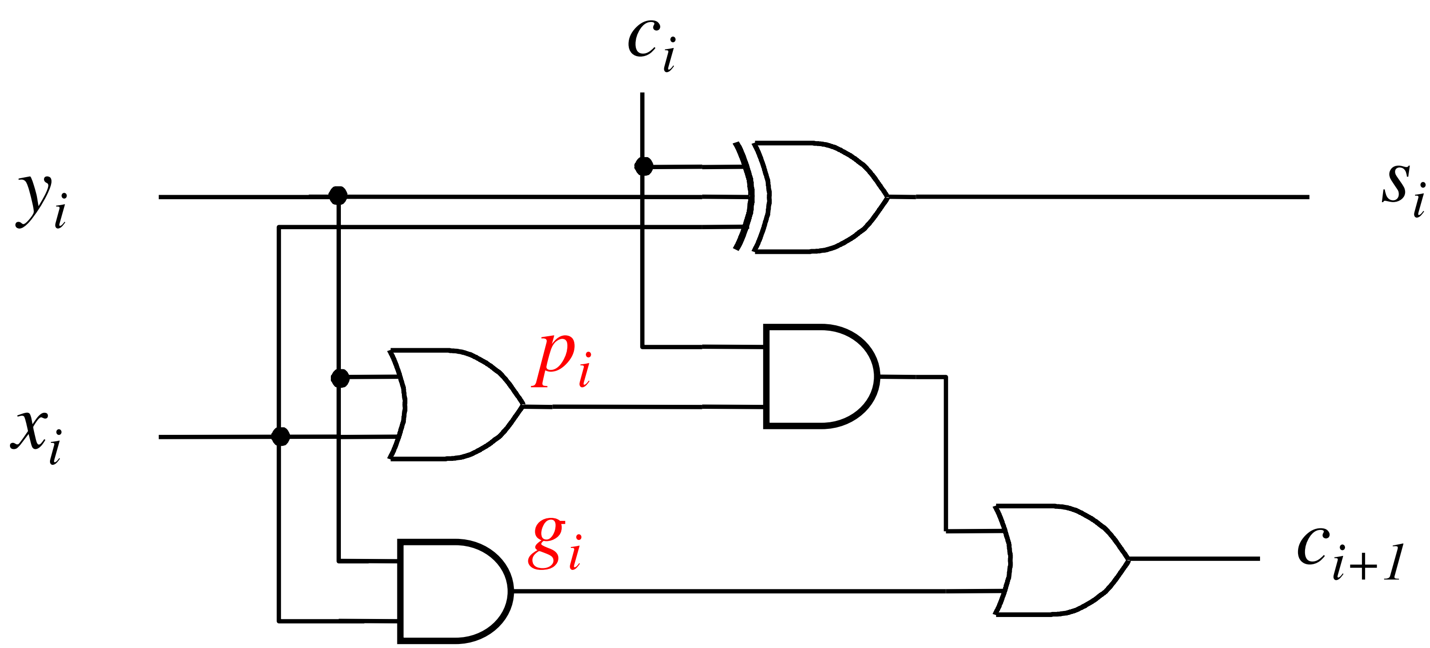
Another Way to Draw the Full-Adder Circuit

$$c_{i+1} = \underbrace{x_i\, y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

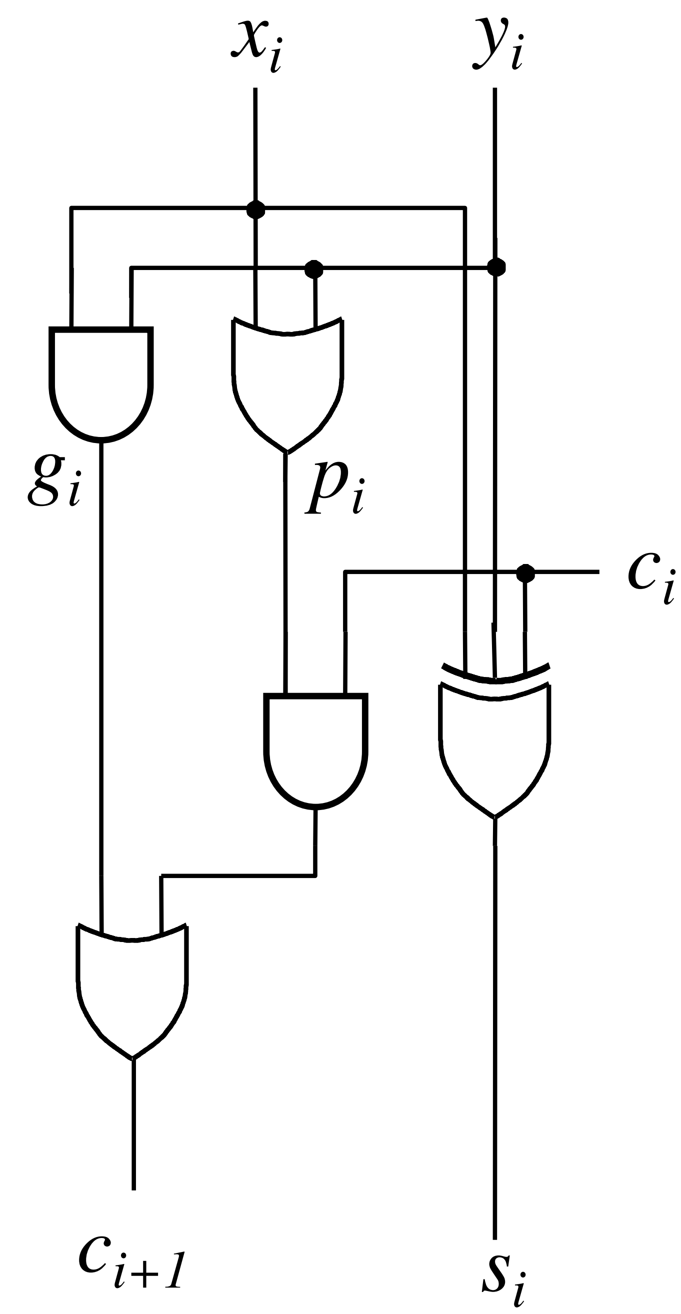


Another Way to Draw the Full-Adder Circuit

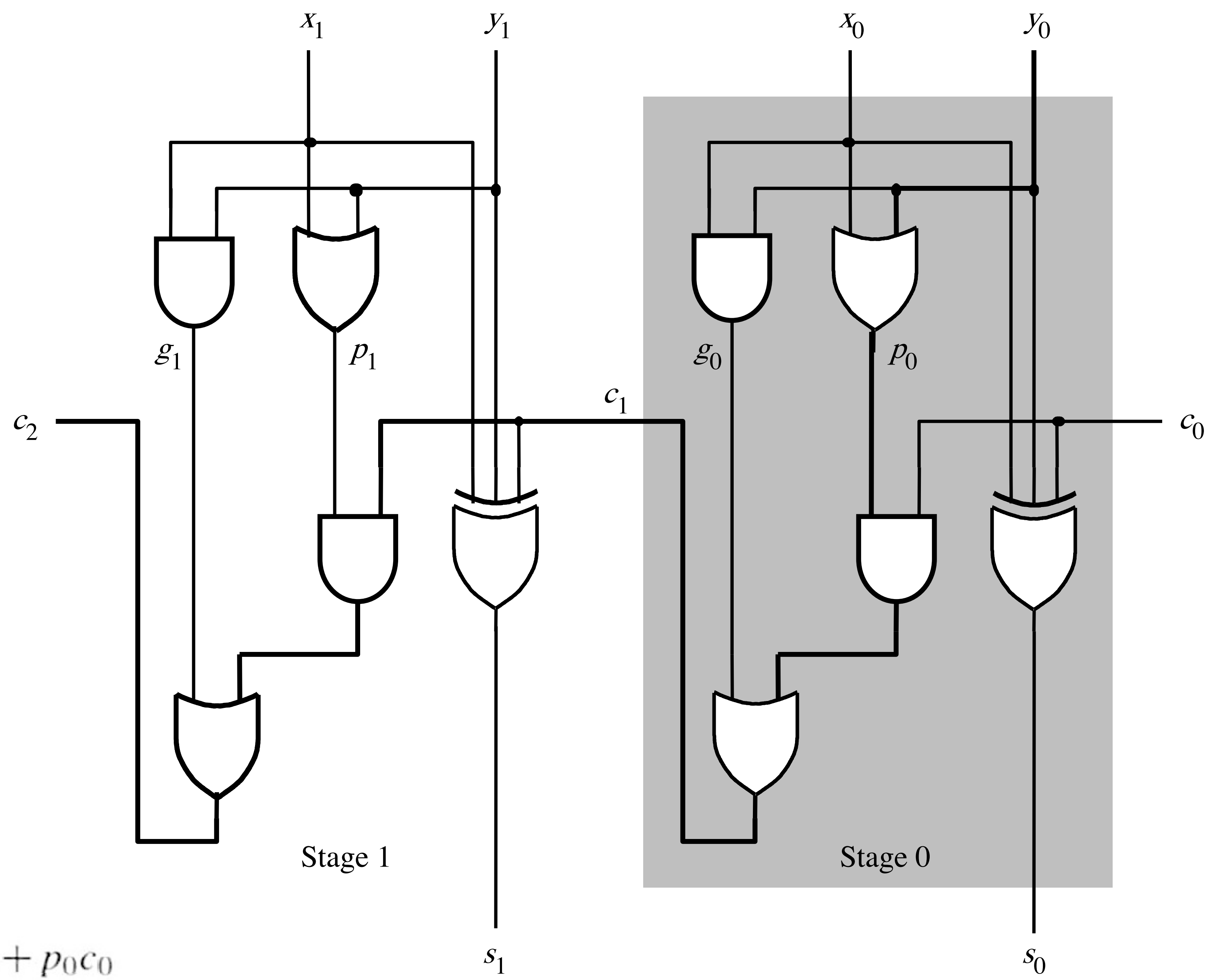
$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$



Yet Another Way to Draw It (Just Rotate It)



Now we can Build a Ripple-Carry Adder

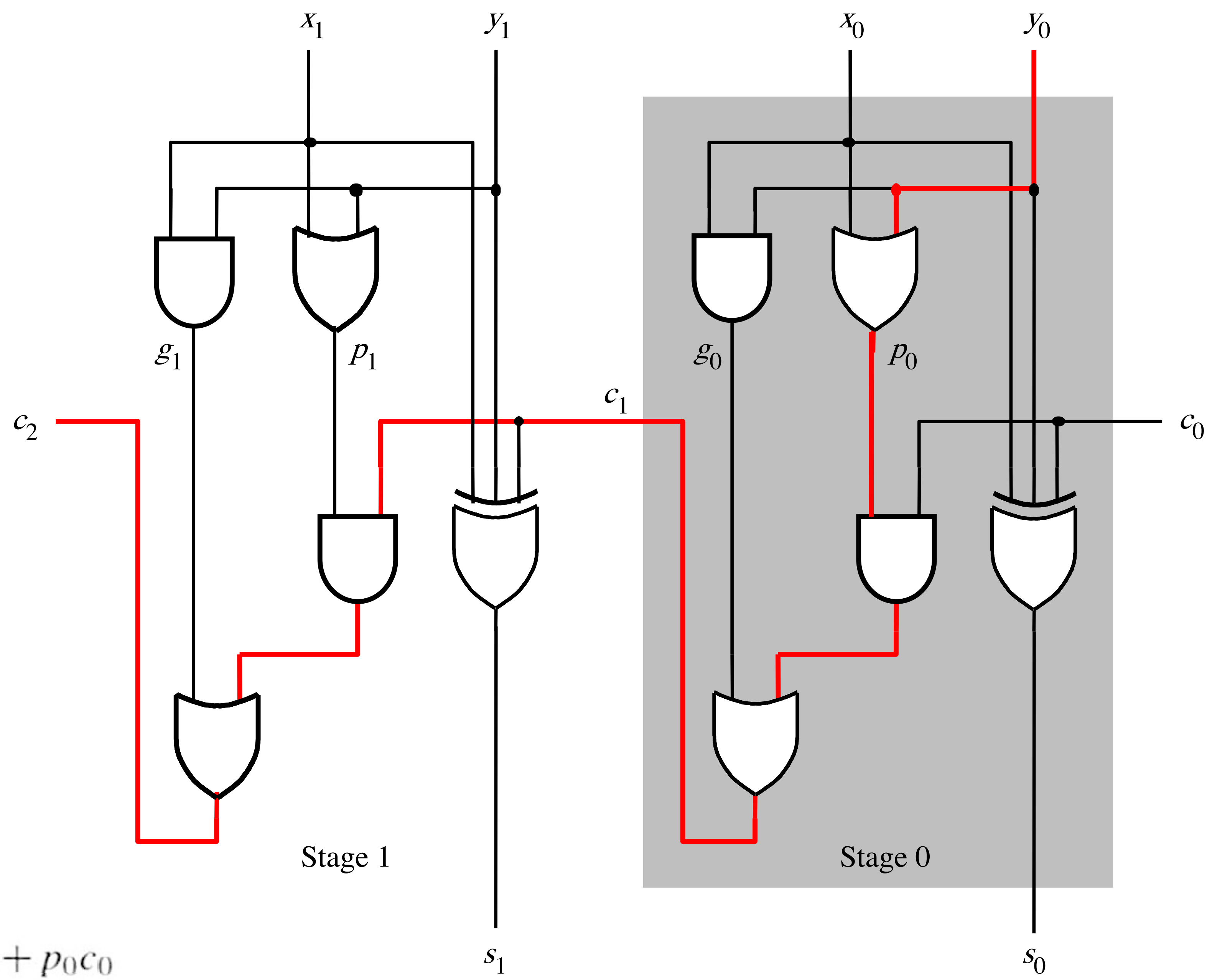


$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

[Figure 3.14 from the textbook]

Now we can Build a Ripple-Carry Adder

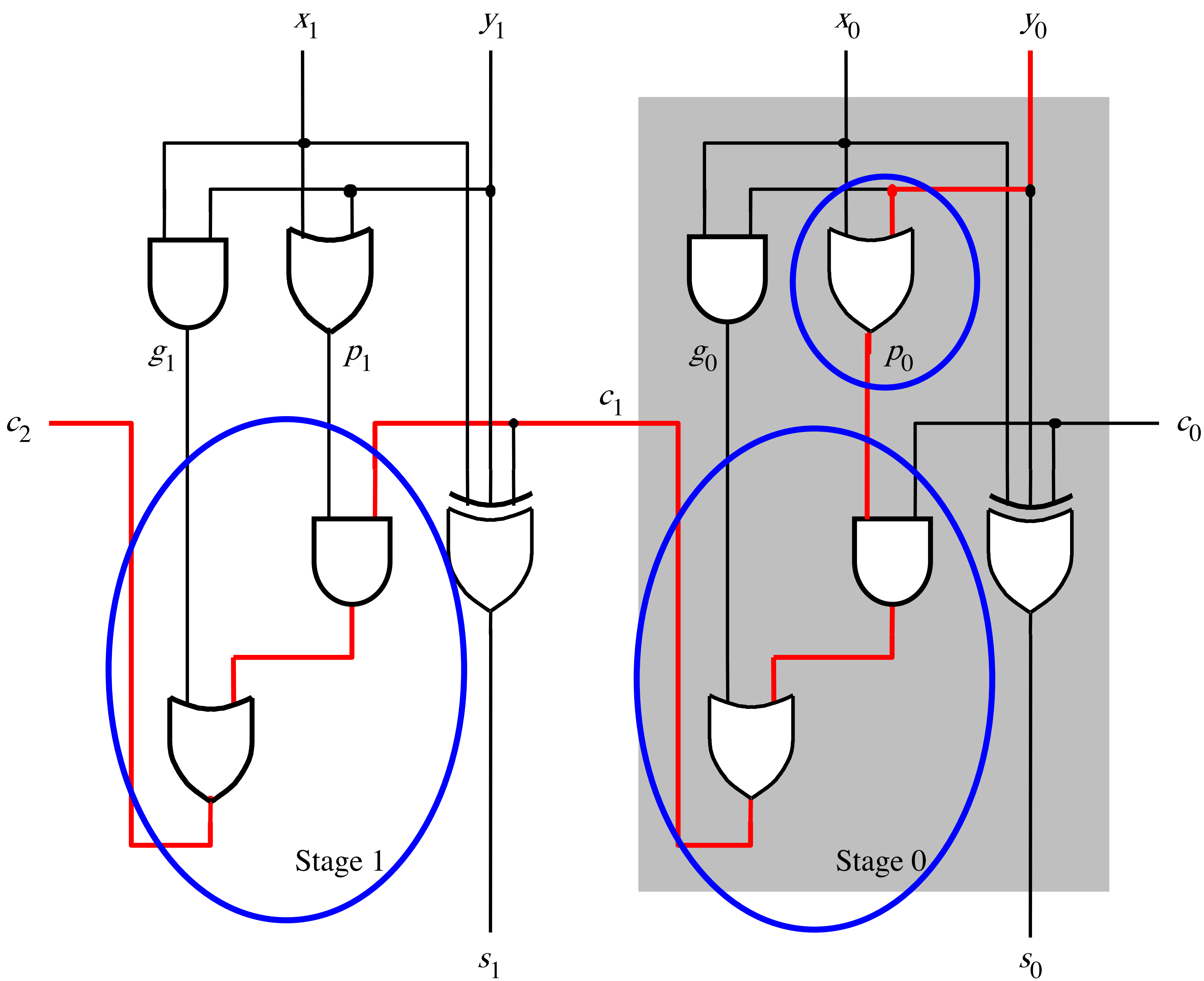


$$c_1 = g_0 + p_0 c_0$$

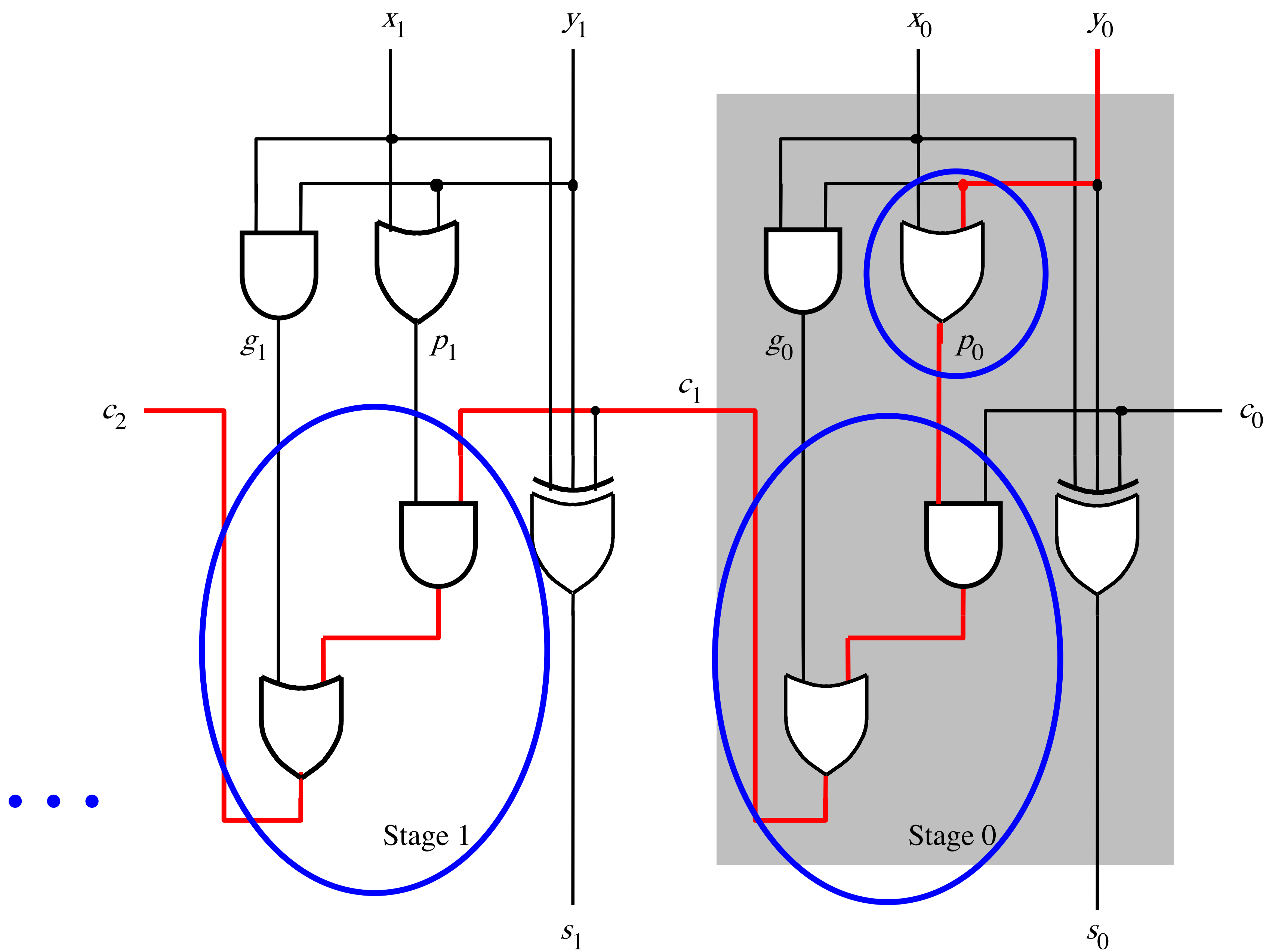
$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

[Figure 3.14 from the textbook]

The delay is 5 gates (1+2+2)



n-bit ripple-carry adder: 2n+1 gate delays



Decomposing the Carry Expression

$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

$$c_{i+1} = \underbrace{x_i\, y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

$$c_{i+1} = g_i \;+ p_i\, c_i$$

$$c_{i+1} = g_i \;+ p_i (g_{i\text{-}1} \;+ p_{i\text{-}1}\, c_{i\text{-}1})$$

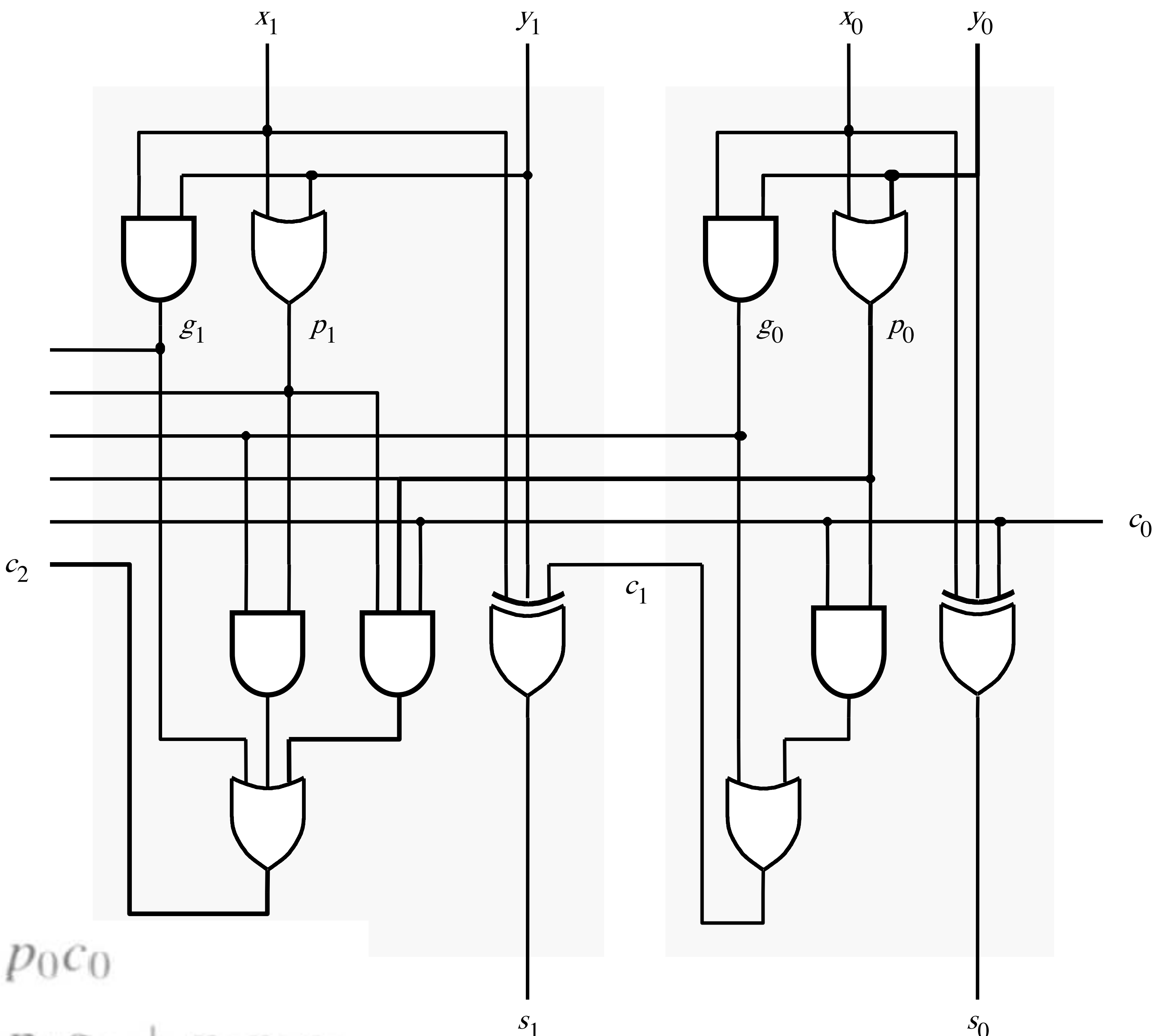
$$= g_i \;+ p_i\, g_{i\text{-}1} \;+ p_i\, p_{i\text{-}1}\, c_{i\text{-}1}$$

Carry for the first two stages

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

The first two stages of a carry-lookahead adder

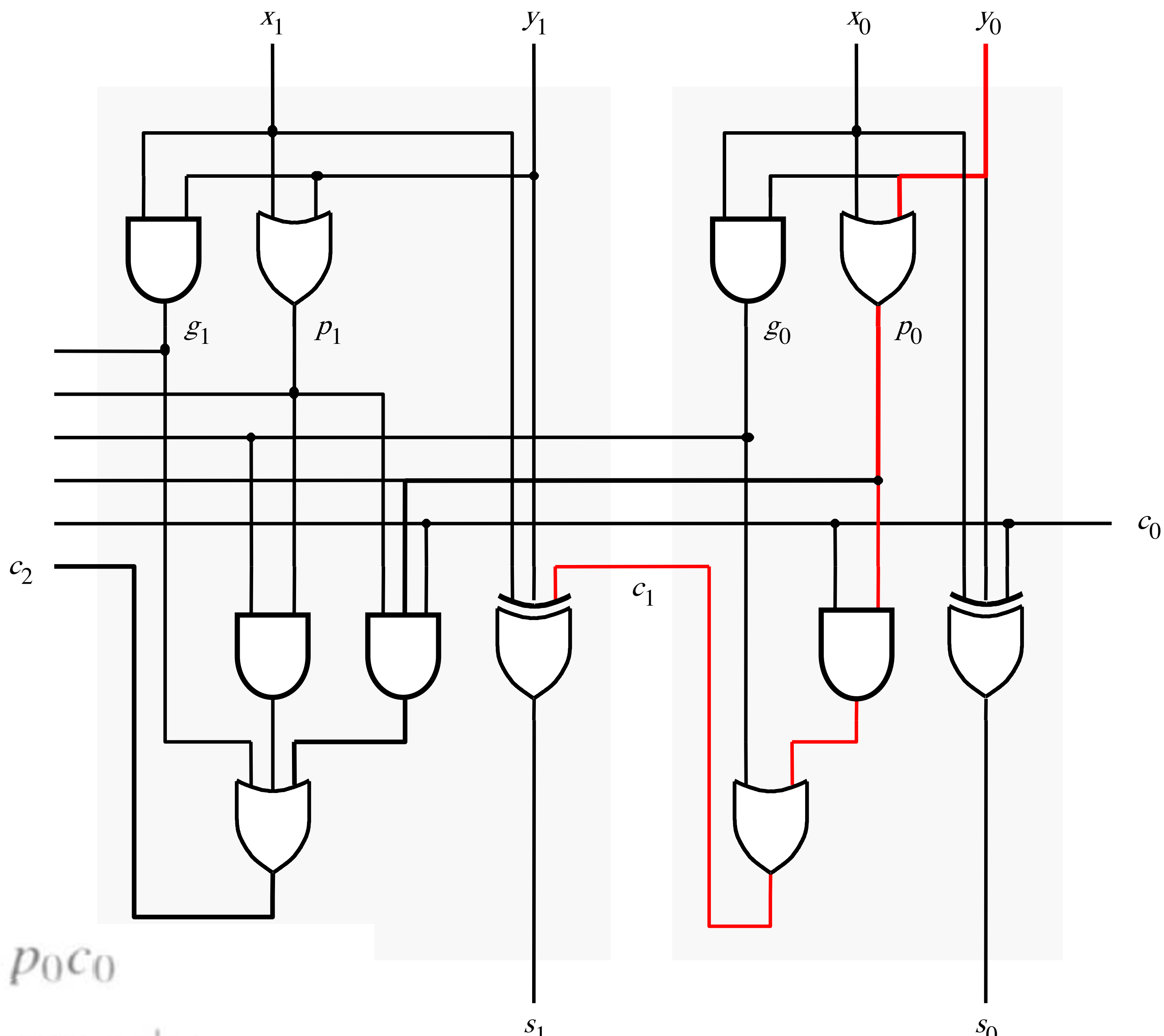


$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

[Figure 3.15 from the textbook]

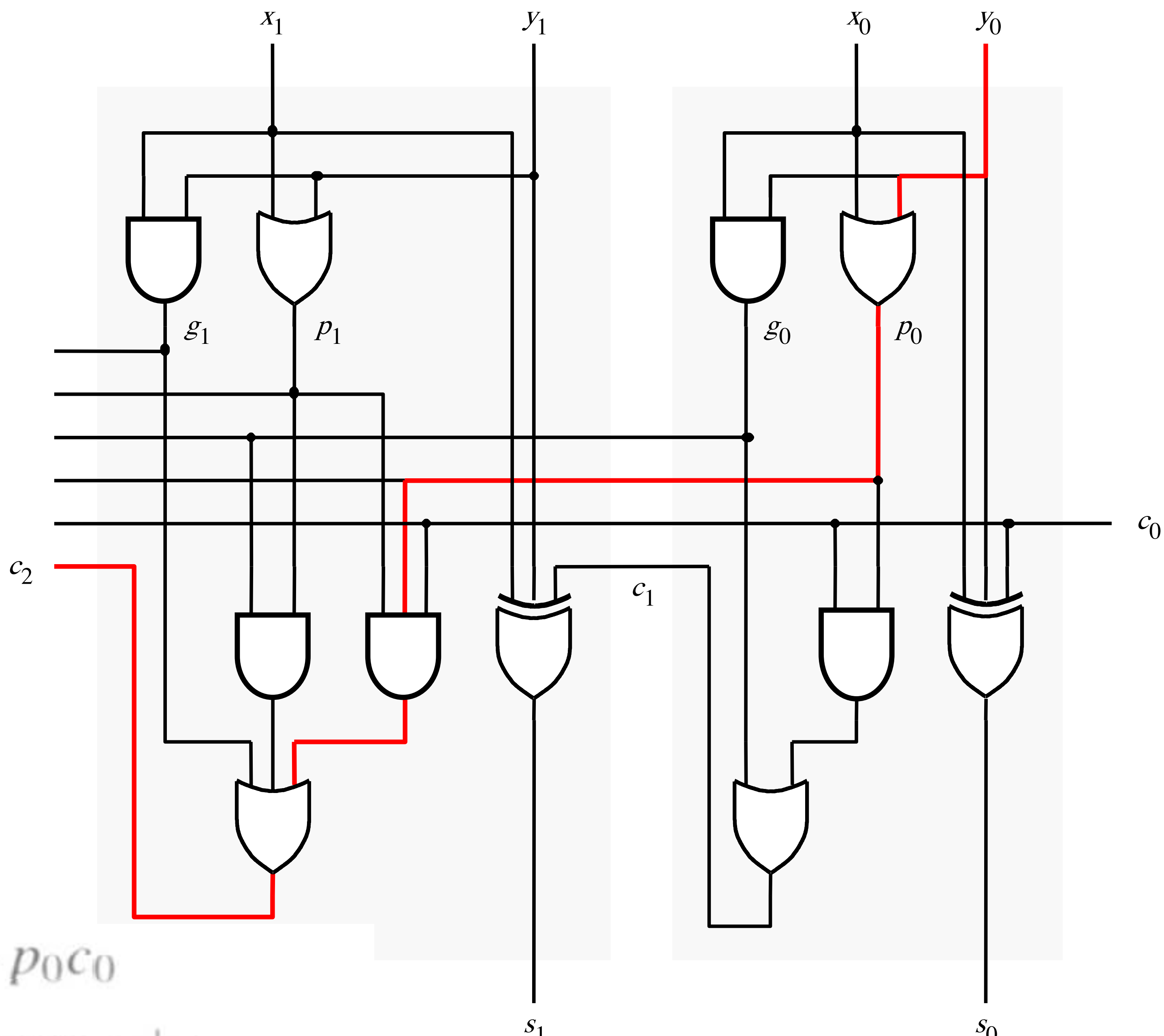
It takes 3 gate delays to generate c_1



$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

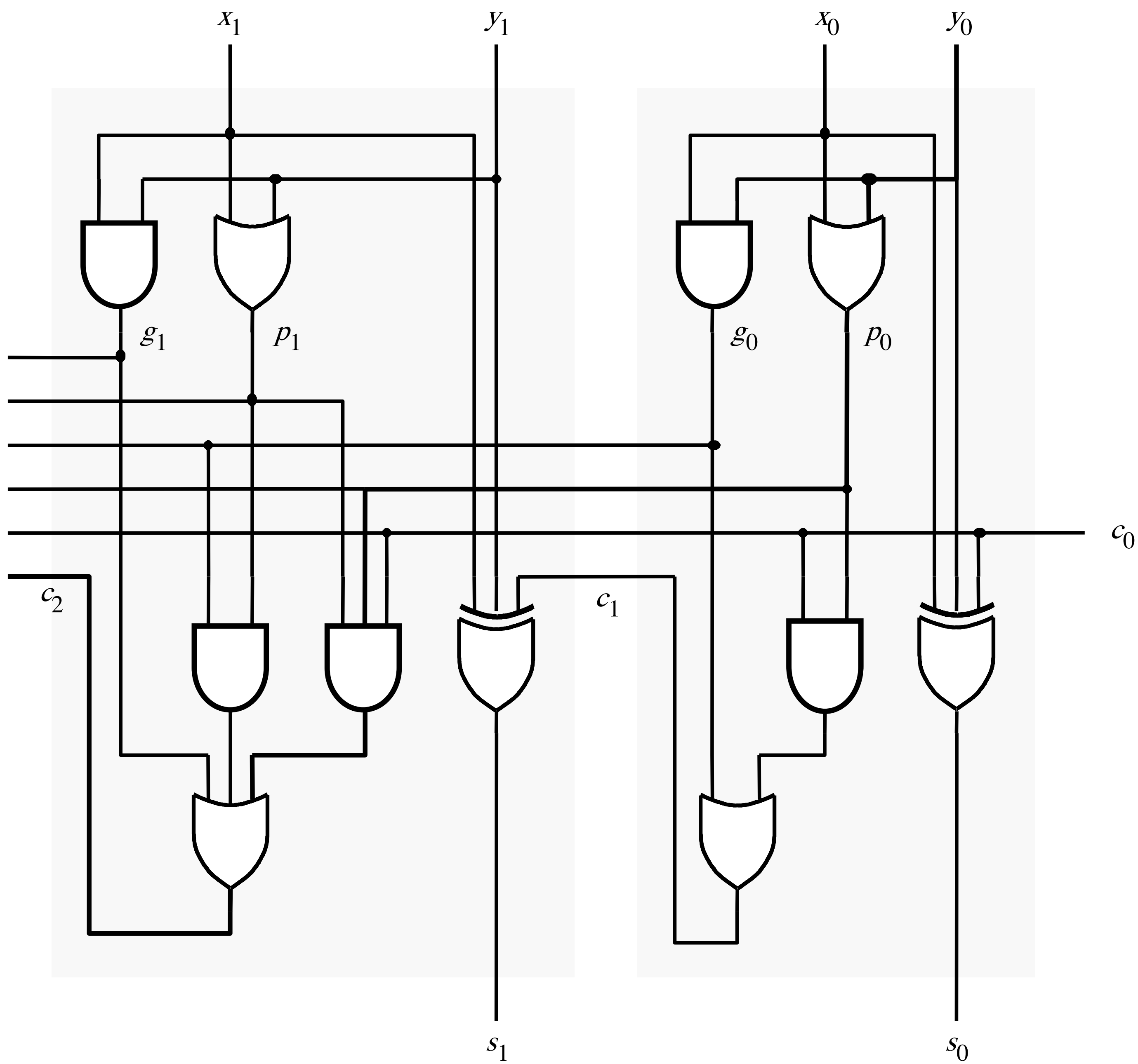
It takes 3 gate delays to generate c_2



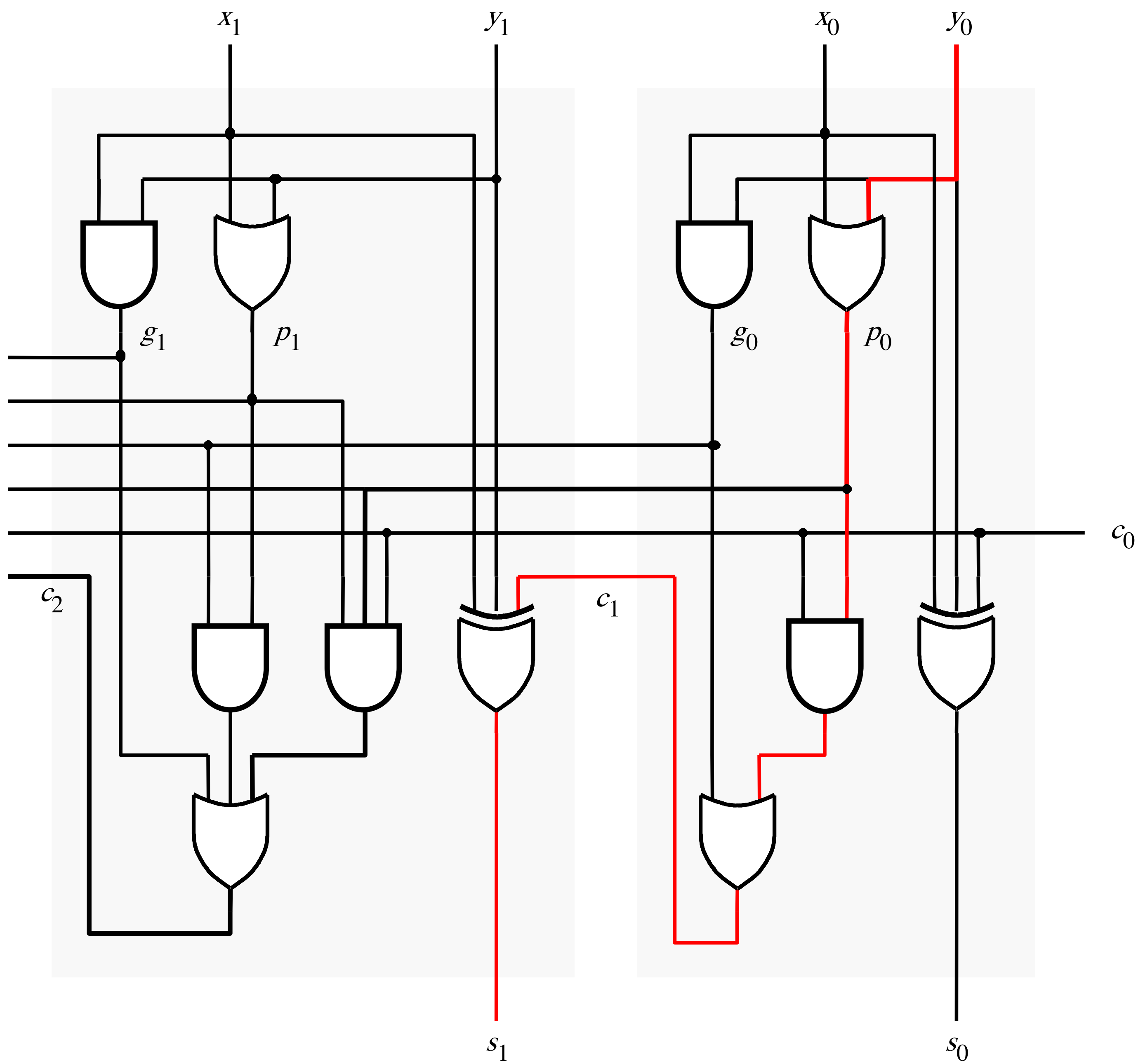
$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

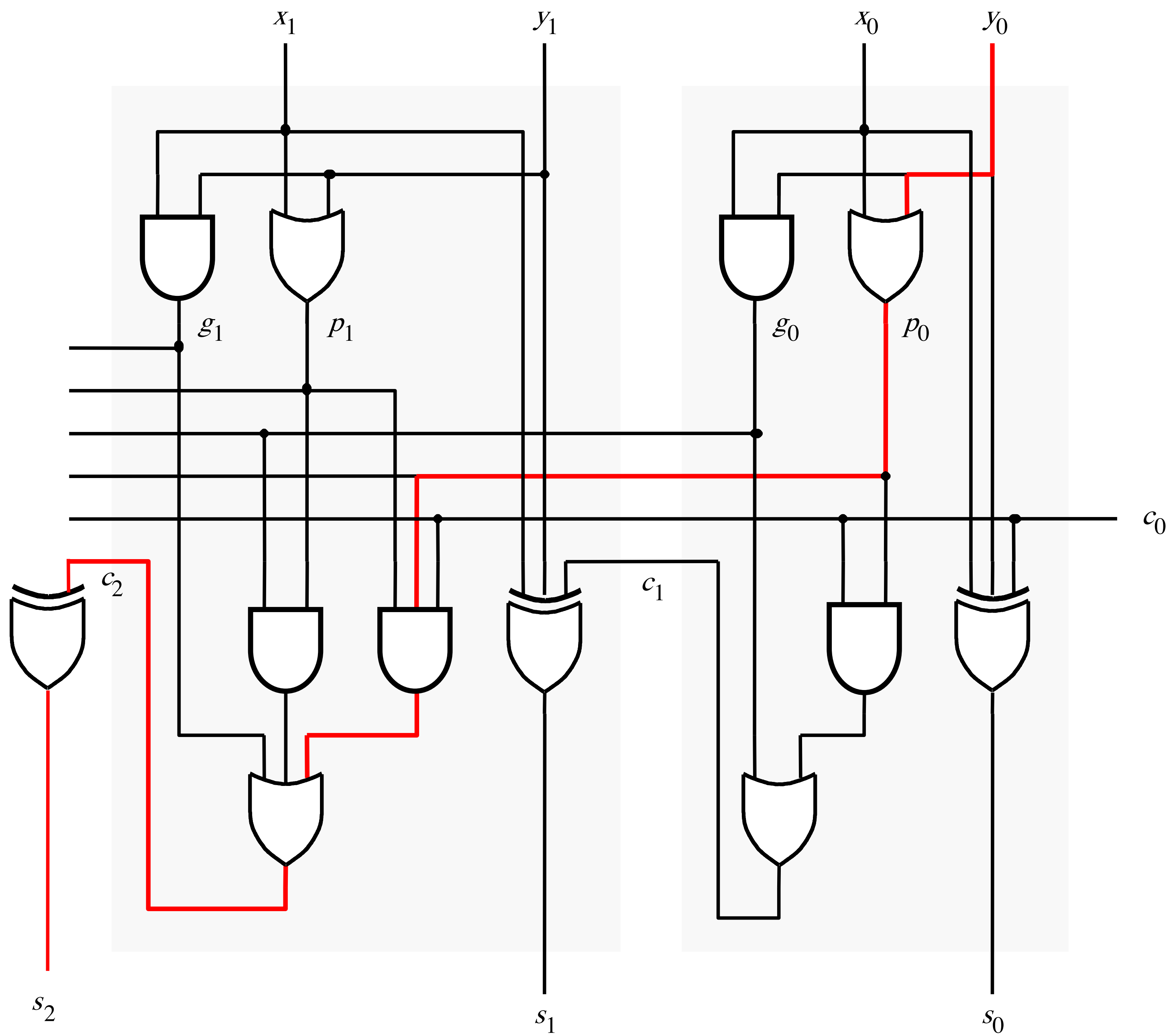
The first two stages of a carry-lookahead adder



It takes 4 gate delays to generate s_1



It takes 4 gate delays to generate s_2



N-bit Carry-Lookahead Adder

- It takes 3 gate delays to generate all carry signals
- It takes 1 more gate delay to generate all sum bits
- Thus, the total delay through an n-bit carry-lookahead adder is only 4 gate delays!

Expanding the Carry Expression

$$c_{i+1} = g_i + p_i c_i$$

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

$$c_3 = g_2 + p_2 g_1 + p_2 p_1 g_0 + p_2 p_1 p_0 c_0$$

$$\cdots$$

$$\begin{aligned} c_8 = \; & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0 c_0 \end{aligned}$$

Expanding the Carry Expression

$$c_{i+1} = g_i + p_i c_i$$

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

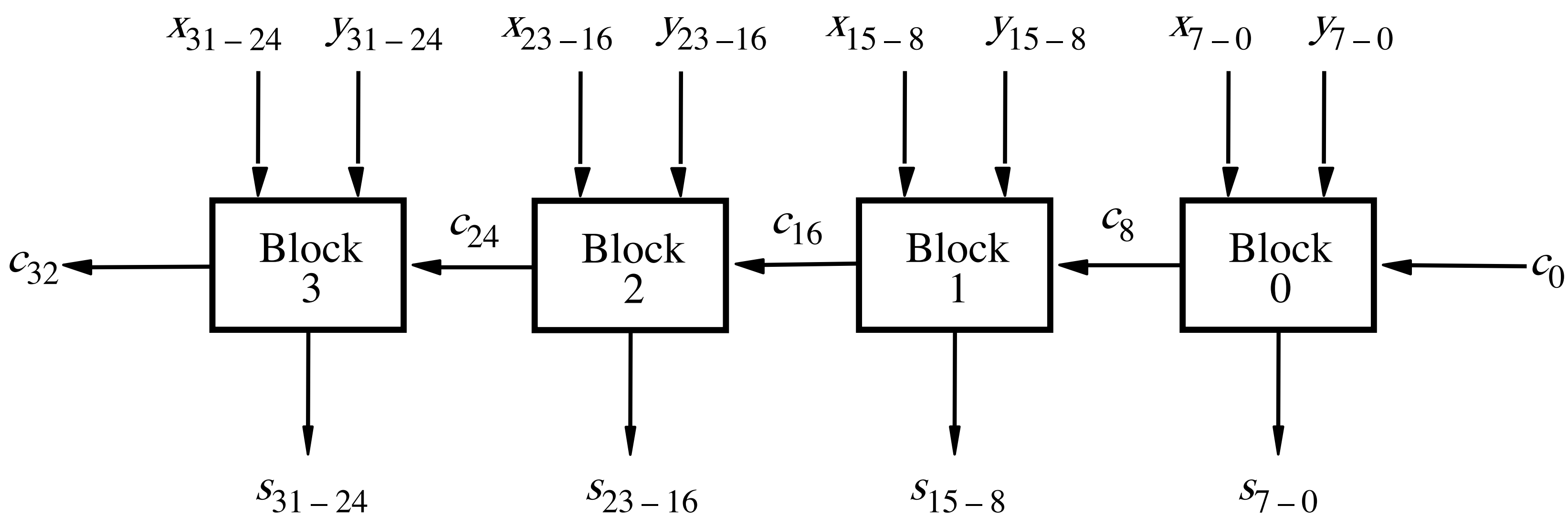
$$c_3 = g_2 + p_2 g_1 + p_2 p_1 g_0 + p_2 p_1 p_0 c_0$$

$$\ldots$$

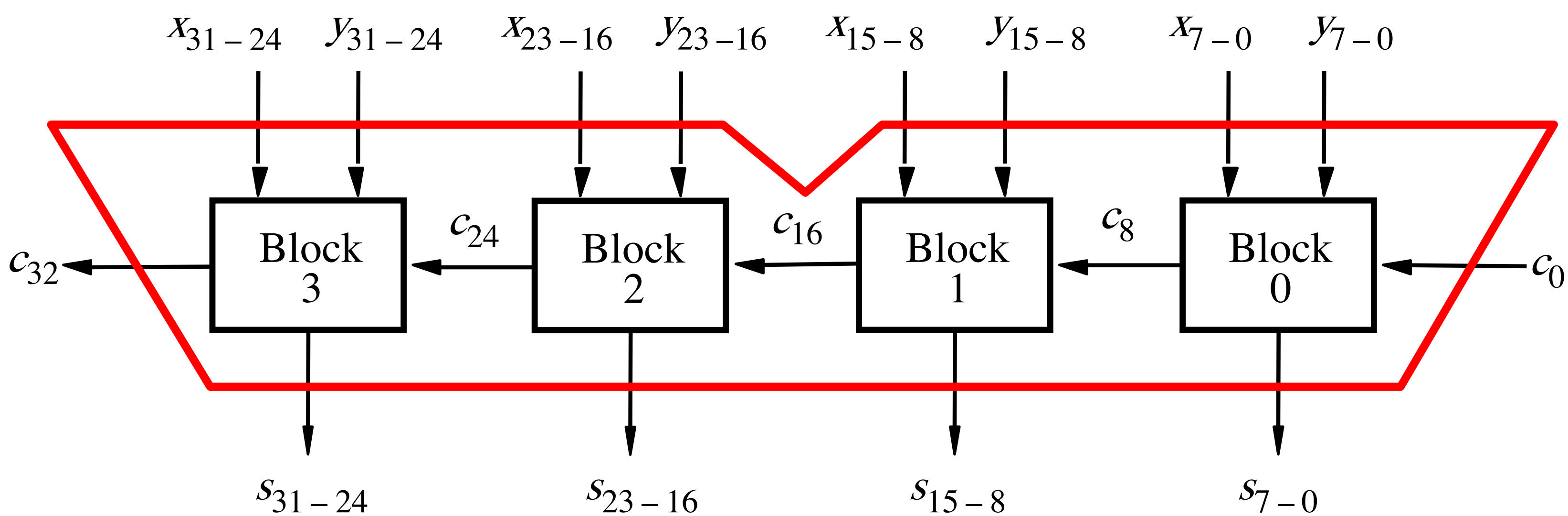
$$\begin{aligned} c_8 = {} & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0 c_0 \end{aligned}$$

Even this takes
only 3 gate delays

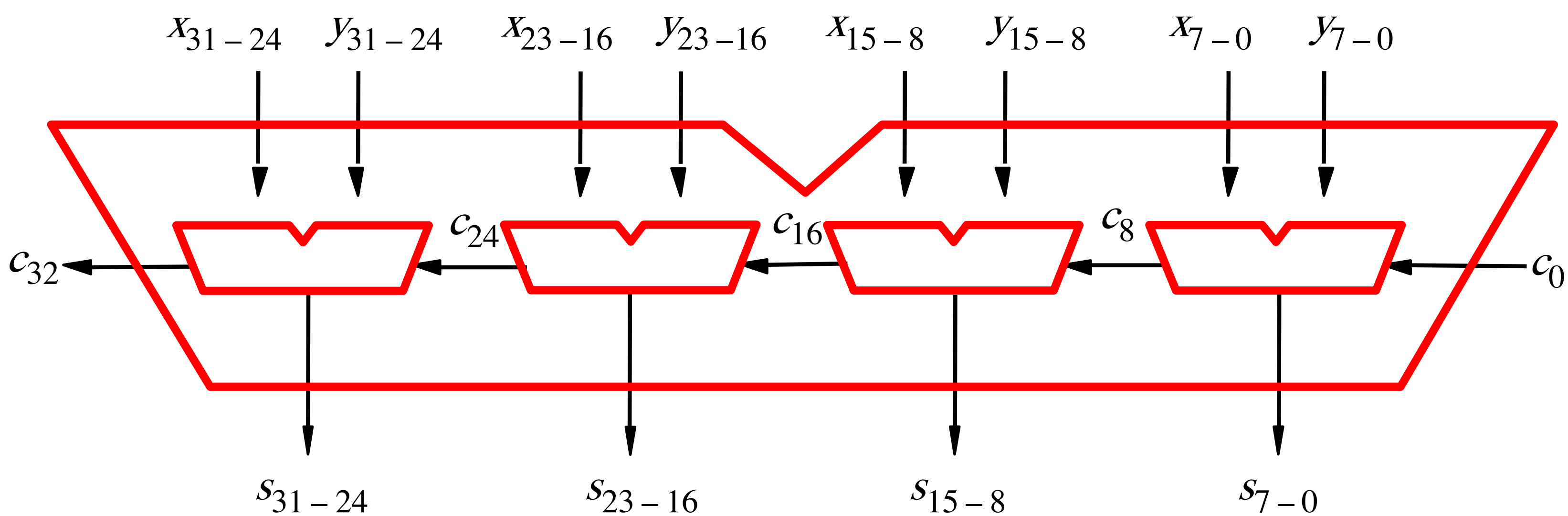
A hierarchical carry-lookahead adder with ripple-carry between blocks



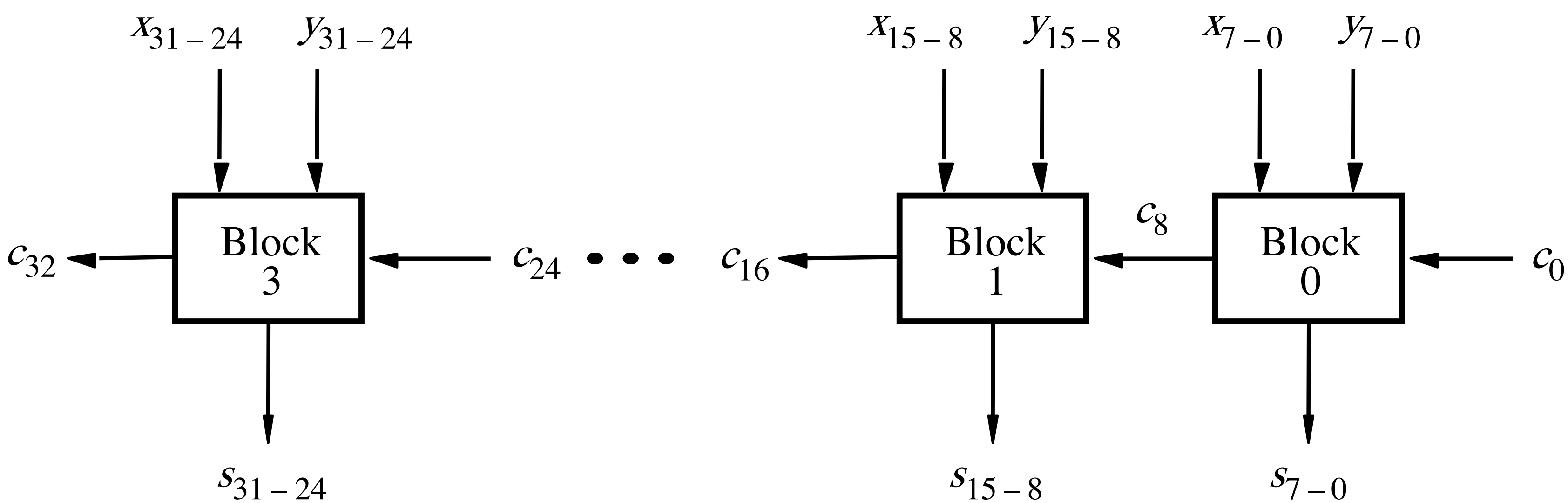
A hierarchical carry-lookahead adder with ripple-carry between blocks



A hierarchical carry-lookahead adder with ripple-carry between blocks

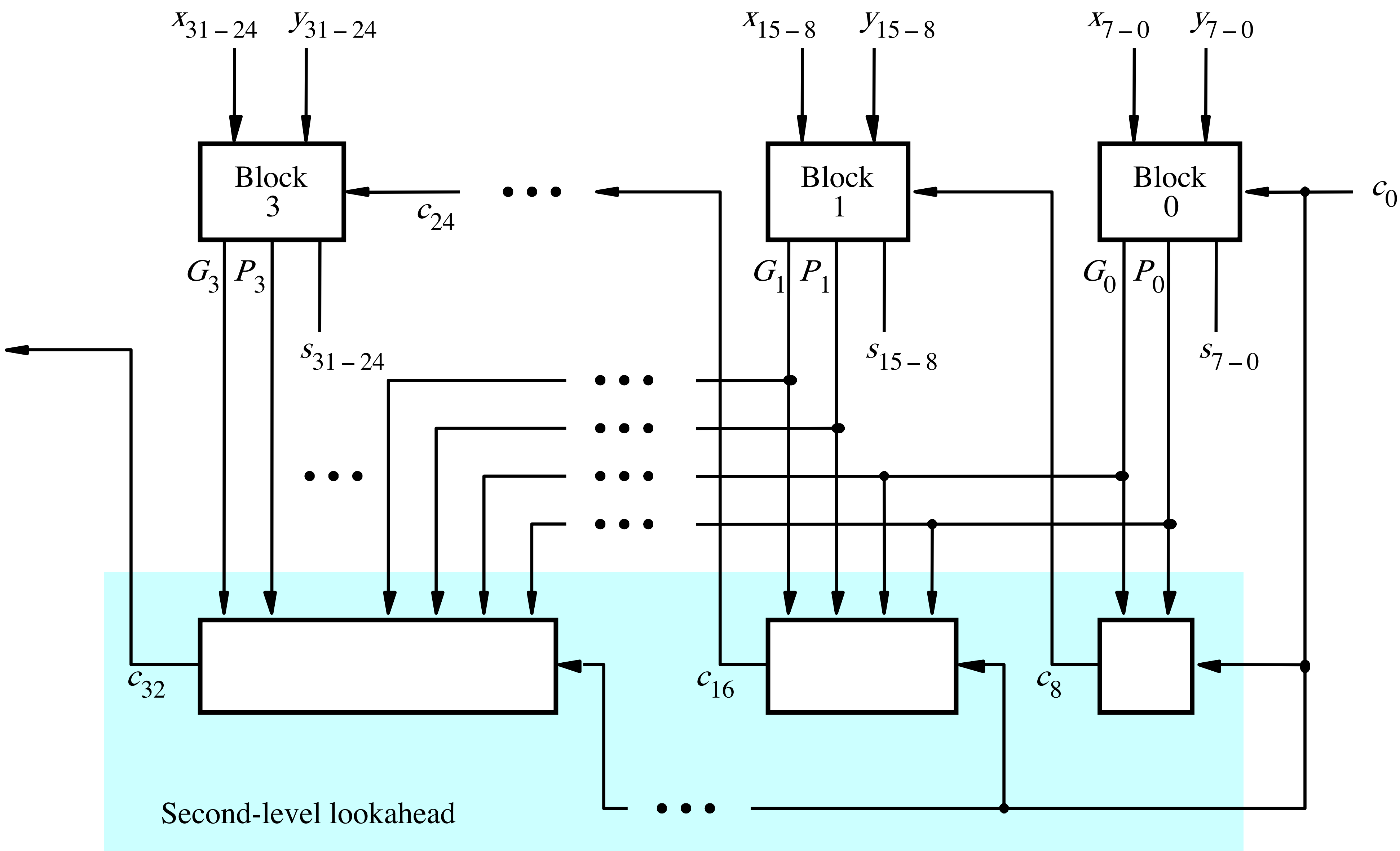


A hierarchical carry-lookahead adder with ripple-carry between blocks



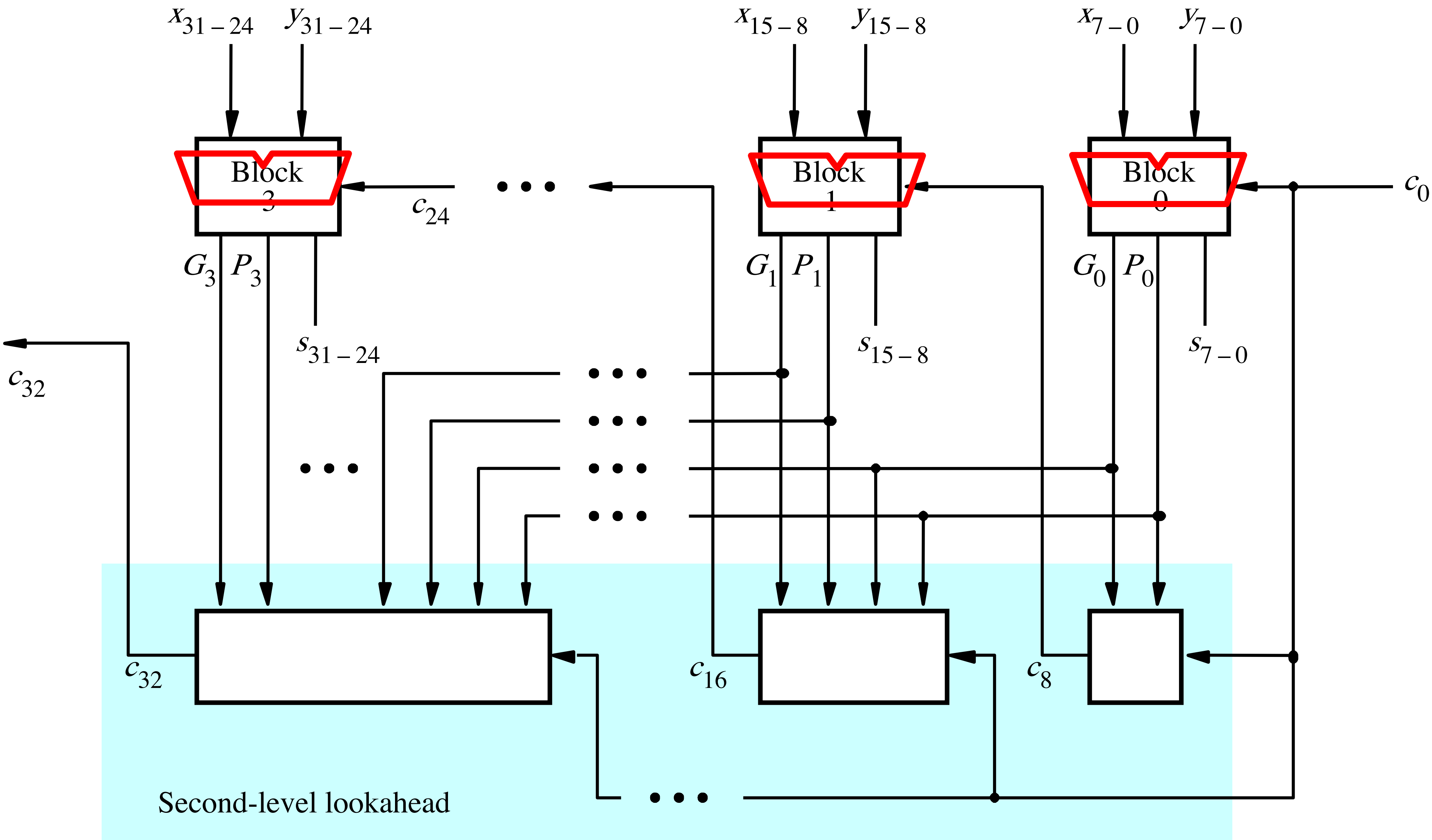
[Figure 3.16 from the textbook]

A hierarchical carry-lookahead adder

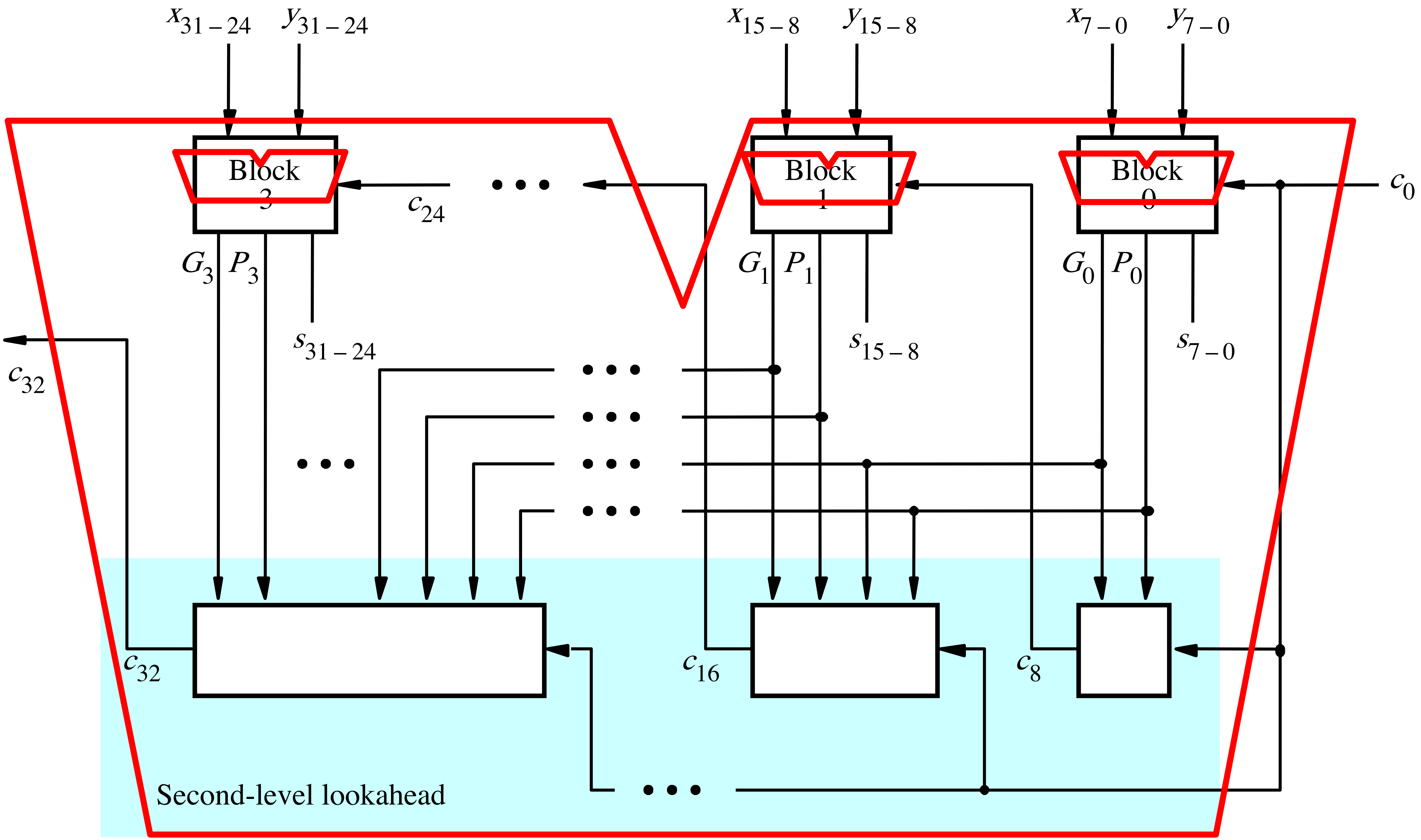


[Figure 3.17 from the textbook]

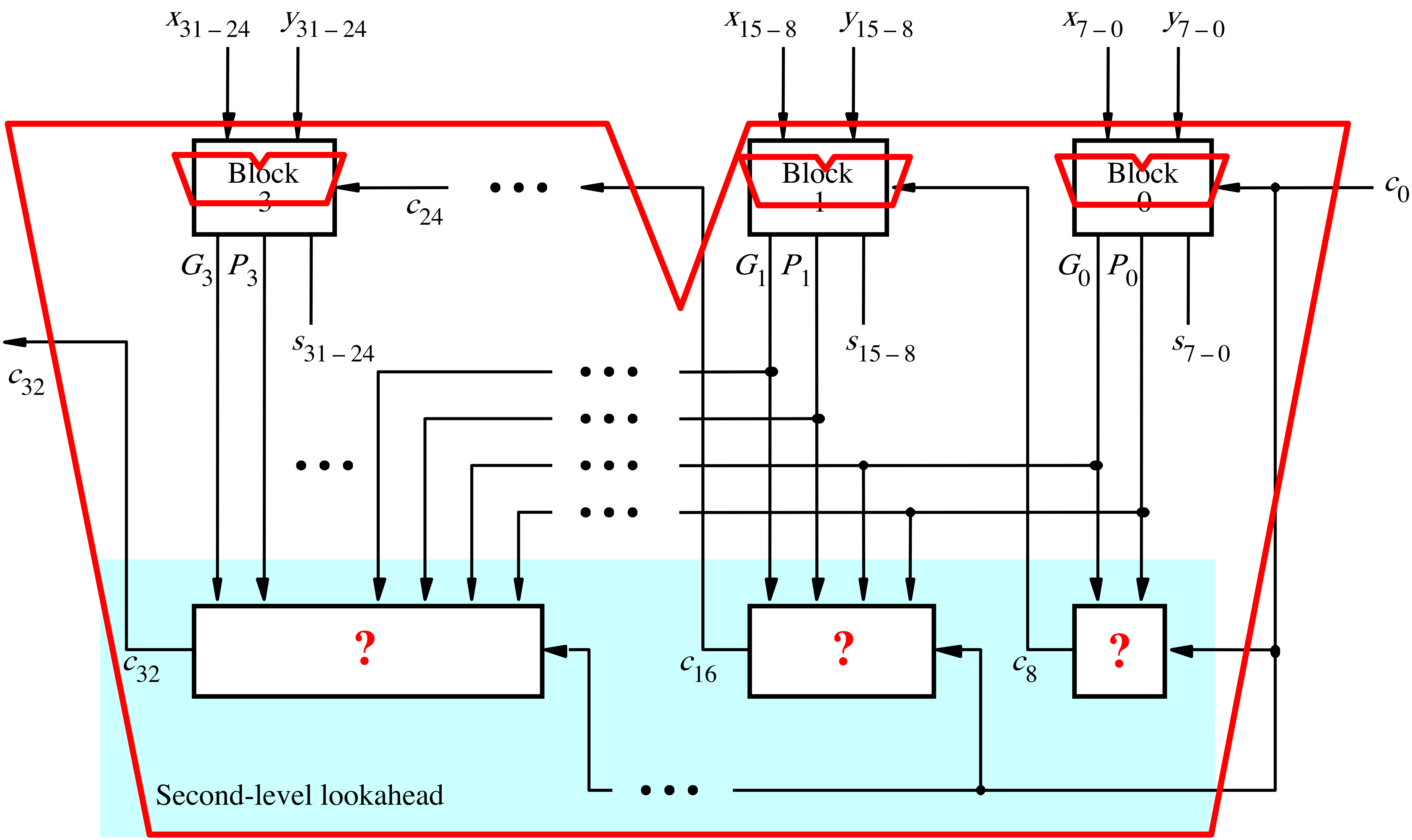
A hierarchical carry-lookahead adder



A hierarchical carry-lookahead adder



A hierarchical carry-lookahead adder



The Hierarchical Carry Expression

$$
\begin{aligned}
c_8 = \; & g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 \\
& + p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 \\
& + p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0 \\
& + p_7p_6p_5p_4p_3p_2p_1p_0c_0
\end{aligned}
$$

The Hierarchical Carry Expression

$$
\begin{aligned}
c_8 = {} & \boxed{\begin{aligned} & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \end{aligned}} \\
& + \boxed{p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0} c_0
\end{aligned}
$$

The Hierarchical Carry Expression

$$c_8 = \underbrace{g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 + p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 + p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0}_{G_0} + \underbrace{p_7p_6p_5p_4p_3p_2p_1p_0}_{P_0}c_0$$

The Hierarchical Carry Expression

$$c_8 = \underbrace{\boxed{\begin{aligned} & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \end{aligned}}}_{\mathbf{G_0}}$$

$$+ \underbrace{\boxed{p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0}}_{\mathbf{P_0}} c_0$$

$$c_8 = G_0 + P_0 c_0$$

The Hierarchical Carry Expression

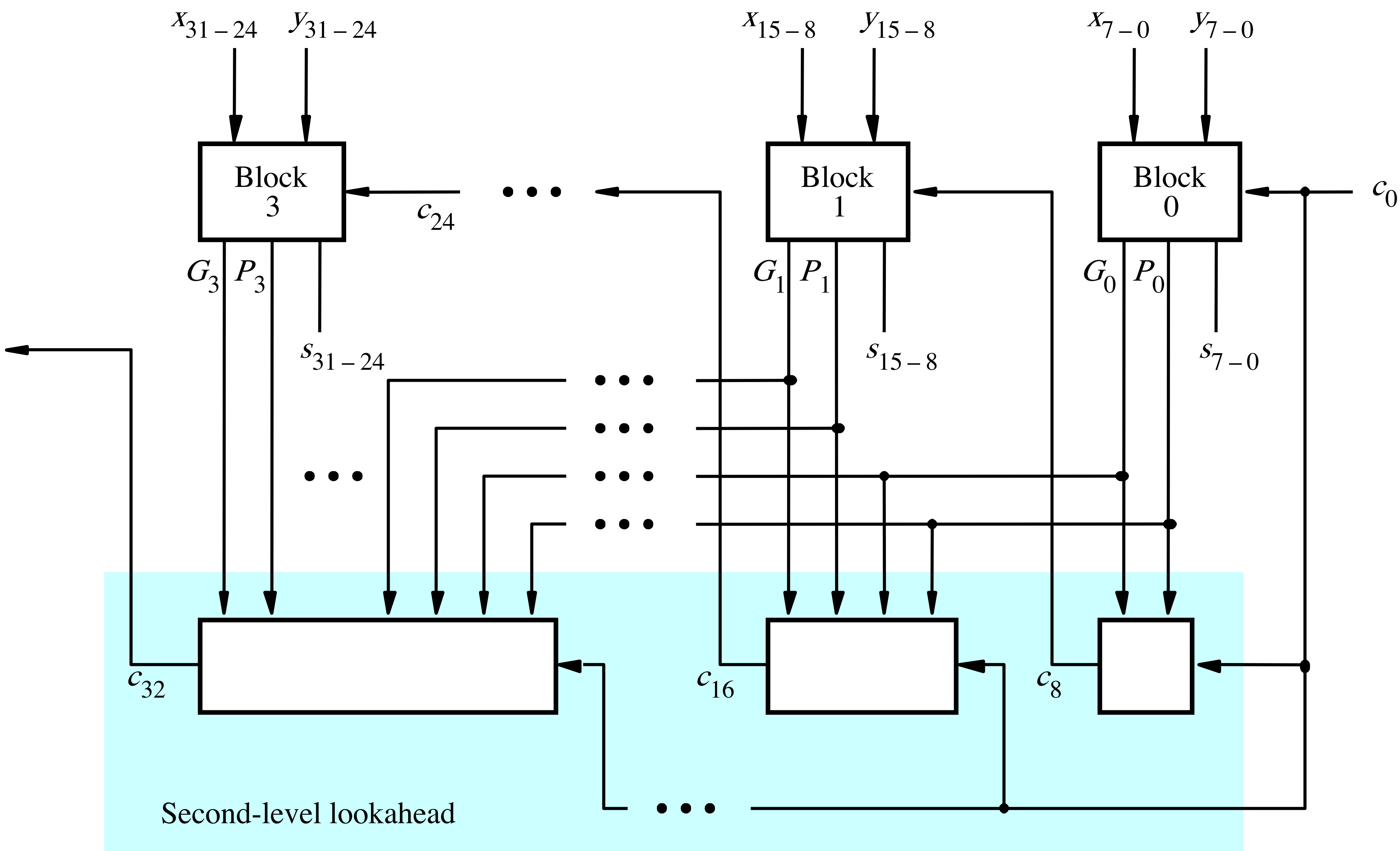
$$c_8 = G_0 + P_0 c_0$$

$$\begin{aligned} c_{16} &= G_1 + P_1 c_8 \\ &= G_1 + P_1 G_0 + P_1 P_0 c_0 \end{aligned}$$

$$c_{24} = G_2 + P_2 G_1 + P_2 P_1 G_0 + P_2 P_1 P_0 c_0$$

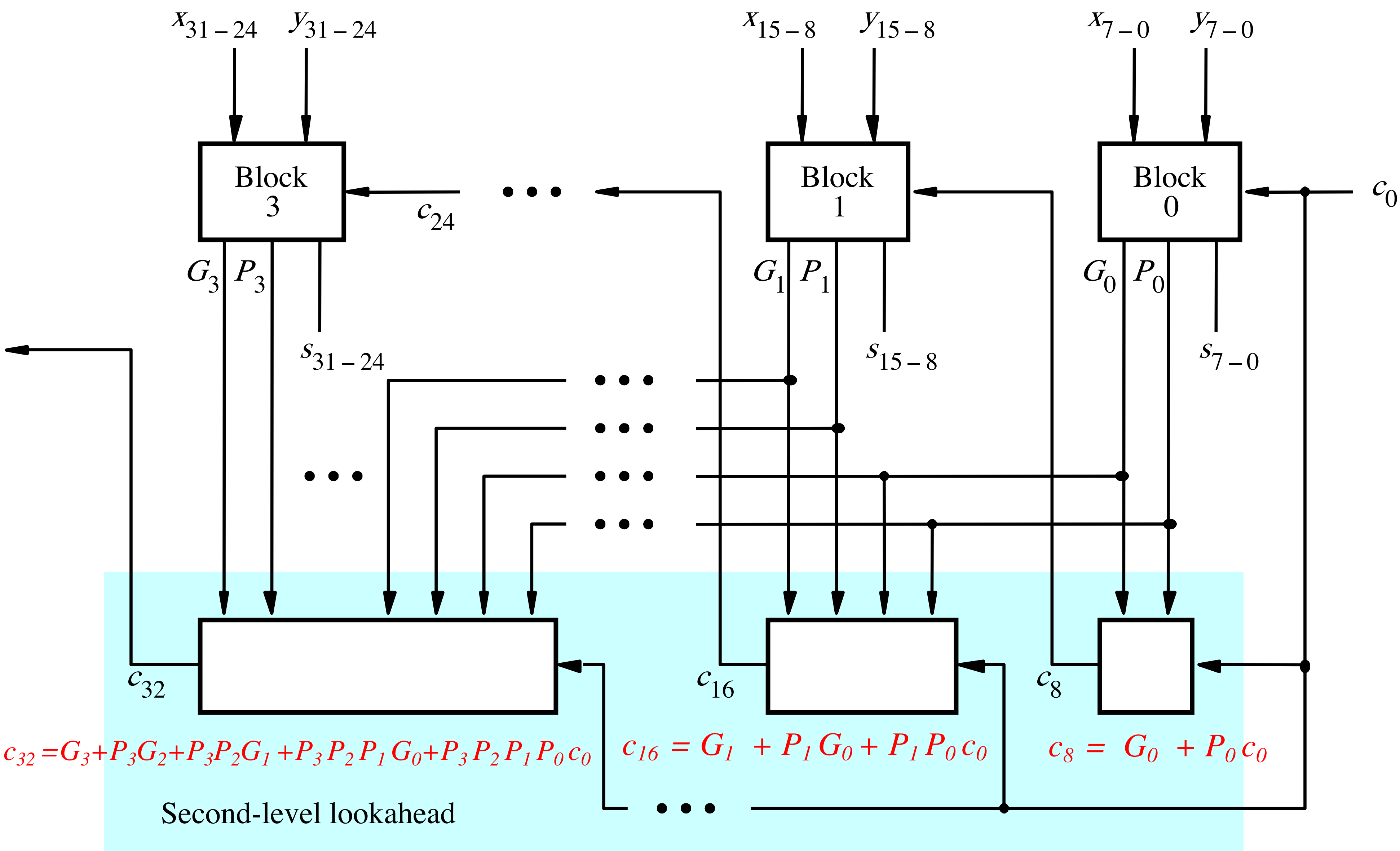
$$c_{32} = G_3 + P_3 G_2 + P_3 P_2 G_1 + P_3 P_2 P_1 G_0 + P_3 P_2 P_1 P_0 c_0$$

A hierarchical carry-lookahead adder



[Figure 3.17 from the textbook]

A hierarchical carry-lookahead adder

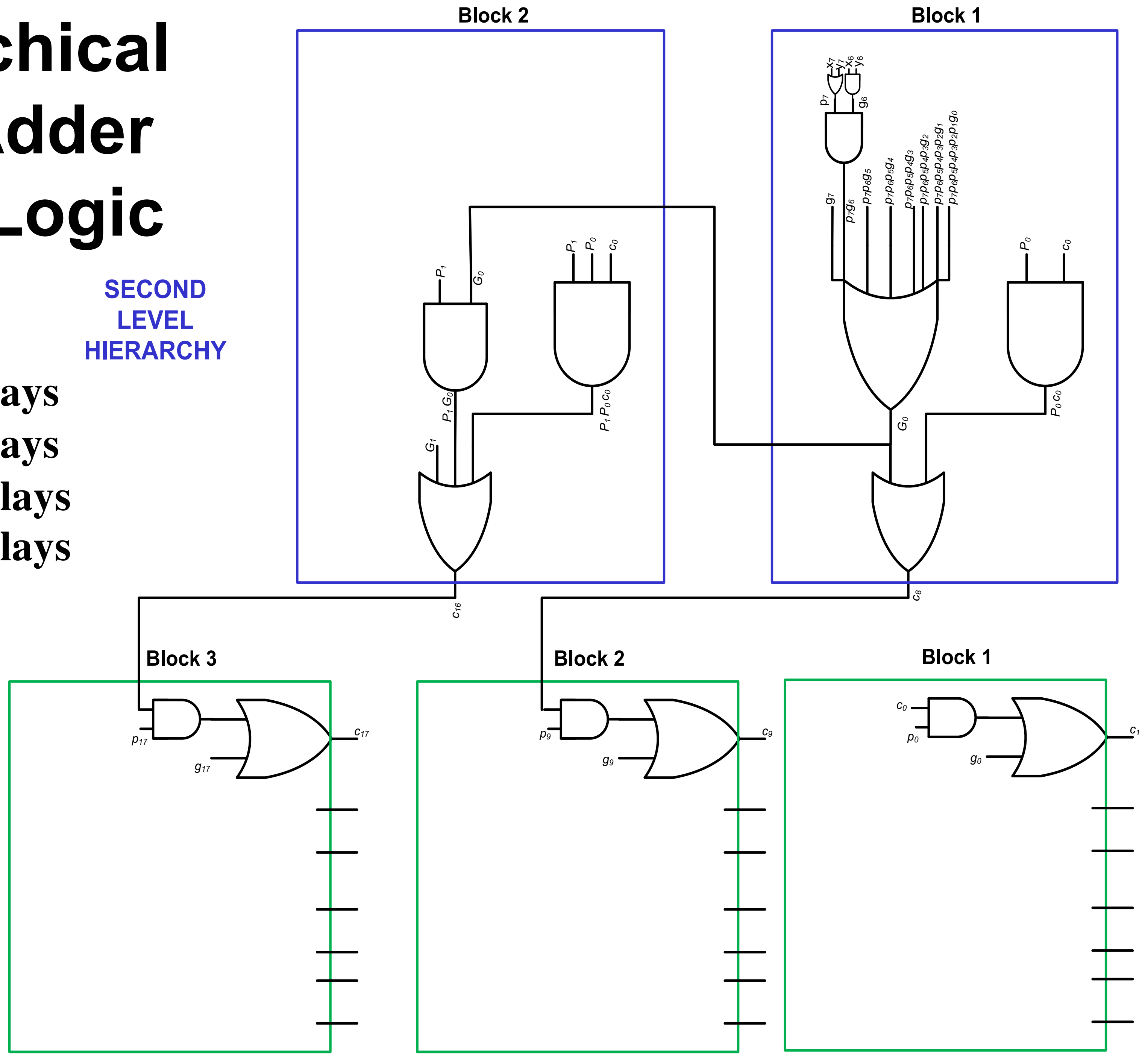


[Figure 3.17 from the textbook]

Hierarchical CLA Adder Carry Logic

SECOND LEVEL HIERARCHY

C8 – 5 gate delays
C16 – 5 gate delays
C24 – 5 Gate delays
C32 – 5 Gate delays



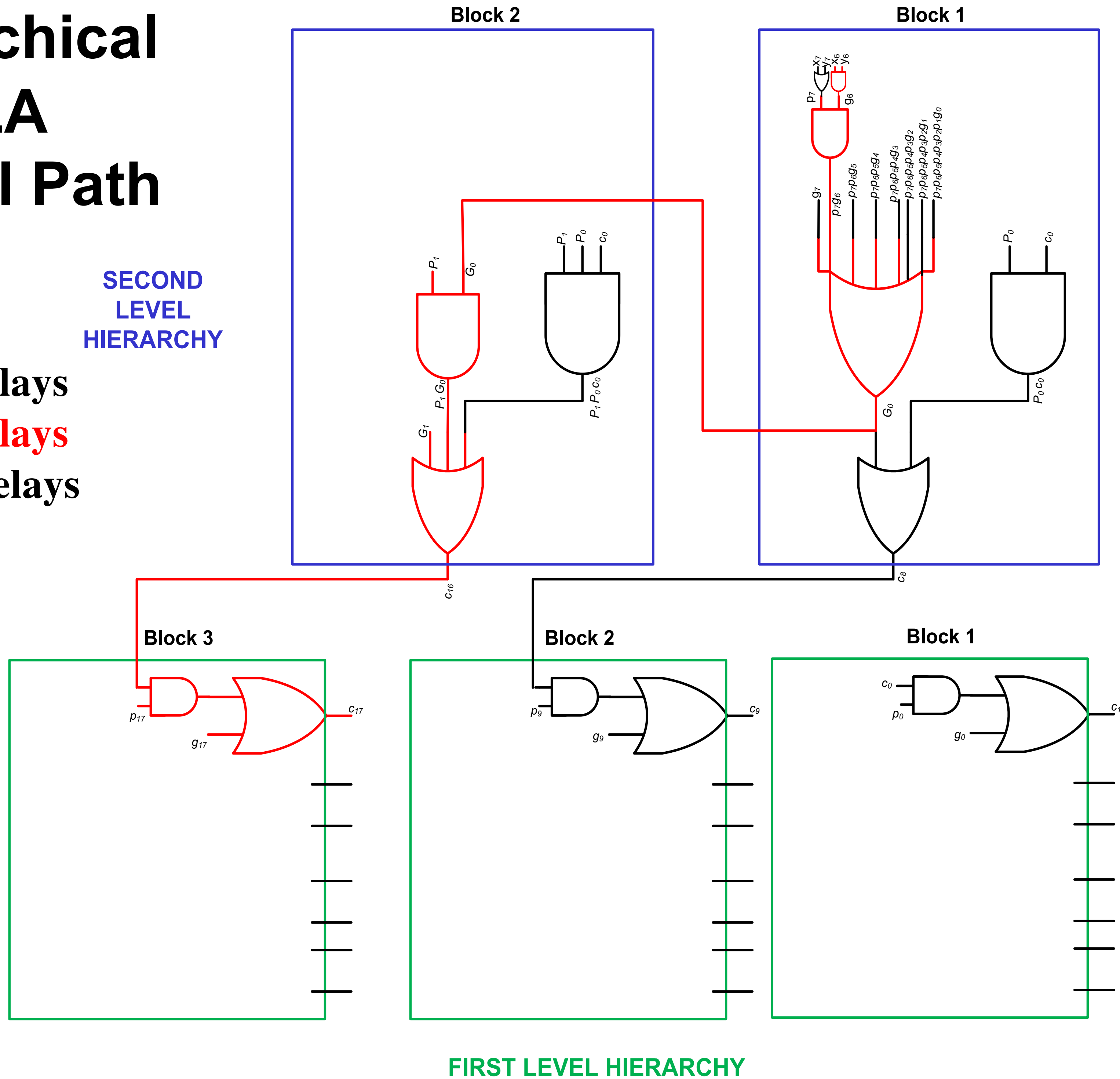
FIRST LEVEL HIERARCHY

Hierarchical CLA Critical Path

C9 – 7 gate delays

C17 – 7 gate delays

C25 – 7 Gate delays



Total Gate Delay Through a Hierarchical Carry-Lookahead Adder

- Is 8 gates
  - 3 to generate all Gj and Pj
  - +2 to generate c8, c16, c24, and c32
  - +2 to generate internal carries in the blocks
  - +1 to generate the sum bits (one extra XOR)

Decimal Multiplication by 10

What happens when we multiply a number by 10?

4 x 10 = ?

542 x 10 = ?

1245 x 10 = ?

Decimal Multiplication by 10

What happens when we multiply a number by 10?

4 x 10 = 40

542 x 10 = 5420

1245 x 10 = 12450

Decimal Multiplication by 10

What happens when we multiply a number by 10?

4 x 10 = 40

542 x 10 = 5420

1245 x 10 = 12450

You simply add a zero as the rightmost number

Decimal Division by 10

What happens when we divide a number by 10?

14 / 10 = ?

540 / 10 = ?

1240 x 10 = ?

Decimal Division by 10

What happens when we divide a number by 10?

14 / 10 = 1 //integer division

540 / 10 = 54

1240 x 10 = 124

You simply delete the rightmost number

Binary Multiplication by 2

What happens when we multiply a number by 2?

011 times 2 = ?

101 times 2 = ?

110011 times 2 = ?

Binary Multiplication by 2

What happens when we multiply a number by 2?

011 times 2 = 0110

101 times 2 = 1010

110011 times 2 = 1100110

You simply add a zero as the rightmost number

Binary Multiplication by 4

What happens when we multiply a number by 4?

011 times 4 = ?

101 times 4 = ?

110011 times 4 = ?

Binary Multiplication by 4

What happens when we multiply a number by 4?

011 times 4 = 01100

101 times 4 = 10100

110011 times 4 = 11001100

add two zeros in the last two bits and shift everything else to the left

Binary Multiplication by 2^N

What happens when we multiply a number by 2^N?

011 times 2^N = 01100…0 // add N zeros

101 times 4 = 10100…0 // add N zeros

110011 times 4 = 11001100…0 // add N zeros

Binary Division by 2

What happens when we divide a number by 2?

0110 divided by 2 = ?

1010 divides by 2 = ?

110011 divides by 2 = ?

Binary Division by 2

What happens when we divide a number by 2?

0110 divided by 2 = 011

1010 divides by 2 = 101

110011 divides by 2 = 11001

You simply delete the rightmost number

Decimal Multiplication By Hand

$$
\begin{array}{r}
5127 \\
\times\ 4265 \\
\hline
25635 \\
307620 \\
1025400 \\
20508000 \\
\hline
21866655
\end{array}
$$

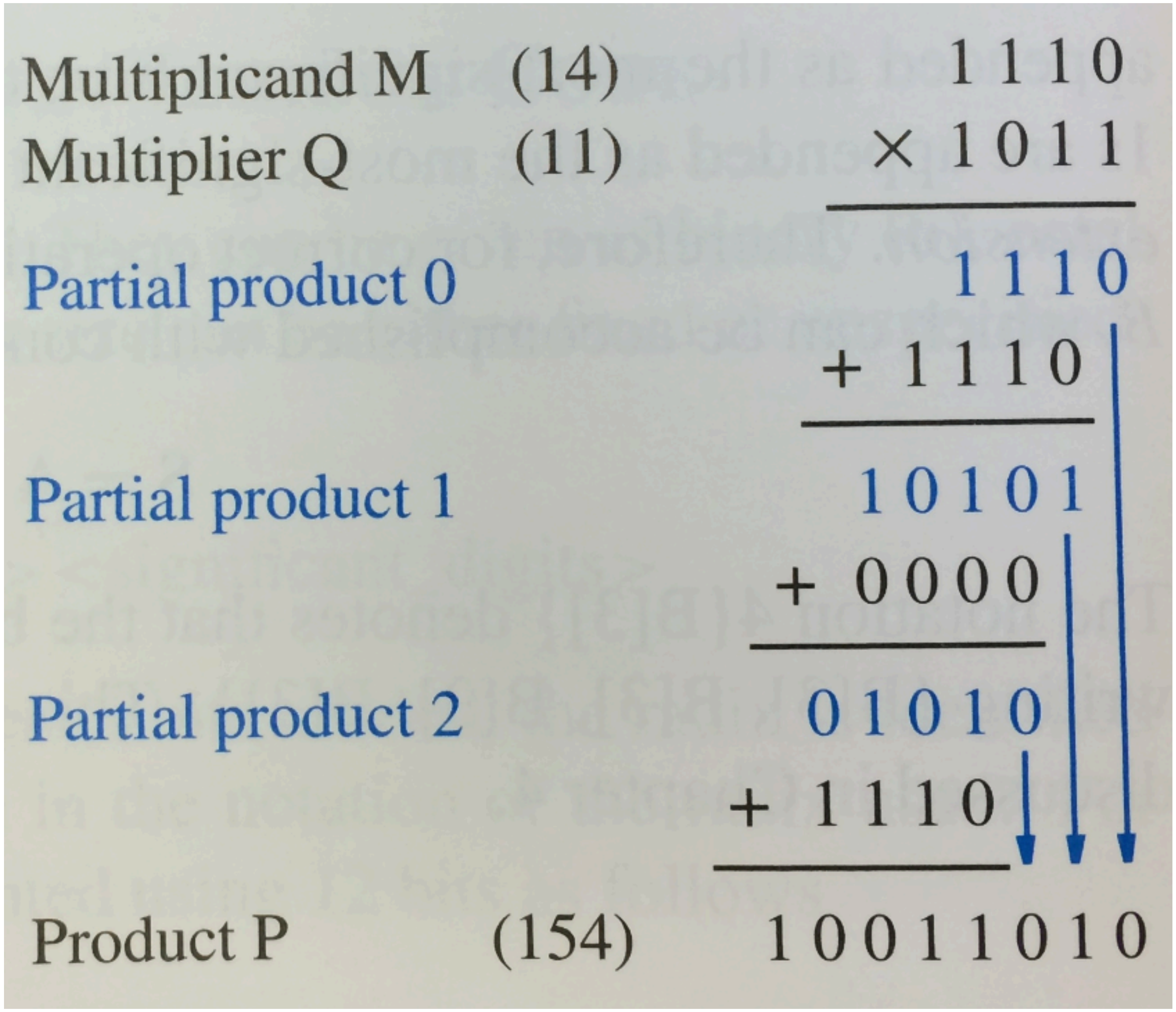
[http://www.ducksters.com/kidsmath/long_multiplication.php]

Binary Multiplication By Hand

Multiplicand M	(14)					1	1	1	0
Multiplier Q	(11)				x	1	0	1	1
						1	1	1	0
					1	1	1	0	
				0	0	0	0		
			1	1	1	0			
Product P	(154)	1	0	0	1	1	0	1	0

[Figure 3.34a from the textbook]

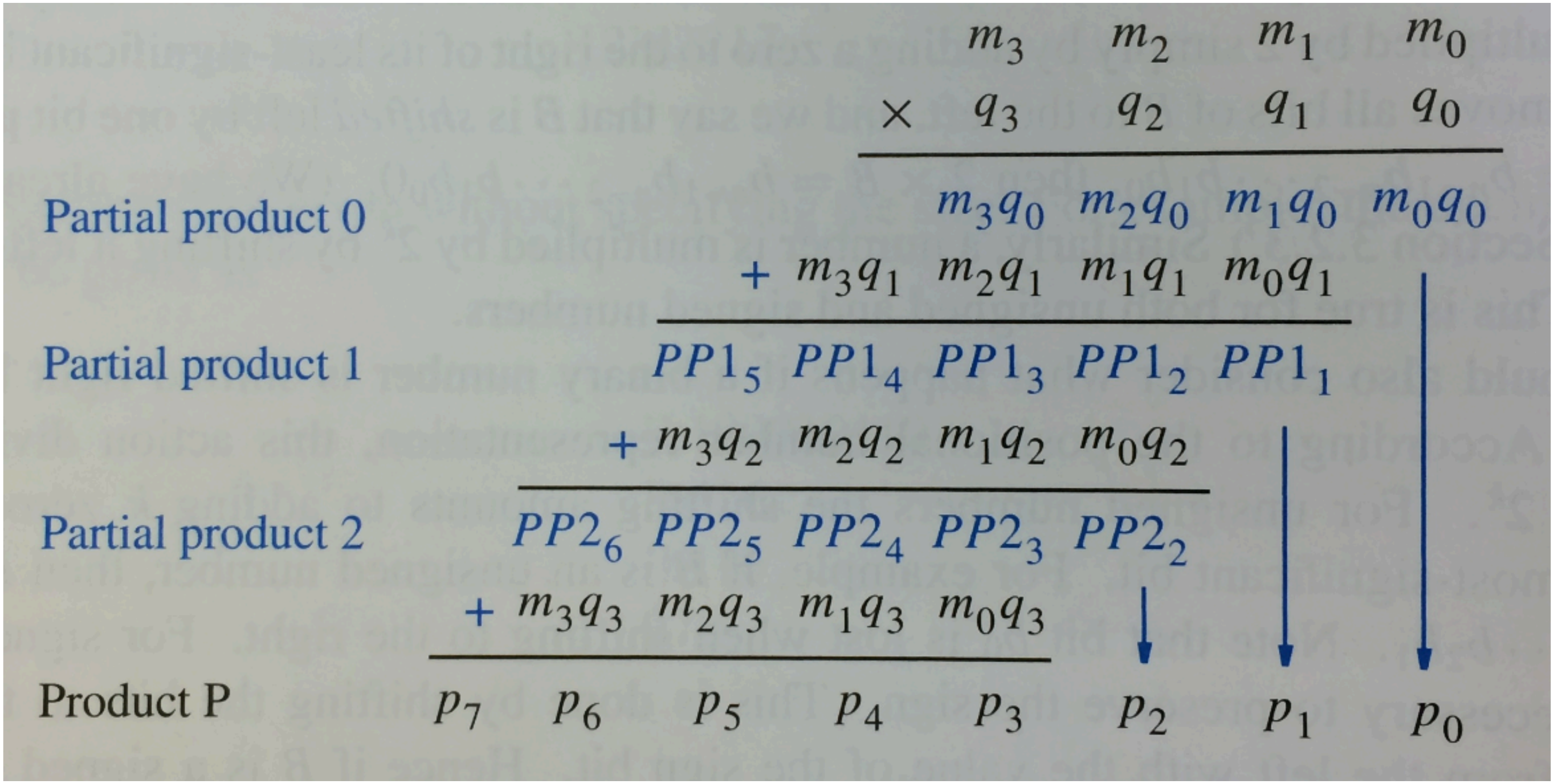
Binary Multiplication By Hand



Multiplicand M	(14)	1110
Multiplier Q	(11)	× 1011
Partial product 0		1110
		+ 1110
Partial product 1		10101
		+ 0000
Partial product 2		01010
		+ 1110
Product P	(154)	10011010

[Figure 3.34b from the textbook]

Binary Multiplication By Hand



[Figure 3.34c from the textbook]

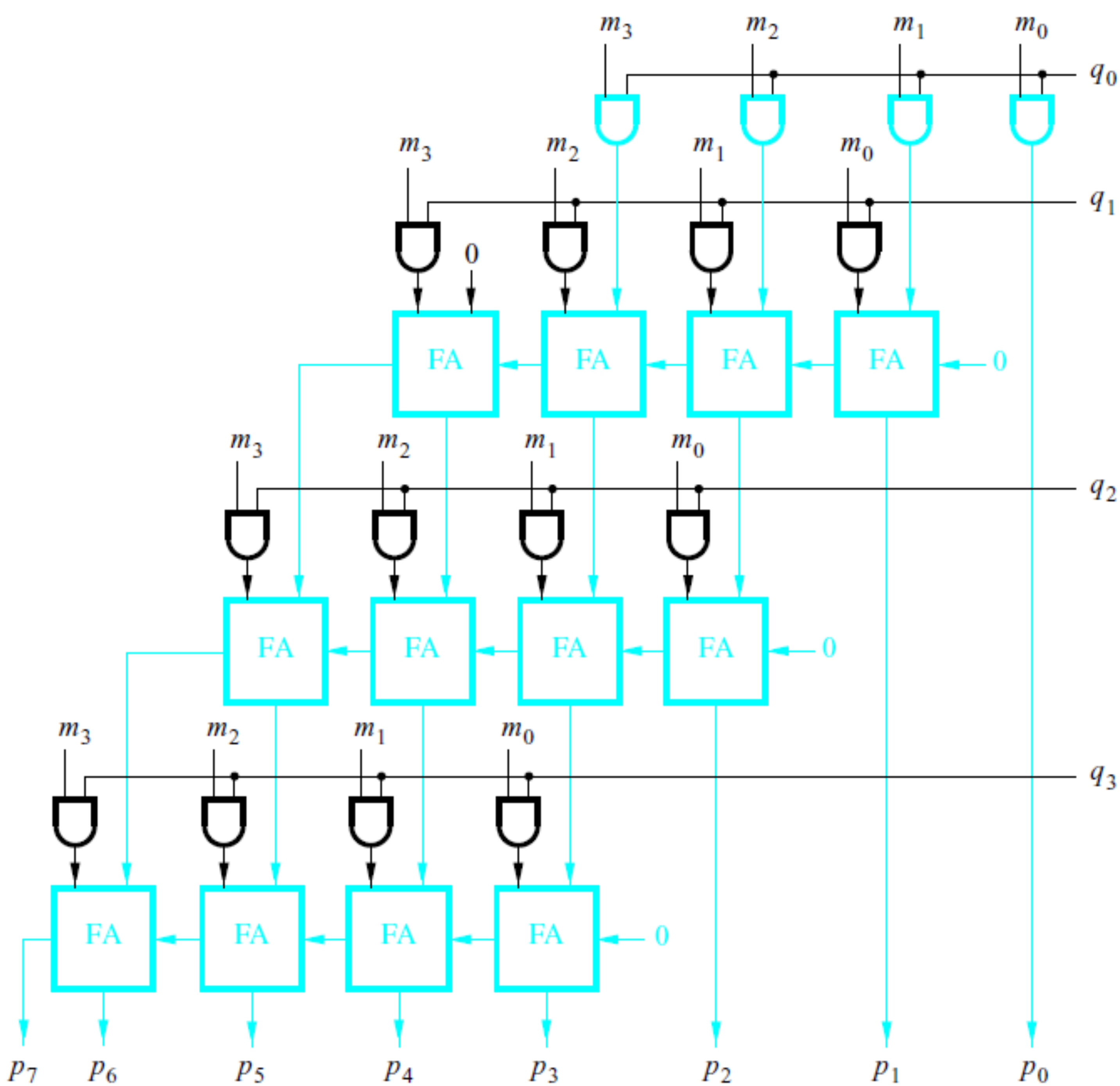


Figure 3.35. A 4x4 multiplier circuit.

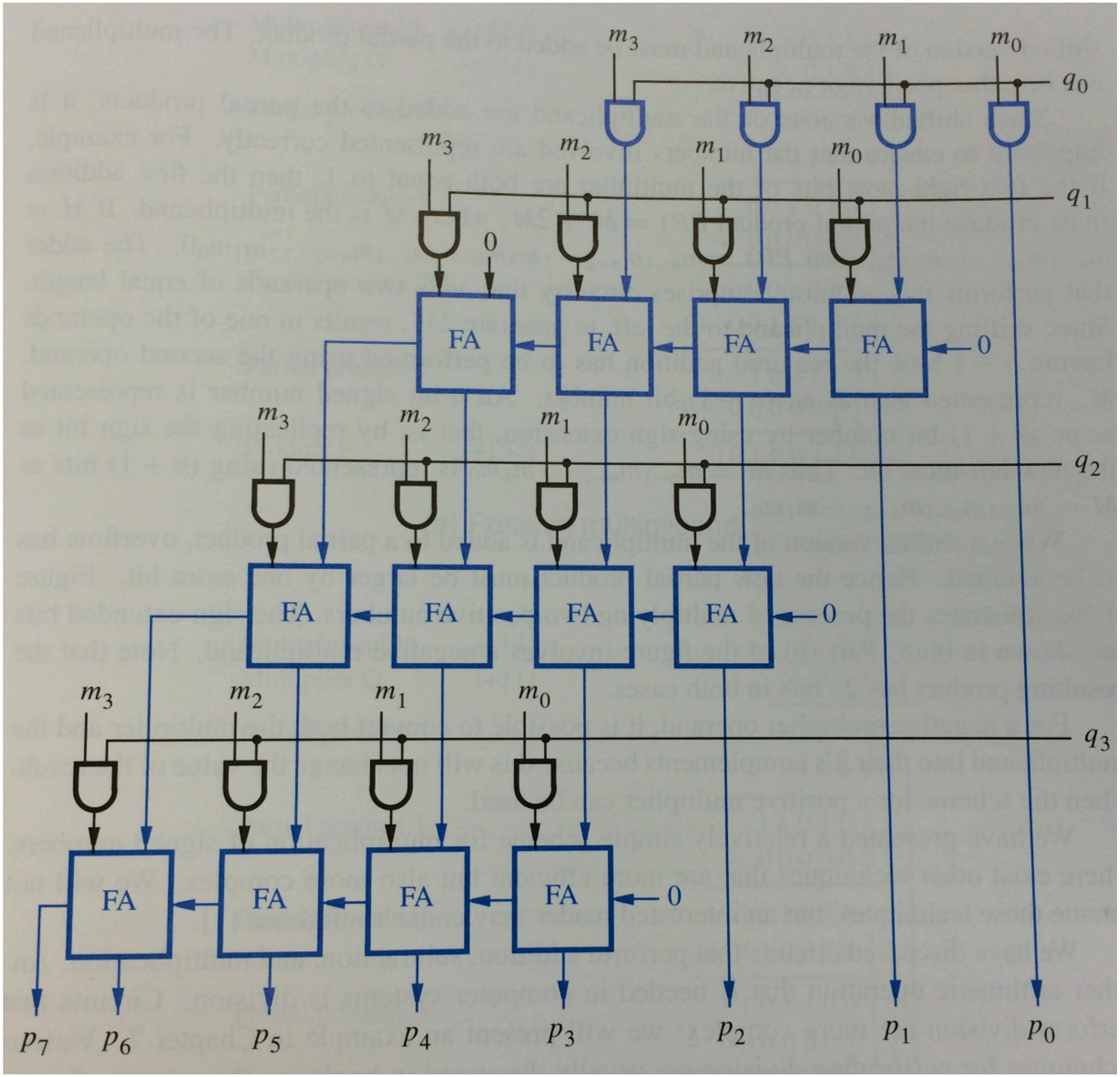


Figure 3.35. A 4x4 multiplier circuit.

Positive Multiplicand Example

Multiplicand M	(+14)	0 1 1 1 0
Multiplier Q	(+11)	x 0 1 0 1 1
Partial product 0		0 0 0 1 1 1 0
		+ 0 0 1 1 1 0
Partial product 1		0 0 1 0 1 0 1
		+ 0 0 0 0 0 0
Partial product 2		0 0 0 1 0 1 0
		+ 0 0 1 1 1 0
Partial product 3		0 0 1 0 0 1 1
		+ 0 0 0 0 0 0
Product P	(+154)	0 0 1 0 0 1 1 0 1 0

[Figure 3.36a in the textbook]

Positive Multiplicand Example

Multiplicand M	(+14)	0 1 1 1 0
Multiplier Q	(+11)	x 0 1 0 1 1
Partial product 0		0 0 0 1 1 1 0
	add an extra bit to avoid overflow	+ 0 0 1 1 1 0
Partial product 1		0 0 1 0 1 0 1
		+ 0 0 0 0 0 0
Partial product 2		0 0 0 1 0 1 0
		+ 0 0 1 1 1 0
Partial product 3		0 0 1 0 0 1 1
		+ 0 0 0 0 0 0
Product P	(+154)	0 0 1 0 0 1 1 0 1 0

[Figure 3.36a in the textbook]

Negative Multiplicand Example

Multiplicand M	(–14)	1 0 0 1 0
Multiplier Q	(+11)	× 0 1 0 1 1
Partial product 0		1 1 1 0 0 1 0
		+ 1 1 0 0 1 0
Partial product 1		1 1 0 1 0 1 1
		+ 0 0 0 0 0 0
Partial product 2		1 1 1 0 1 0 1
		+ 1 1 0 0 1 0
Partial product 3		1 1 0 1 1 0 0
		+ 0 0 0 0 0 0
Product P	(–154)	1 1 0 1 1 0 0 1 1 0

[Figure 3.36b in the textbook]

Negative Multiplicand Example

Multiplicand M	(–14)	1 0 0 1 0
Multiplier Q	(+11)	× 0 1 0 1 1
Partial product 0		1 1 1 0 0 1 0
		+ 1 1 0 0 1 0
Partial product 1		1 1 0 1 0 1 1
		+ 0 0 0 0 0 0
Partial product 2		1 1 1 0 1 0 1
		+ 1 1 0 0 1 0
Partial product 3		1 1 0 1 1 0 0
		+ 0 0 0 0 0 0
Product P	(–154)	1 1 0 1 1 0 0 1 1 0

add an extra bit
to avoid overflow
but now it is 1

[Figure 3.36b in the textbook]

What if the Multiplier is Negative?

- Convert both to their 2's complement version
- This will make the multiplier positive
- Then Proceed as normal
- This will not affect the result
- Example: 5*(-4) = (-5)*(4)= -20

Binary Coded Decimal

Table of Binary-Coded Decimal Digits

Decimal digit	BCD code
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

Addition of BCD digits

X	0111	7
$+ Y$	+ 0101	+ 5
Z	1100	12

[Figure 3.38a in the textbook]

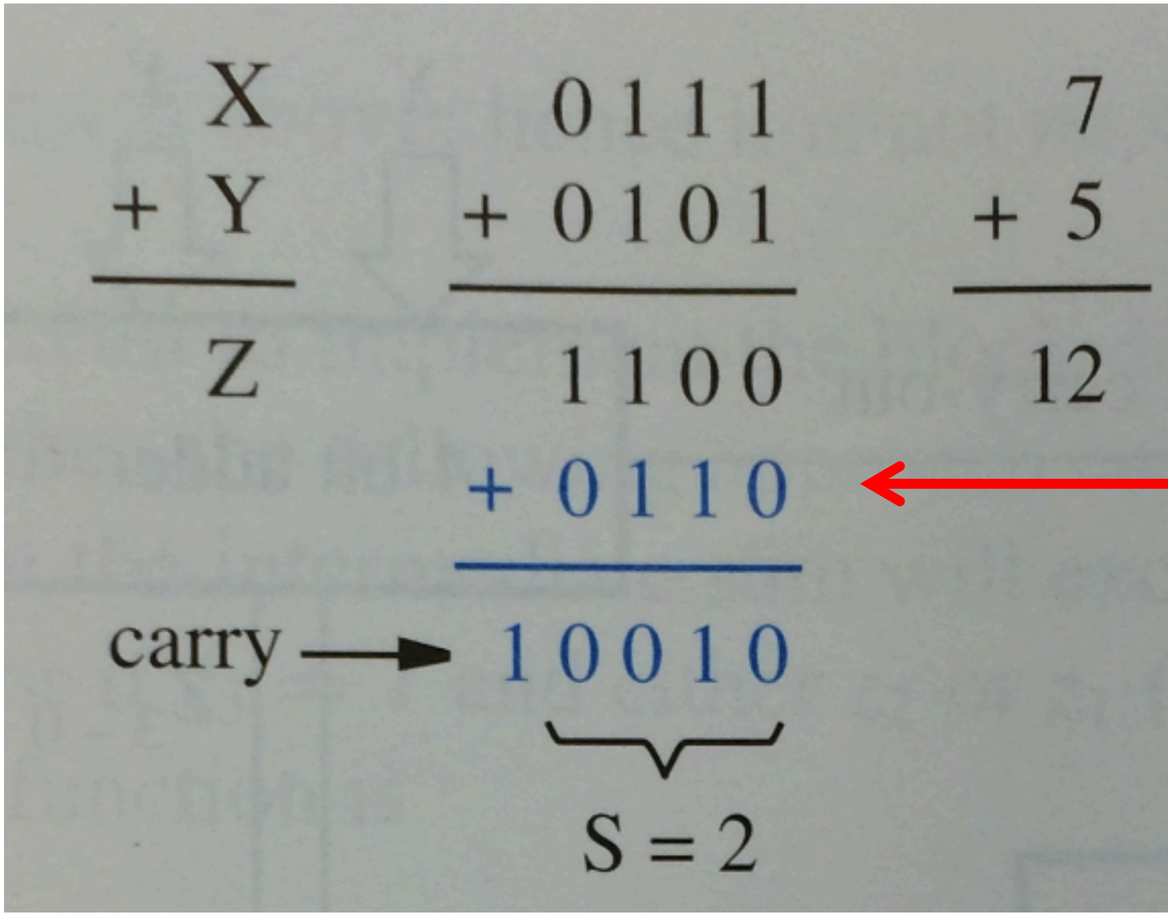
Addition of BCD digits

X	0111	7
$+\ Y$	$+\ 0101$	$+\ 5$
Z	1100	12

The result is greater than 9, which is not a valid BCD number

[Figure 3.38a in the textbook]

Addition of BCD digits



add 6

[Figure 3.38a in the textbook]

Addition of BCD digits

$$\begin{array}{rcrcr}
X & \qquad & 1000 & \qquad & 8 \\
+\ Y & & +\ 1001 & & +\ 9 \\
\hline
Z & & 10001 & & 17
\end{array}$$

[Figure 3.38b in the textbook]

Addition of BCD digits

$$
\begin{array}{r}
X \\
+\ Y \\
\hline
Z
\end{array}
\qquad\qquad
\begin{array}{r}
1\,0\,0\,0 \\
+\ 1\,0\,0\,1 \\
\hline
1\,\underbrace{0\,0\,0\,1}
\end{array}
\qquad\qquad
\begin{array}{r}
8 \\
+\ 9 \\
\hline
17
\end{array}
$$

The result is 1, but it should be 7

[Figure 3.38b in the textbook]

Addition of BCD digits

$$\begin{array}{r} X \\ +\ Y \\ \hline Z \end{array} \qquad \begin{array}{r} 1\,0\,0\,0 \\ +\ 1\,0\,0\,1 \\ \hline 1\,0\,0\,0\,1 \\ +\ 0\,1\,1\,0 \\ \hline 1\,\underbrace{0\,1\,1\,1}_{S\,=\,7} \end{array} \qquad \begin{array}{r} 8 \\ +\ 9 \\ \hline 17 \end{array}$$

$\text{carry} \longrightarrow 1$

add 6 $\longleftarrow$ $+\ 0110$

[Figure 3.38b in the textbook]

Why add 6?

- Think of BCD addition as a mod 16 operation

- Decimal addition is mod 10 operation

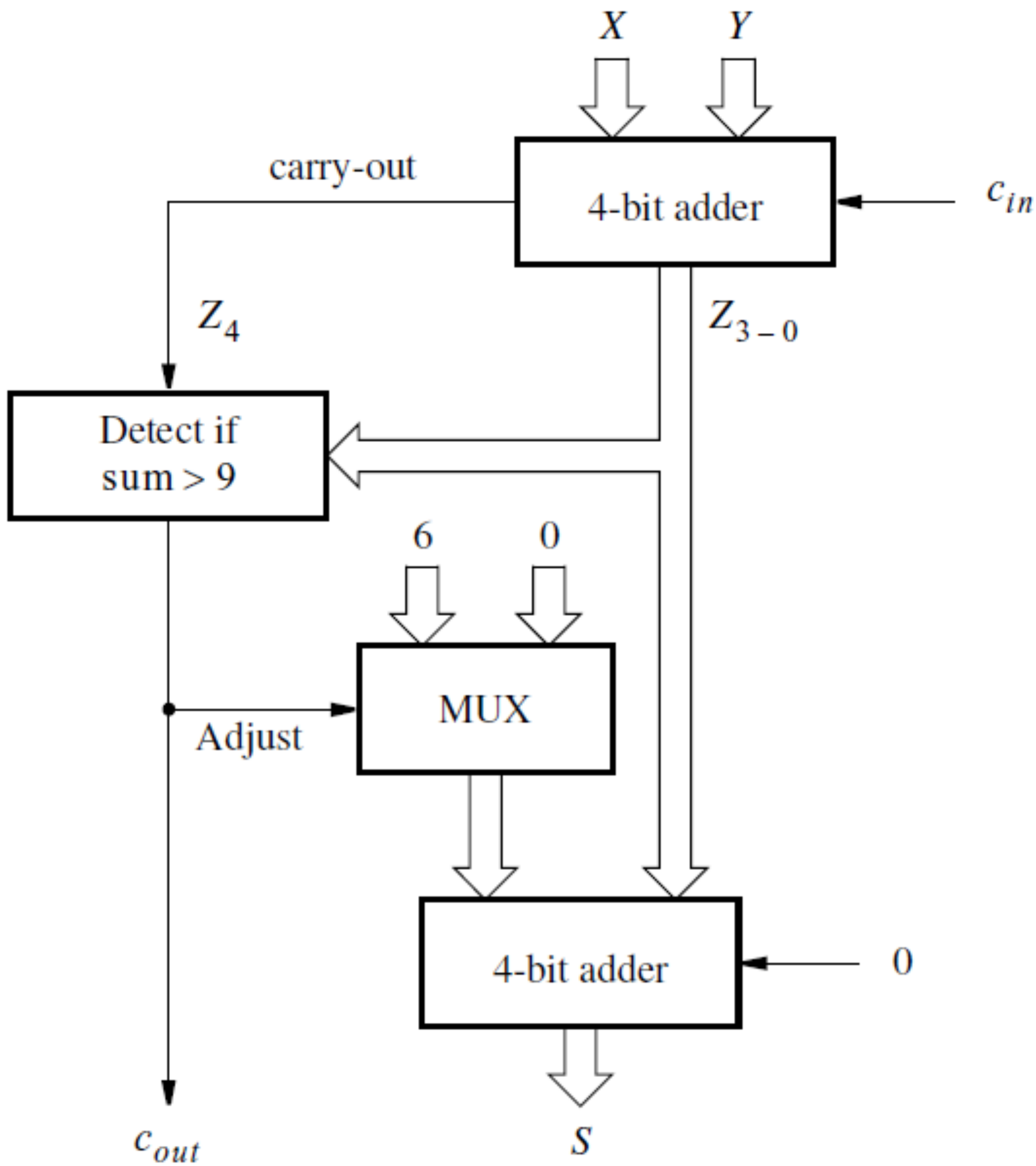
BCD Arithmetic Rules

$Z = X + Y$

If $Z \le 9$, then $S=Z$ and carry-out = 0

If $Z > 9$, then $S=Z+6$ and carry-out =1

Block diagram for a one-digit BCD adder



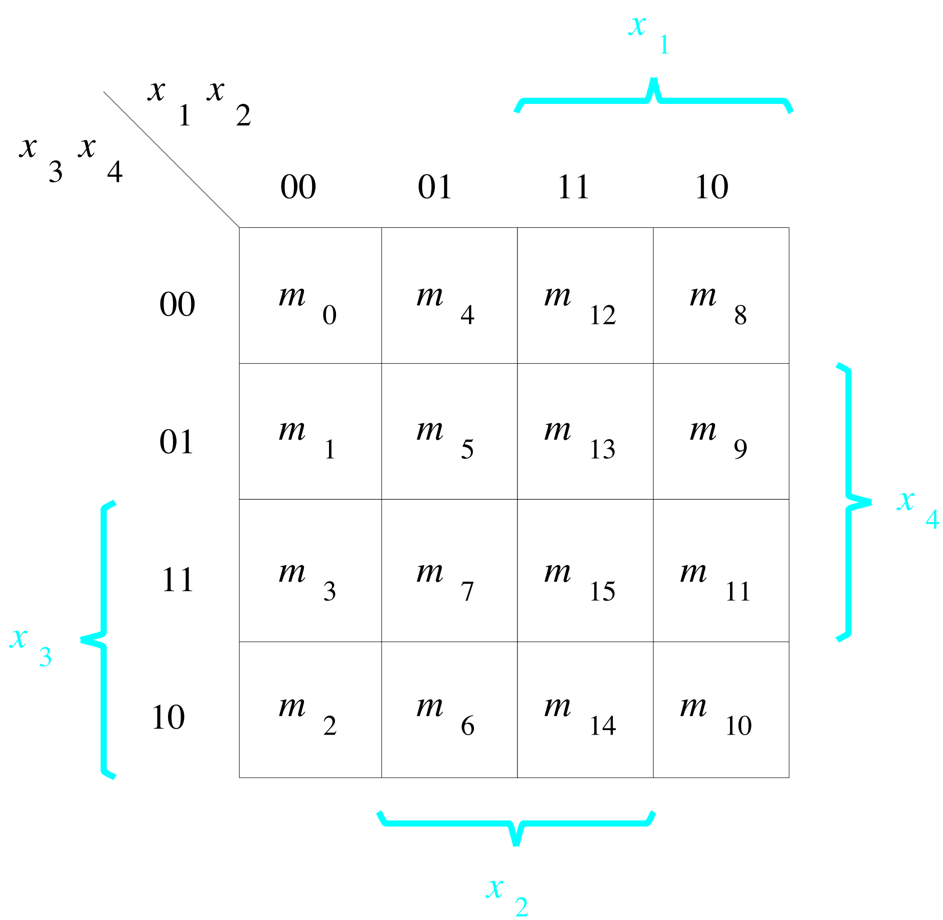
[Figure 3.39 in the textbook]

How to check if the number is > 9?

7	-	0111
8	-	1000
9	-	1001
10	-	1010
11	-	1011
12	-	1100
13	-	1101
14	-	1110
15	-	1111

A four-variable Karnaugh map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15



How to check if the number is > 9?

z3	z2	z1	z0	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

$z_1 z_0$ \ $z_3 z_2$	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	0	0	1	1
10	0	0	1	1

How to check if the number is > 9?

z3	z2	z1	z0	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

$z_1 z_0$ \ $z_3 z_2$	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	0	0	1	1
10	0	0	1	1

$$f = z_3 z_2 + z_3 z_1$$

How to check if the number is > 9?

z3	z2	z1	z0	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

z_1z_0 \ z_3z_2	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	0	0	1	1
10	0	0	1	1

$$f = z_3z_2 + z_3z_1$$

In addition, also check if there was a carry

$$f = \text{carry-out} + z_3z_2 + z_3z_1$$

Verilog code for a one-digit BCD adder

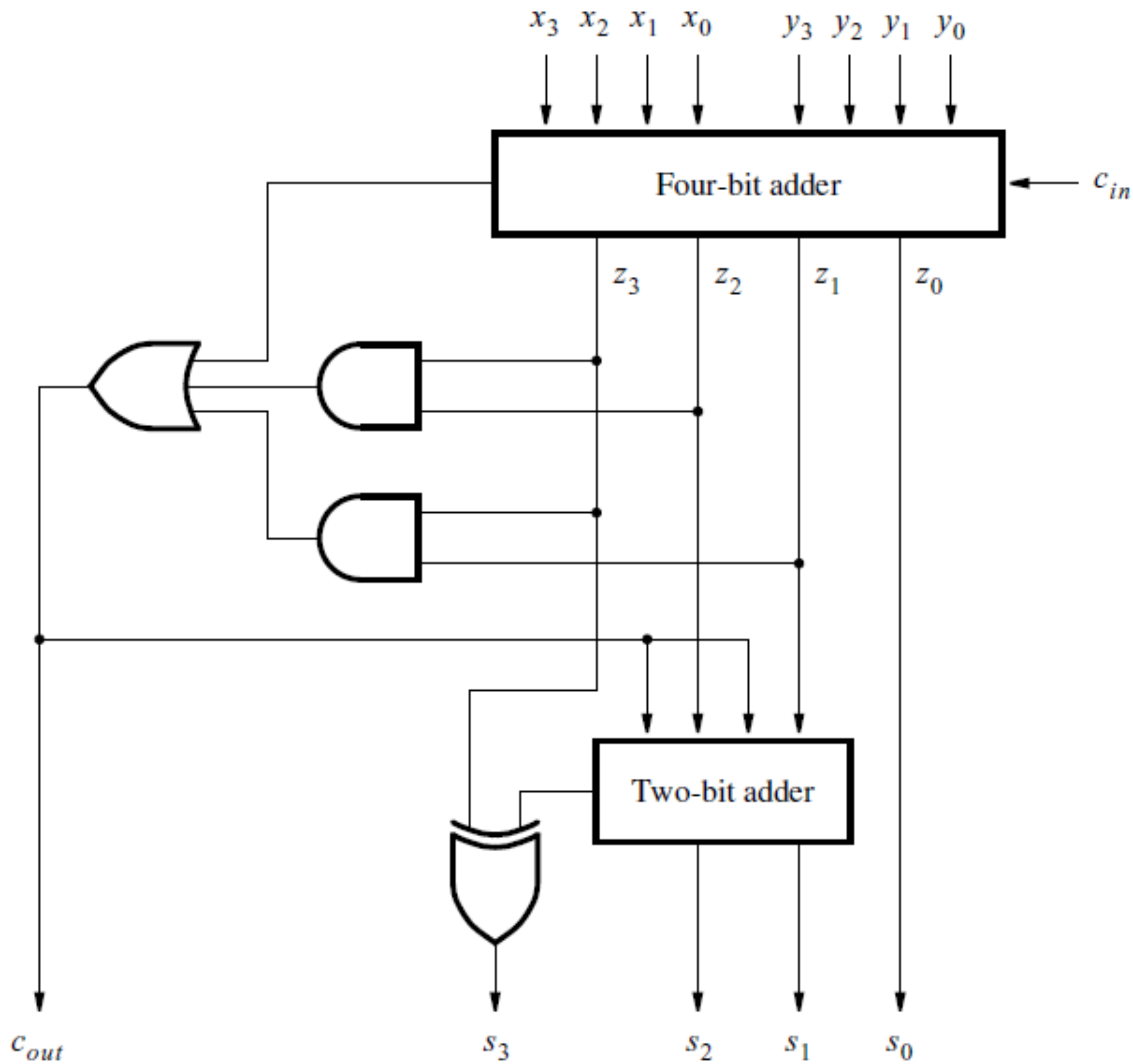
```
module bcdadd(Cin, X, Y, S, Cout);
    input Cin;
    input [3:0] X,Y;
    output reg [3:0] S;
    output reg Cout;
    reg [4:0] Z;

    always@ (X, Y, Cin)
    begin
        Z = X + Y + Cin;
        if (Z < 10)
            {Cout, S} = Z;
        else
            {Cout, S} = Z + 6;
    end

endmodule
```

[Figure 3.40 in the textbook]

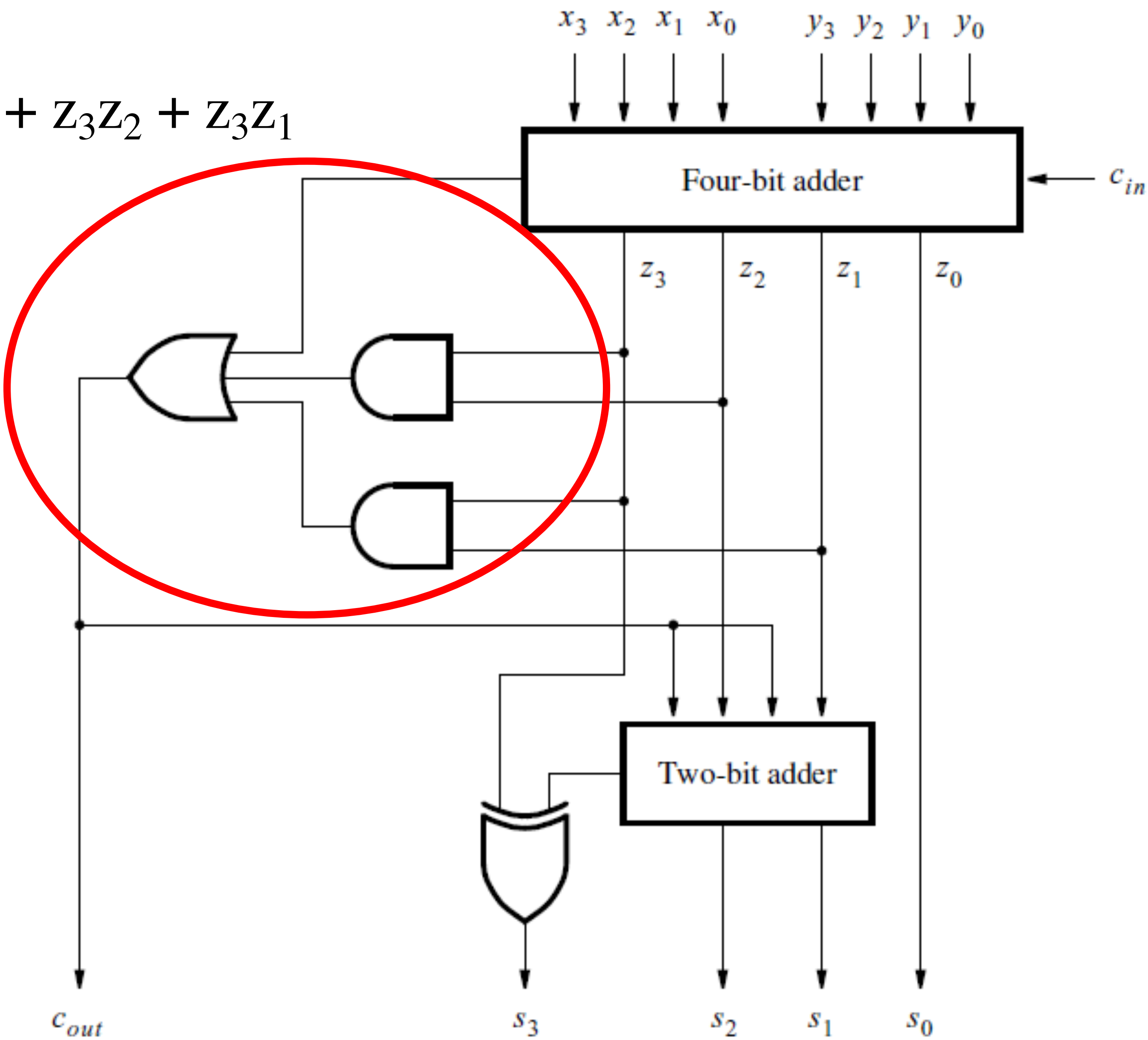
Circuit for a one-digit BCD adder



[Figure 3.41 in the textbook]

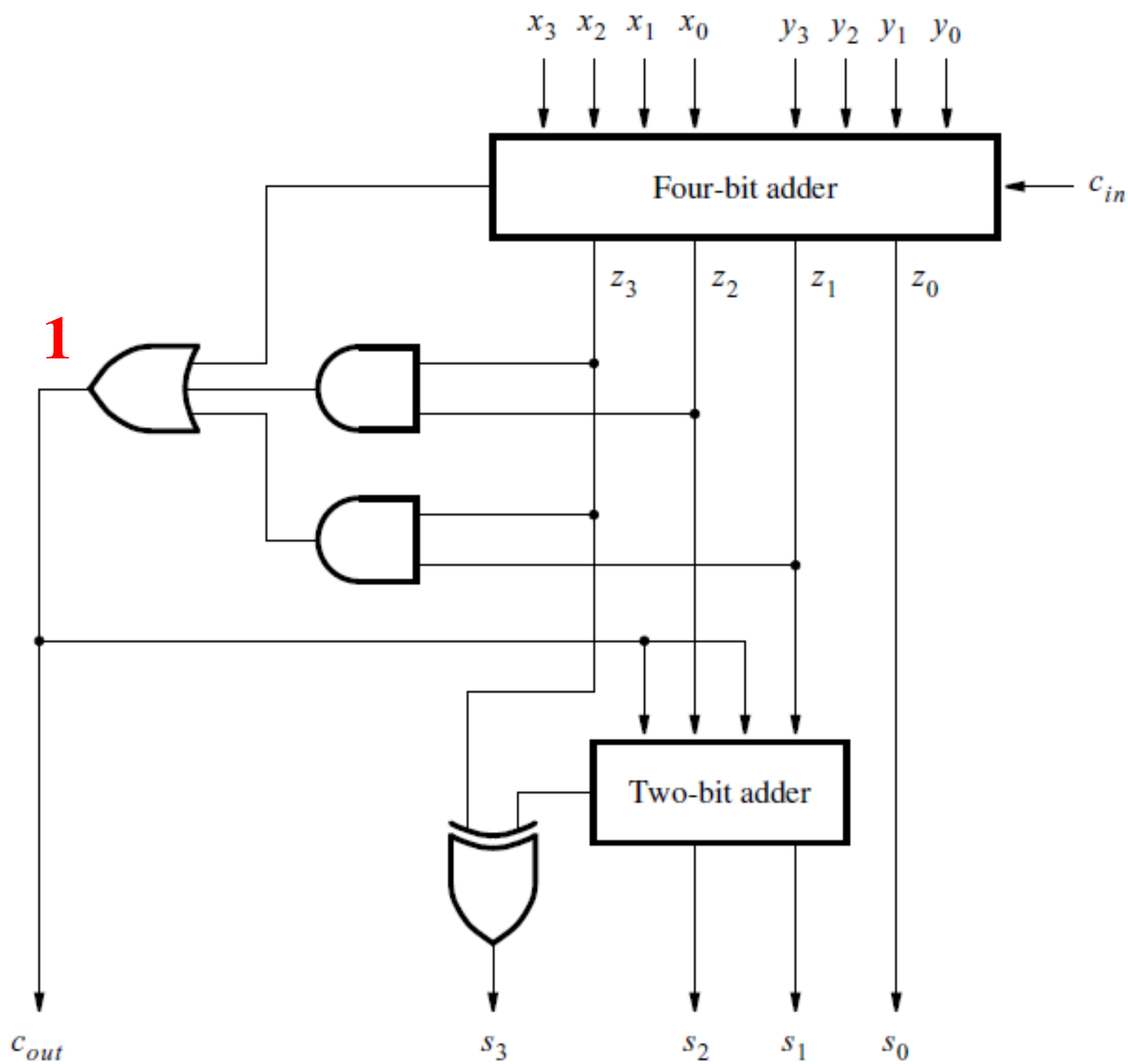
Circuit for a one-digit BCD adder

carry-out + z_3z_2 + z_3z_1



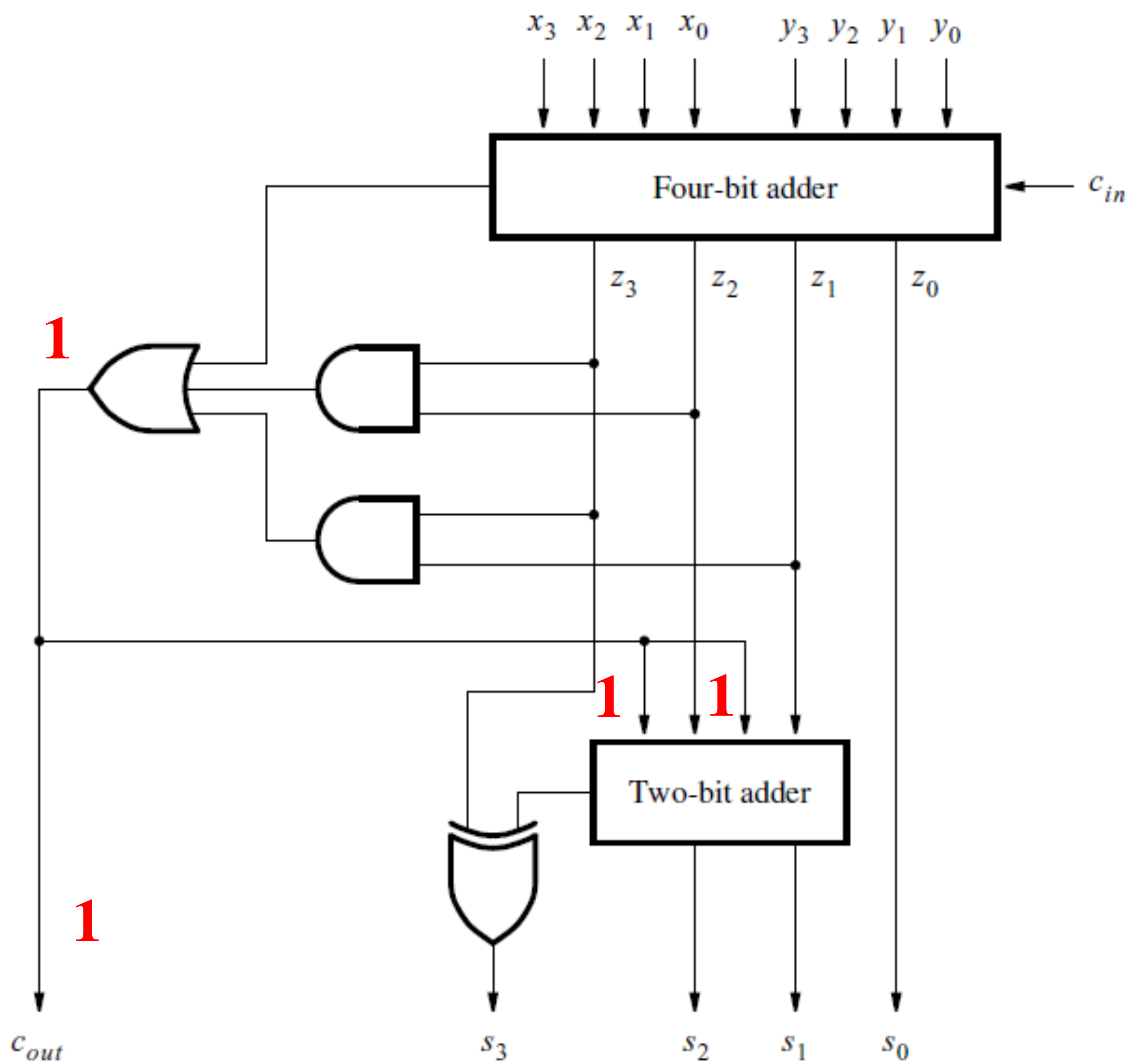
[Figure 3.41 in the textbook]

Circuit for a one-digit BCD adder



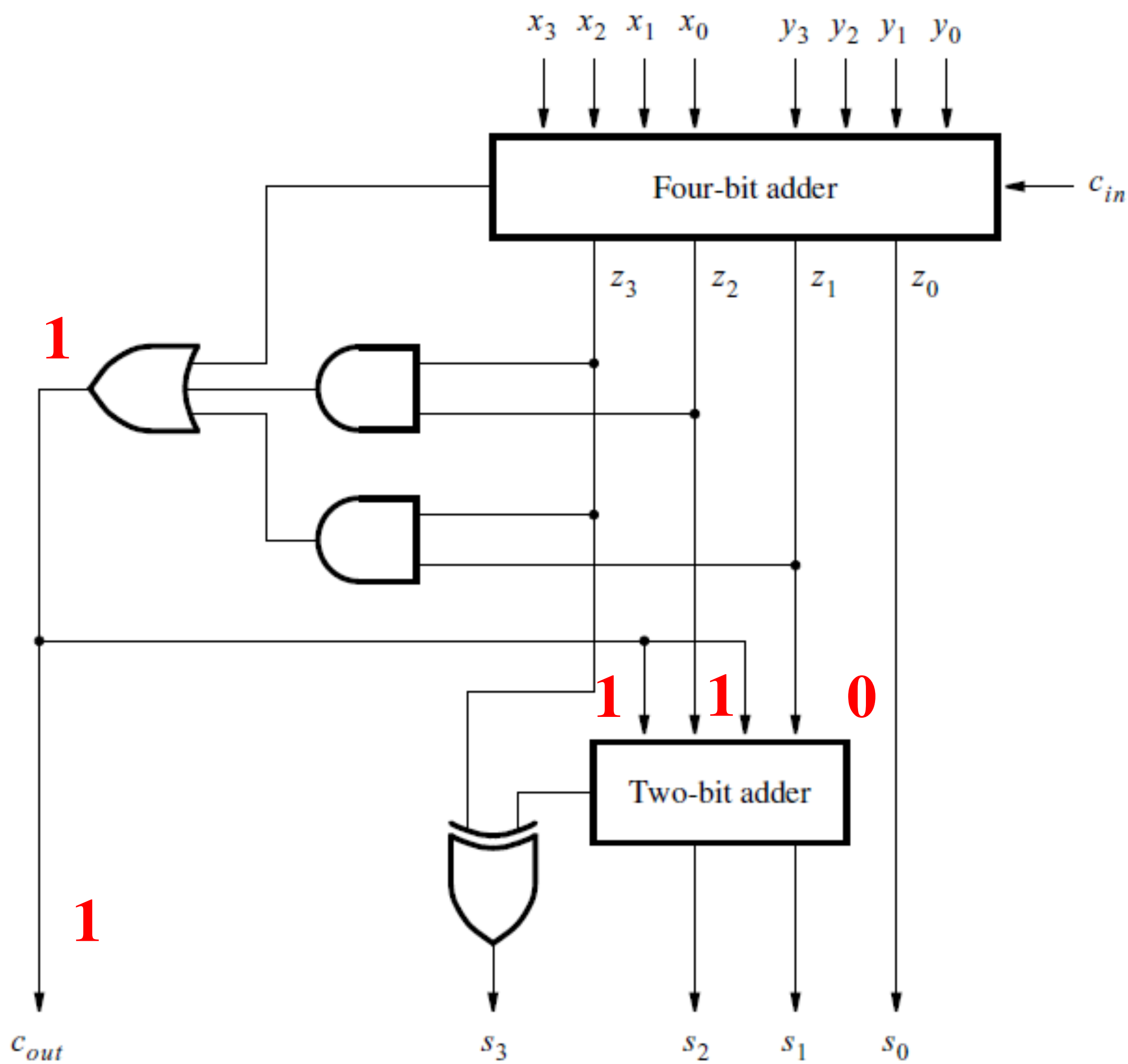
[Figure 3.41 in the textbook]

Circuit for a one-digit BCD adder



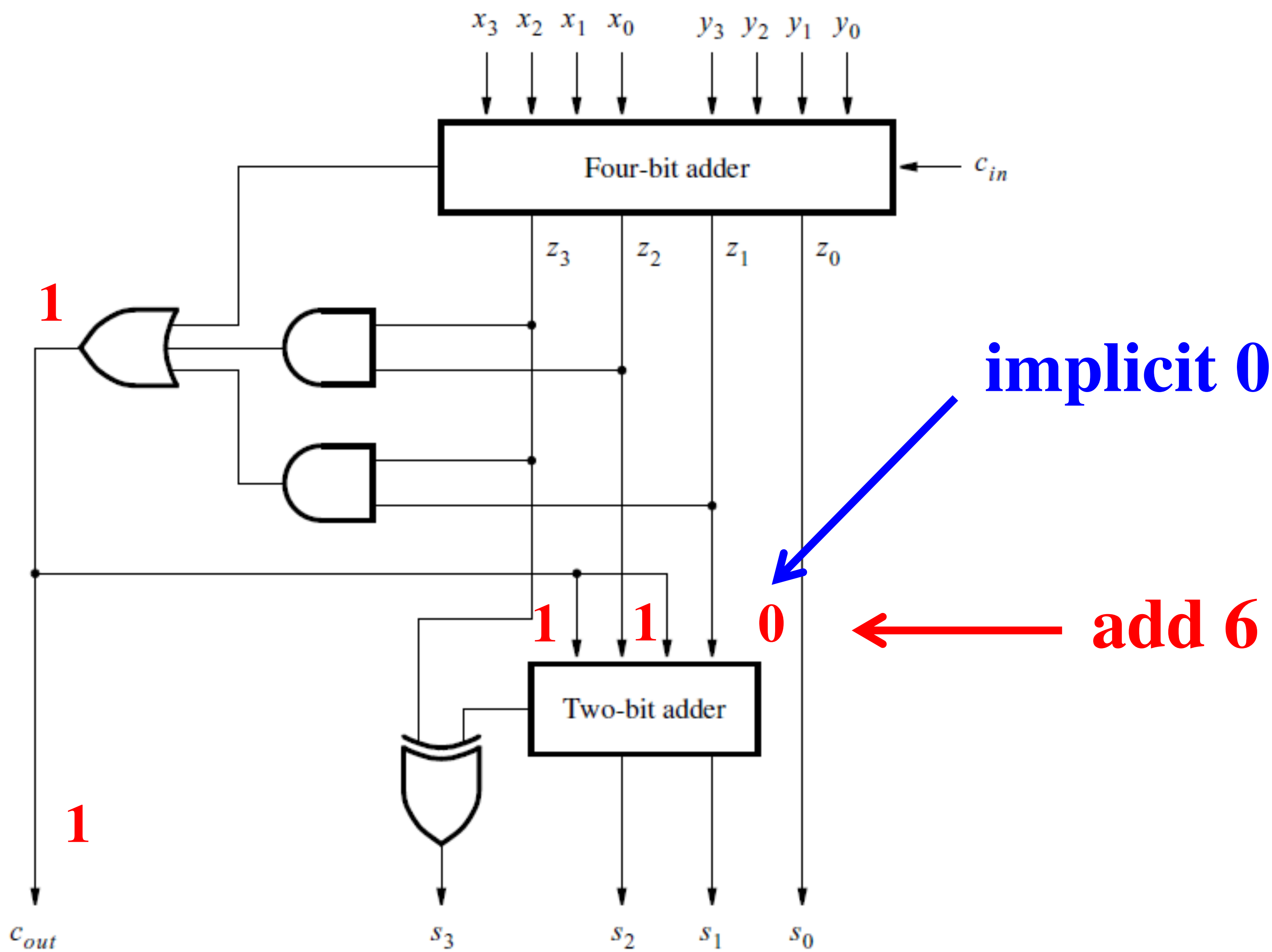
[Figure 3.41 in the textbook]

Circuit for a one-digit BCD adder



[Figure 3.41 in the textbook]

Circuit for a one-digit BCD adder



[Figure 3.41 in the textbook]

Questions?

THE END