Sensitivities

1.0 Introduction

Operation of the Eastern Interconnection relies on using the so-called Interchange Distribution Calculator (IDC) [1]. This is an internet-accessed system that interfaces with OASIS and allows market participants and network operators to efficiently, but approximately, determine the change in MW flow on a *flowgate* given a set of changes in MW bus injections.

A flowgate is a circuit or set of circuits that interconnect different regions of a network that can be limiting under some condition.

The IDC does not represent buses but rather represents control areas, and there are about 100 of them in the eastern interconnection. Therefore the flowgates mostly represent interconnections between these control areas; however, a flowgate may also be internal to a single control area as well.

For purposes of the IDC, a control area is a bus, and the flowgates are interconnections between the buses.

One of the most important uses of the IDC is in the coordination of Transmission Loading Relief (TLR) actions. TLR procedures are in place to guide operators in mitigating flows that exceed operational security limits. TLR *levels*, summarized in Table 1 [2] have been defined that correspond to different types of actions that may be taken for which curtailments must be made. When a TLR level 5 is declared, all ongoing transactions including those with firm transmission service are subject to curtailment.

What we desire to obtain, then, is an expression for computing the change in flow on a branch in a network for a given change in MW bus injection.

Table 1: Summary of TLR Levels [1]

TLR Level	RELIABILITY COORDINATOR Action	Comments	
1	**Notify** RELIABILITY COORDINATORS of potential OPERATING SECURITY LIMIT violations		System Secure
2	**Hold** INTERCHANGE TRANSACTIONS at current levels to prevent OPERATING SECURITY LIMIT violations	Of those transactions at or above the CURTAILMENT THRESHOLD, only those under existing Transmission Service reservations will be allowed to continue, and only to the level existing at the time of the hold. Transactions using Firm Point-to-Point Transmission Service are not held. See Section B.1.	
3a	**Reallocation** Transactions using Non-firm Point-to-Point Transmission Service are curtailed to allow Transactions using higher priority Point-to-Point Transmission Service	Curtailment follows Transmission Service priorities. Higher priority transactions are enabled to start by the REALLOCATION process. See Section B.3.	
3b	**Curtail** Transactions using Non-firm Point-to-Point Transmission Service to mitigate Operating Security Limit Violation	Curtailment follows Transmission Service priorities. There are special considerations for handling Transactions using Firm Point-to-Point Transmission Service. See Section B.4.	Security Limit Violation
4	**Reconfigure** transmission system to allow Transactions using Firm Point-to-Point Transmission Service to continue	There may or may not be an OPERATING SECURITY LIMIT violation. There are special considerations for handling Transactions using Firm Point-to-Point Transmission Service. See Section B.5.	
5a	**Reallocation** Transactions using Firm Point-to-Point Transmission Service are curtailed (pro rata) to allow new Transactions using Firm Point-to-Point Transmission Service to begin (pro rata).	Attempts to accommodate all Transactions using Firm Point-to-Point Transmission Service, though at a reduced ("pro rata") level. Pro forma tariff also requires curtailment / REALLOCATION on pro rata basis with Network Integration Transmission Service and Native Load. See Section B.6.	System Secure
5b	**Curtail** Transactions using Firm Point-to-Point Transmission Service to mitigate Operating Security Limit Violation	Pro forma tariff requires curtailment on pro rata basis with Network Integration Transmission Service and Native Load. See Section B.7.	Security Limit Violation
6	**Emergency Action**	Could include demand-side management, re-dispatch, voltage reductions, interruptible and firm load shedding. See Section B.8.	
0	**TLR Concluded**	**Restore** transactions. See Section B.9.	System Secure

TLR Lev	"Risk" Criteria		Transaction criteria	Reliability Coord Action	Comments
	Imminence	State			
1	Forsee possible condition resulting in violation	Secure		Notify	
2	Expected to approach, is approaching, SOL			Hold	Not > 30 minutes before going to higher levels so xactions may be made based on priority.
3a	Expected to approach is approaching, SOL		Some non-firm ptp at or above curtailment thres - holds, higher priority ptp reservation approved	Reallocate	Curtailments made at top of hour.
3b	Existing or imminent SOL violation or will occur on element removal	Insecure or about to be	Some non-firm ptp at or above their curtailment thresholds.	Hold and Curtail	Hold on nonfirm; Curtailments made immediately.
4	Existing or imminent SOL violation	Insecure or about to be		Hold and Reconfigur e	Hold on nonfirm.
5a	At SOL, no further reconfig possible	Secure	All non-firm ptp at or above curtailment thresholds curtailed; xaction request for previously arranged firm xmission service.	Reallocate	Curtailments made at top of upcoming hour.
5b	Existing or imminent SOL violation or one will occur on element removal, no further reconfig possible	Insecure or about to be	All non-firm ptp at or above curtailment thresholds curtailed.	Curtail	Curtailments made immediately.
6	Existing SOL violation or one will occur upon element removal	Insecure or about to be		Emergency Action	Could include redispatch, reconfiguration, voltage reductions, interruptible and firm load shedding.

2.0 Calculation of Generation Shift Factors

The desired quantity for computing the change in flow on a branch for a given change in generation is referred to as the generation shift factor and will be denoted by $t_{b,k}$. It gives the fraction of a change in injection at bus k that appears on branch b. The Power Transfer Distribution Factor (PTDF) is a generalization of the generation shift factor. This quantity is not only useful for the IDC, it is also useful in obtaining fast (but approximate) answers for many other different kinds of planning and operating problems.

This calculation of generation shift factors is relatively straightforward based on what we have done using the DC power flow model.

Recall the DC power flow equations and the corresponding matrix relation for flows across branches.

$$\underline{P} = \underline{B}' \underline{\theta} \tag{1}$$

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \tag{2}$$

Inverting eq (1) yields:

$$\underline{\theta} = [\underline{B}']^{-1} \underline{P} \tag{3}$$

Substitution of (3) into (2) yields:

$$\underline{P}_B = (\underline{D} \times \underline{A})[\underline{B}']^{-1} \underline{P} \tag{4}$$

As we have defined in the notes on DC PowerFlow:

- $\underline{P}_B$ is the vector of branch flows. It has dimension of M x 1, where M is the number of branches. Branches are ordered arbitrarily, but whatever order is chosen must also be used in $\underline{D}$ and $\underline{A}$.
- $\underline{D}$ is an M x M matrix having non-diagonal elements of zeros; the diagonal element in position row k, column k contains the negative of the susceptance of the k^{th} branch.
- $\underline{A}$ is the M x (N-1) *node-arc incidence matrix*.

- $\underline{B}'$ is the DC power flow matrix of dimension (N-1)x(N-1), where N is the number of buses in the network, obtained from the Y-bus as follows:
  1. Replace diagonal element $\underline{B}'_{kk}$ with the sum of the non-diagonal elements in row k. Alternatively, subtract b_k (the shunt term) from B_{kk}, and multiply by -1.
  2. Multiply all off-diagonals by -1.
  3. Remove row 1 and column 1.
- $\underline{P}$ is the vector of nodal injections for buses 2, …, N

The calculation of eq. (4) provides the flows on all circuits given the injections at all buses.
Bus this is not what we want. What we want is the change in flow on all circuits given a change in injection at one bus.

Here is a "change in injection vector," $\Delta\underline{P}$:

$$\Delta\underline{P} = \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_k \\ \vdots \\ P_N \end{bmatrix} - \begin{bmatrix} P_2^0 \\ P_3^0 \\ \vdots \\ P_k^0 \\ \vdots \\ P_N^0 \end{bmatrix} = \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \vdots \\ \Delta P_k \\ \vdots \\ \Delta P_N \end{bmatrix} = \underline{P} - \underline{P}^0 \tag{5}$$

The change in circuit flows can then be expressed as

$$\begin{aligned} \Delta\underline{P}_B &= \underline{P}_B - \underline{P}_B^0 \\ &= (\underline{D}\times\underline{A})[\underline{B}']^{-1}\underline{P} - (\underline{D}\times\underline{A})[\underline{B}']^{-1}\underline{P}^0 \\ &= (\underline{D}\times\underline{A})[\underline{B}']^{-1}\left(\underline{P} - \underline{P}^0\right) \\ &= (\underline{D}\times\underline{A})[\underline{B}']^{-1}\Delta\underline{P} \end{aligned} \tag{6}$$

Now let the $\Delta\underline{P}$ vector be all zeros except for the element corresponding to the kth bus, and assign this bus an injection change of 1.

$$\Delta\underline{P} = \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \vdots \\ \Delta P_k \\ \vdots \\ \Delta P_N \end{bmatrix} = \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \vdots \\ \Delta P_k \\ \vdots \\ \Delta P_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad (7)$$

Then

$$\Delta\underline{P}_B = \begin{bmatrix} \Delta P_{B1} \\ \Delta P_{B2} \\ \vdots \\ \Delta P_{Bb} \\ \vdots \\ \Delta P_{BM} \end{bmatrix} = \begin{bmatrix} t_{1,k} \\ t_{2,k} \\ \vdots \\ t_{b,k} \\ \vdots \\ t_{M,k} \end{bmatrix} = (\underline{D} \times \underline{A})[\underline{B}']^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \Delta P_k \\ \\ \\ \end{matrix} \qquad (8)$$

Question: Does the above equation imply that the injection is changed at only one bus? Explain.

Definition: The generation shift factor $t_{b,k}$ is defined as

$$t_{b,k} = \left. \frac{\Delta P_{Bb}}{\Delta P_k} \right|_{\substack{\text{Reallocation} \\ \text{Policy}}}$$

This is denoted as $a_{\ell i}$ in W&W text (see eq. 11.1).

Example 1:

We consider an example used in the "DCPowerFlowEquations" notes, illustrated below. Compute the generation shift factors for all branches corresponding to an increase in bus 2 injection and a decrease in bus 3 injection.

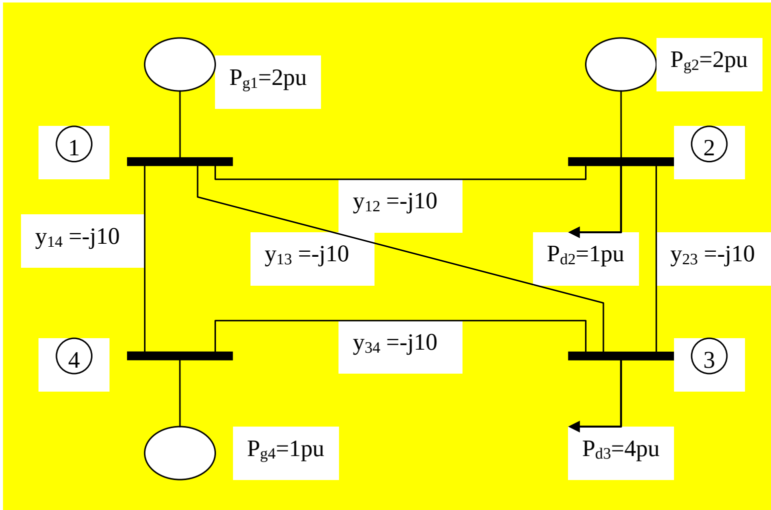


Fig. 2: Four-bus network used in example

$$\begin{bmatrix} t_{1,23} \\ t_{2,23} \\ t_{3,23} \\ t_{4,23} \\ t_{5,23} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -10 \\ -10 & 0 & 0 \\ 10 & -10 & 0 \\ 0 & -10 & 10 \\ 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0.0625 & 0.025 & 0.0125 \\ 0.025 & 0.05 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -10 \\ -10 & 0 & 0 \\ 10 & -10 & 0 \\ 0 & -10 & 10 \\ 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0.0375 \\ -0.025 \\ -0.0125 \end{bmatrix} = \begin{bmatrix} 0.125 \\ -0.375 \\ 0.625 \\ 0.125 \\ 0.25 \end{bmatrix}$$

Note that the above generation shift factors are for a “double shift.”

You can think of it like this. A generation shift factor for branch b, bus k would be $t_{b,k}$ and another generation shift factor for branch b, bus j would be $t_{b,j}$. If we have an injection increase at bus k of ΔP_k and an injection increase at bus j of ΔP_j (negative), then

$$\Delta P_b = t_{b,k}\Delta P_k + t_{b,j}\Delta P_j \tag{9}$$

Increase P_k, Decrease P_1

Decrease P_j, Increase P_1

Therefore, if $\Delta P_k = -\Delta P_j$, then

$$\Delta P_b = \left(t_{b,k} - t_{b,j}\right)\Delta P_k \tag{10}$$

Note that once ΔP_b is obtained, then it must be added to the original flow on branch b to get the resulting total flow following the generation shift, i.e.,

$$\hat{P}_b = P_b^0 + \Delta P_b$$

The last equation is the same as eq. (11.2) in W&W text.

3.0 Generation Shift Factors with Distributed Slack

Equation (8) shows how to compute the generation shift factors for the case when a single specified slack bus corresponds to bus 1.

Example 1 above shows how to compute the generation shift factors for the case when a single specified slack bus corresponds to some other bus in the network (not the bus corresponding to the reference by way of omission from its corresponding row and column in the B' matrix).

What we are interested in here is computation of generation shift factors for the case when we would like to distribute the slack, or the compensation, throughout the network. <u>The key criterion to guide this is that the elements in the nodal injection vector should correspond to the percentage of desired compensation for each bus</u>.

This criterion is illustrated below:

$$\Delta \underline{\boldsymbol{P}}_{\boldsymbol{B}} = \begin{bmatrix} \Delta \boldsymbol{P}_{\boldsymbol{B}1} \\ \Delta \boldsymbol{P}_{\boldsymbol{B}2} \\ \vdots \\ \Delta \boldsymbol{P}_{\boldsymbol{Bb}} \\ \vdots \\ \Delta \boldsymbol{P}_{\boldsymbol{BM}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}_{1,\boldsymbol{k}} \\ \boldsymbol{t}_{2,\boldsymbol{k}} \\ \vdots \\ \boldsymbol{t}_{\boldsymbol{b},\boldsymbol{k}} \\ \vdots \\ \boldsymbol{t}_{\boldsymbol{M},\boldsymbol{k}} \end{bmatrix} = (\underline{\boldsymbol{D}} \times \underline{\boldsymbol{A}}) [\underline{\boldsymbol{B}}']^{-1} \begin{bmatrix} \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \\ \vdots \\ 1 \\ \vdots \\ \boldsymbol{c}_N \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{bus } k \\ \\ \\ \end{matrix} \tag{11}$$

where

$$c_1 = -\sum_{i=2}^{N} c_i = -\left(1 + \sum_{\substack{i=2 \\ i \neq k}}^{N} c_i \right) \tag{12}$$

is the allocation desired for the reference bus.

One way to distribute the slack is to distribute equally to all buses. In this case,

$$c_i = \frac{-1}{N-1} \tag{13}$$

where we use $N\text{-}1$ in the denominator because one bus, bus k, is the bus for which the computation is being made (and therefore $c_k=1$). If we use (13), then we can substitute into (12) to obtain:

$$c_1 = -\sum_{i=2}^{N} c_i = -\left(1 + \sum_{\substack{i=2 \\ i \neq k}}^{N} c_i \right) = -\left(1 + \sum_{\substack{i=2 \\ i \neq k}}^{N} \frac{-1}{N-1} \right)$$

$$= -\left(1 + (N-2) \frac{-1}{N-1} \right) = -\left(\frac{N-1}{N-1} - \frac{N-2}{N-1} \right) = \frac{-1}{N-1}$$

Example 2:

Using the system from Example 1 above, compute generation shift factors for all branches corresponding to an increase in bus 2 injection, when the slack is equally distributed to **all** buses.

$$\begin{bmatrix} t_{1,2all} \\ t_{2,2all} \\ t_{3,2all} \\ t_{4,2all} \\ t_{5,2all} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -0.333 \\ -0.333 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -10 \\ -10 & 0 & 0 \\ 10 & -10 & 0 \\ 0 & -10 & 10 \\ 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0.0625 & 0.025 & 0.0125 \\ 0.025 & 0.05 & 0.025 \\ 0.0125 & 0.025 & 0.0625 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ -0.333 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -10 \\ -10 & 0 & 0 \\ 10 & -10 & 0 \\ 0 & -10 & 10 \\ 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0 \\ -0.0166 \end{bmatrix} = \begin{bmatrix} 0.1664 \\ -0.5001 \\ 0.4999 \\ -0.1666 \\ -0.0002 \end{bmatrix}$$

It is of interest to compare the answer from the example where the slack was distributed entirely to bus 3 and the example where the slack was distributed to all buses.

$$\begin{bmatrix} t_{1,23} \\ t_{2,23} \\ t_{3,23} \\ t_{4,23} \\ t_{5,23} \end{bmatrix} = \begin{bmatrix} 0.125 \\ -0.375 \\ 0.625 \\ 0.125 \\ 0.25 \end{bmatrix} \qquad \begin{bmatrix} t_{1,2all} \\ t_{2,2all} \\ t_{3,2all} \\ t_{4,2all} \\ t_{5,2all} \end{bmatrix} = \begin{bmatrix} 0.1664 \\ -0.5001 \\ 0.4999 \\ -0.1666 \\ -0.0002 \end{bmatrix}$$

Clearly the assumption on slack distribution is important!

There are other ways to distribute the slack. For example, we may distribute the slack equally to all generation buses. Or we may distribute the slack equally to all load buses. Or we may distribute the slack to all generation buses in proportion to the MVA rating of the generation that is located there (this approach conforms best to reality, as we will see when we study AGC).

4.0 Generation Shift Factor Matrix

Given a specified slack distribution, we may compute a matrix of generation shift factors according to

$$\underline{T} = \underbrace{\begin{bmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,k} & \cdots & t_{1,N} \\ t_{2,1} & t_{2,2} & \cdots & t_{2,k} & \cdots & t_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{b,1} & t_{b,2} & \cdots & t_{b,k} & \cdots & t_{b,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M,1} & t_{M,2} & \cdots & t_{M,k} & \cdots & t_{M,N} \end{bmatrix}}_{M\times N}$$

$$= (\underbrace{D}_{M\times M} \times \underbrace{A}_{M\times(N-1)}) \underbrace{[B']^{-1}}_{(N-1)\times(N-1)} \underbrace{\begin{bmatrix} c_{2,1} & 1 & c_{2,3} & c_{2,4} & \cdots & c_{2,N-1} & c_{2,N} \\ c_{3,1} & c_{3,2} & 1 & c_{2,4} & \cdots & c_{3,N-1} & c_{3,N} \\ c_{4,1} & c_{4,2} & c_{4,3} & 1 & \cdots & c_{4,N-1} & c_{4,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{N-1,1} & c_{N-1,2} & c_{N-1,3} & c_{N-1,4} & \cdots & c_{N-1,N-1} & c_{N-1,N} \\ c_{N,1} & c_{N,2} & c_{N,3} & c_{N,4} & \cdots & c_{N,N-1} & 1 \end{bmatrix}}_{(N-1)\times N}$$

The last matrix on the right is called the reallocation matrix and is the matrix for all considered {*injection changes with corresponding reallocation policies*}, where an element $c_{i,k}$ is the percent allocation to bus i when an injection of 1.0 is made at bus k. For example,

- first column $c_{i,1}$, $i=2,\ldots,N$ provides the reallocation policy at all buses $i=2,\ldots,N$ when an injection of 1.0 is made at bus 1.
- second column $c_{i,2}$, $i=2,\ldots,N$ provides the reallocation policy at all buses $i=2,\ldots,N$ when an injection of 1.0 is made at bus 2.
- …
- last column $c_{i,N}$ provides the reallocation policy at all buses $i=2,\ldots N$, when an injection of 1.0 is made at bus N.

The number of columns in the reallocation matrix is equal to the number of considered {*injection changes with corresponding reallocation policies*}.

In the reallocation matrix, the sum of all elements in a column k (corresponding to the injection of 1.0 being made at bus k) is the negative of the allocation made to bus 1 (a generalization of eq. (12)), as follows:

$$c_{1,k} = -\sum_{i=2}^{N} c_{i,k} \qquad \forall k = 1,\ldots,N$$

The formulation above assumes that we desire generation shift factors for every branch (a row of $\underline{T}$) and every bus (a column of $\underline{T}$). The first column of $\underline{T}$ is for a shift at bus 1, which is the bus for which a column and a row are deleted from the $\underline{B}'$ matrix.

However, we need not include every branch. There may be some branches that we know from experience will never overload, or there may be policy that requires a particular application to only monitor certain branches. The latter is the case for NERC's IDC described at the beginning of this document.

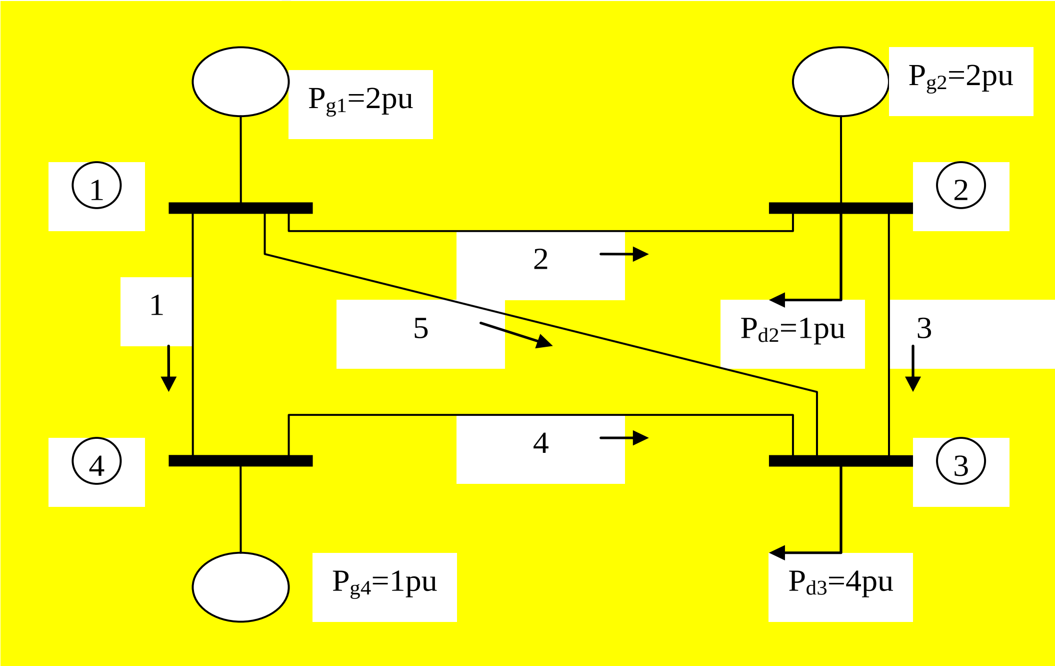
<u>Example 3</u>: Let's compute the T-matrix for Example 2. We assume a distributed slack bus, where, $c_{i,k}$ =-1/3. Therefore

$$\underline{T} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} -0.333 & 1 & -0.333 & -0.333 \\ -0.333 & -0.333 & 1 & -0.333 \\ -0.333 & -0.333 & -0.333 & 1 \end{bmatrix}$$

$$\underline{T} = \begin{bmatrix} 0.3333 & 0.1667 & 0 & -0.5 \\ 0.3333 & -0.5 & 0 & 0.1667 \\ 0 & 0.5 & -0.3333 & -0.1667 \\ 0 & -0.1667 & -0.3333 & 0.5 \\ 0.3333 & 0 & -0.3333 & 0 \end{bmatrix}$$

Remember: each column is the set of shift factors for a unit increase in injection (generation) at a certain bus. Column 1 is when the injection at bus 1 is increased (there is no "1" in that column because that is the one corresponding to the bus that was deleted in the B' matrix). Column 2 is when the injection at bus 2 is increased, and so on.

The one-line diagram is shown below to facilitate understanding of the relation between increased injection at bus k (which identifies a particular column) and how branch flows are affected (the elements in that particular column).



For example, the first column indicates that if we increase bus 1 injection by 1 pu, we get 0.3333 flowing over branches 1, 2, 5, with 0 pu on branches 3, 4[1].

One caution: It is possible to obtain a complete power flow calculation using the shift factors. That is

$$P_{Bb} = \sum_{k=2}^{N} \Delta P_k t_{b,k} \qquad *$$

where ΔP_k is the total injection at bus k. In this case, however, the shift factors $t_{b,k}$, defined according to

[1] It is interesting that we get 0 pu flowing over branches 3 and 4. The reason is due to network symmetry (all branch impedances have Z=j0.1 pu). This symmetry is most clearly understood by using superposition. Begin by applying 0.3333 pu injection at bus 1, -0.3333 pu injection at bus 2, and compute the flows. Then apply 0.3333 pu injection at bus 1, -0.3333 injection at bus 3, and compute the flows. Then apply 0.3333 pu at bus 1, -0.3333 pu injection at bus 4, and compute the flows. Then add the 3 sets of flows for each branch, and one will observe the exact cancelation of the three flows in branch 3, and the three flows on branch 4.

$$t_{b,k} = \left.\frac{\Delta P_{Bb}}{\Delta P_k}\right|_{\substack{\text{Reallocation}\\ \text{Policy}}}$$

must be computed according to a consistent reallocation policy. Thus, we should not compute (*) above using the T-matrix values based on the distributed slack. If we did, for our 4 bus system:

> For column 1, "bus k" = bus 1. It means that a 1 unit change at bus 1 gets compensated by a -1/3 unit change at buses 2, 3, 4.
>
> For column 2, "bus k" = bus 2. It means that a 1 unit change at bus 2 gets compensated by a -1/3 unit change at buses 1, 3, 4.
>
> And so on.

This will result in a balanced dispatch (power balance will be satisfied), but it will be a different dispatch than what was intended. As a simple example, try a three bus system having injections -6, 4, 2, at buses 1, 2, and 3, respectively. With "c"=0.5, then we will get the following distribution:

Bus 1	Bus 2	Bus 3	
-6	+3	+3	←Each row here shows injected
-2	**+4**	-2	←bus in bold and the allocation
-1	-1	**+2**	←to other buses based on c=0.5.
-9	+6	+3	←This last row shows total injection at each bus from the 3 calculations, i.e., the resulting bus distribution from applying the "dist slack" a*ij*'s, which does not match with the intended distribution of -6, 4, 2. Thus, you will get a different set of flows.

So, for this example, in computing the "T" matrix, the "c" matrix (that post multiplies DAB^{-1}) for our 4 bus system should appear as

0 1 0 0
0 0 1 0
0 0 0 1

which means using bus # 1 as the slack in each case, that is,

$$\underline{T} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.0 Efficient computation of GSFs

In the previous discussion, it was assumed that we would be able to compute $(B')^{-1}$, i.e., that the number of nodes would not be too large, which can be the case under some approximations such as those made by the IDC [1]. However, it is also common that this is not the case, i.e., that we may want to obtain GSFs for a system where the number of nodes is very large.

In such a case, one can obtain the GSFs without matrix inversion but only for one shift at a time, via

$$\Delta \underline{P} = \underline{B}' \Delta \underline{\theta} \tag{14}$$

$$\Delta \underline{P}_B = (\underline{D} \times \underline{A}) \times \Delta \underline{\theta} \tag{15}$$

Equation (14) is solved for $\Delta\underline{\theta}$ via LU factorization for a given $\Delta\underline{P}$, and then the resulting $\Delta\underline{\theta}$ is used in (15) to obtain the line flow shifts in $\Delta\underline{P}_B$.

Example 4: Repeat example 1, which is to obtain the GSF for all branches corresponding to an increase in bus 2 injection and a decrease in bus 3 injection.

$$\Delta\underline{P} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{bmatrix} = \underline{B}'\Delta\underline{\theta}$$

Using LU factorization:

$$\begin{bmatrix} 1 & -0.5 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & & \\ 0 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 25 & -10 \\ 0 & -10 & 20 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & \\ 0 & -10 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & -10 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 16 \end{bmatrix} \qquad L = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & 0 \\ 0 & -10 & 16 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now use backwards/forwards substitution to obtain $\Delta\underline{\theta}$, resulting in

$$Lw = b$$

$$\Rightarrow \begin{bmatrix} 20 & 0 & 0 \\ -10 & 25 & 0 \\ 0 & -10 & 16 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$w_1 = 0.05$$

$$w_2 = (-1 + 10 * 0.05) / 25 = -0.02$$

$$w_3 = (10/16) * (-0.02) = -0.0125$$

$$U = \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ux = w$$

$$\Rightarrow \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.05 \\ -0.02 \\ -0.0125 \end{bmatrix}$$

$$x_3 = -0.0125$$

$$x_2 = -0.02 + 0.4 * (-0.0125) = -0.0250$$

$$x_1 = 0.05 + 0.5 * (-0.0250) = 0.0375$$

And this gives our angle changes as

$$\begin{bmatrix} \Delta\boldsymbol{\theta}_2 \\ \Delta\boldsymbol{\theta}_3 \\ \Delta\boldsymbol{\theta}_4 \end{bmatrix} = \begin{bmatrix} 0.0375 \\ -0.0250 \\ -0.0125 \end{bmatrix}$$

Now we can use eq. (15) to obtain

$$\Delta \underline{\boldsymbol{P}}_{\boldsymbol{B}} = (\underline{\boldsymbol{D}} \times \underline{\boldsymbol{A}}) \times \Delta \underline{\boldsymbol{\theta}}$$

$$= \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.0375 \\ -0.0250 \\ -0.0125 \end{bmatrix} = \begin{bmatrix} 0.125 \\ -0.375 \\ 0.625 \\ 0.125 \\ 0.25 \end{bmatrix}$$

which is in agreement with the result of example 1.

6.0 Line outage distribution factors

The line outage distribution factor (LODF) is derived in your text (Appendix 11A, pp.440-444).

The LODFs are linear estimates of the ratio:

change in flow on circuit ℓ due to
outage of circuit k, denoted by Δf_ℓ,
to
pre-contingency flow on
circuit k, denoted by f_{k0}.

In other words, it provides the fraction of pre-contingency flow on circuit k that appears on circuit ℓ following outage of circuit k, and is given by

$$d_{\ell,k} = \Delta f_{\ell} / f_{k0} \tag{16}$$

It is then clear that the change in flow on circuit ℓ due to the outage of circuit k is given by

$$\Delta f_{\ell} = d_{\ell,k} \times f_{k0} \tag{17}$$

The derivation in the text is lengthy; we will not go through it here. To understand the result, we define a matrix $\underline{X}'$ such that

$$\underline{X}' = (\underline{B}')^{-1} \tag{18}$$

This means that

$$\Delta \underline{P} = \underline{B}' \Delta \underline{\theta} \Rightarrow \Delta \underline{\theta} = \underline{X}' \Delta \underline{P} \tag{19}$$

Then we define another matrix $\underline{X}$ such that it is the same as $\underline{X}'$ except we append another row at the top and another column to the left, corresponding to the reference bus (assumed bus #1) injection and angle, as shown below:

$$\underline{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \underline{X}' & \\ 0 & & & \end{bmatrix} \tag{20}$$

The line outage distribution factor $d_{\ell,k} = \Delta f_{\ell} / f_{k0}$ corresponding to the additional flow on branch k from outage of branch ℓ is then given by

$$d_{\ell,k} = \frac{\frac{x_k}{x_\ell}\left(X_{in} - X_{jn} - X_{im} + X_{jm}\right)}{x_k - \left(X_{nn} + X_{mm} - 2X_{nm}\right)} \tag{21}$$

In (21),

- x_k and x_ℓ are the reactances of outage branch k and remaining branch ℓ, respectively;
- m and n are bus numbers terminating branch k;
- i and j are bus numbers terminating branch ℓ.

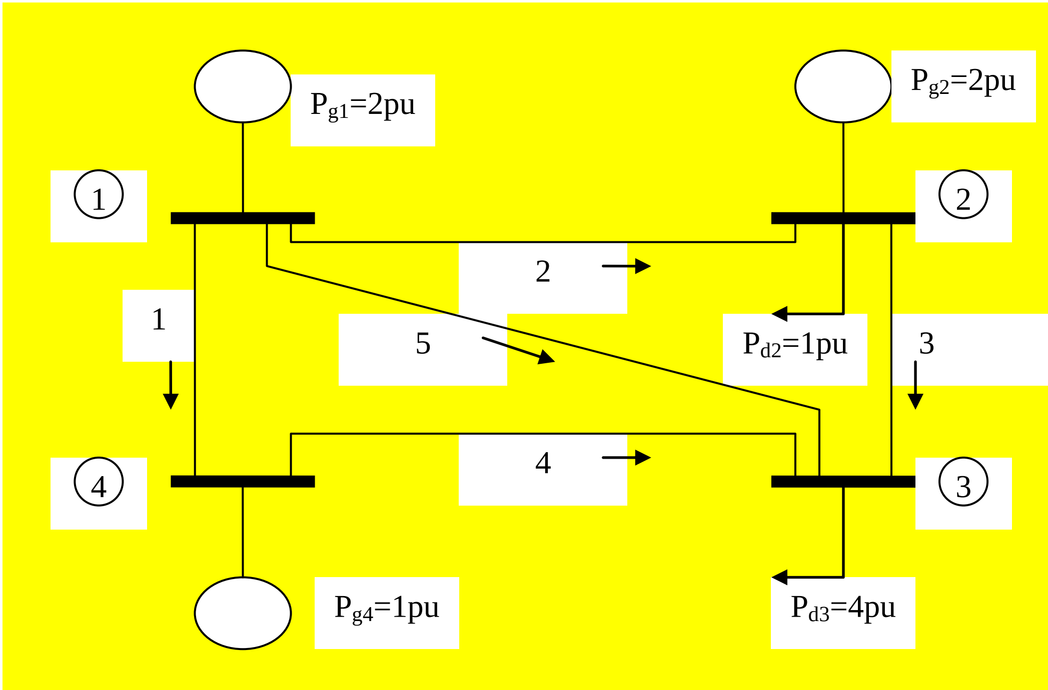
Therefore,

- X_{in} is the element of $\underline{X}$ in row i, column n.
- X_{jn} is the element of $\underline{X}$ in row j, column n.
- X_{im} is the element of $\underline{X}$ in row i, column m.
- X_{jm} is the element of $\underline{X}$ in row j, column m.
- X_{nm} is the element of $\underline{X}$ in row n, column m.
- X_{nn} is the element of $\underline{X}$ in row n, column n.
- X_{mm} is the element of $\underline{X}$ in row m, column m.

7.0 A computationally efficient method to obtain LODFs

A significant problem with W&W's method of obtaining the LODFs is that it requires $\underline{X}=(\underline{B}')^{-1}$, and if the system is very large, then inverting the matrix can be a computationally intense problem. We provide another method in this section. Our treatment is adapted from [3].

Let's reconsider our familiar 4-bus, 5-branch example problem.



The <u>B</u>' matrix for this system is

$$\underline{B}' = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}$$

What happens to B' if we lose the circuit #3 (from bus 2 to bus 3)?

We could re-develop the new <u>B</u>' from the one-line diagram as we are accustomed to doing now. Another way is to discern how the circuit #3 affects the <u>B</u>' matrix, in that it will affect exactly 4 elements, as indicated with the underlines below, corresponding to elements in bus numbered positions (2,2), (2,3), (3,2), and (3,3).

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \underline{B}' = & \begin{bmatrix} \underline{20} & \underline{-10} & 0 \\ \underline{-10} & \underline{30} & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array}$$

Recalling that all branch admittances of our network are –j10, what would these four elements be if branch #3 (between buses 2 and 3) were not there?

$$\underline{B}'^{out} = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \underline{10} & \underline{0} & 0 \\ \underline{0} & \underline{20} & -10 \\ 0 & -10 & 20 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}$$

What is the difference between $\underline{B}$' and $\underline{B}'^{out}$?

$$\Delta\underline{B}' = \underline{B}'^{out} - \underline{B}' = \begin{bmatrix} \underline{10} & \underline{0} & 0 \\ \underline{0} & \underline{20} & -10 \\ 0 & -10 & 20 \end{bmatrix} - \begin{bmatrix} \underline{20} & \underline{-10} & 0 \\ \underline{-10} & \underline{30} & -10 \\ 0 & -10 & 20 \end{bmatrix} = \begin{bmatrix} \underline{-10} & \underline{10} & 0 \\ \underline{10} & \underline{-10} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that the elements in $\Delta\underline{B}$' are all multiples of B'_{23}=-10, i.e.,

$$\Delta\underline{B}' = -10\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice that the above matrix can be expressed as

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

From this simple illustration, we can see a generalization, that whenever we remove a branch between buses i and j, with corresponding $\underline{B}$' matrix element B'_{ij}, the $\underline{B}$' matrix will change as indicated below.

Inconsistency: In Section 6.0, we used (i,j) to indicate terminals of the circuit to be loaded (l) and (m,n) to indicate terminals of the circuit to be outaged (k). In the development of this section, the nomenclature on terminal number has been reversed, i.e., (i,j) becomes the terminals of the circuit to be outaged (k) and (m,n) becomes the terminals of the circuit to be loaded (l).

$$\Delta\underline{\boldsymbol{B}}' = \underline{\boldsymbol{B}}'^{out} - \underline{\boldsymbol{B}}' = \boldsymbol{b_{ij}} \begin{array}{c} \begin{array}{ccccccc} & & \boldsymbol{i} & & \boldsymbol{j} & & \\ & & \downarrow & & \downarrow & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \\ 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -1 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ \leftarrow \boldsymbol{i} \\ \\ \leftarrow \boldsymbol{j} \\ \\ \\ \end{array} \qquad (22)$$

where b_{ij} is the susceptance of branch i-j. We use b_{ij} instead of B'_{ij} in order to ensure we have a defined term even when i or j are the swing bus. Notice that b_{ij} will always be negative.

The previous relation may be expressed as

Caution: The designated positions in the row and column vectors correspond to buses i and j, i.e., they are not the i[th] and j[th] positions.

$$\Delta \underline{B}' = b_{ij} \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \times \begin{array}{c} \begin{bmatrix} 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \end{bmatrix} \\ \begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ & & i & & j & & \end{array} \end{array} \tag{23}$$

If we define

$$\underline{e}_{ij} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \tag{24}$$

then (23) becomes

$$\Delta \underline{B}' = b_{ij} \underline{e}_{ij} \underline{e}_{ij}^T \tag{25}$$

Special case: If the branch to be outaged is connected to the swing bus (in our case, it is bus #1), then,

if i=1, if j=1,

$$\underline{e}_{1j} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow j \\ \\ \\ \end{matrix} \qquad \underline{e}_{i1} = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \\ \\ \\ \end{matrix}$$

From (22), and using (25), we have that

$$\Delta\underline{B}' = \underline{B}'^{out} - \underline{B}' = b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T} \qquad (26)$$

Therefore the post-contingency B' matrix can be expressed as

$$\underline{B}'^{out} = \underline{B}' + \Delta\underline{B}' = \underline{B}' + b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T} \qquad (27)$$

From (1), we recall the DC power flow relation as

$$\underline{P} = \underline{B}'\underline{\theta} \qquad (1)$$

If, when we remove the branch connected between buses i and j, the angles change by $\Delta\underline{\theta}$, then the new (post-contingency) angles will be $\underline{\theta}+\Delta\underline{\theta}$, and (1) becomes

$$\underline{P} = \underline{B}'^{out}(\underline{\theta} + \Delta\underline{\theta}) \qquad (28)$$

Substituting (27) into (28), we obtain

$$\underline{P} = \left(\underline{B}' + b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)(\underline{\theta} + \Delta\underline{\theta}) \qquad (29)$$

We can solve for the new angles according to

$$\underline{\theta} + \Delta\underline{\theta} = \left(\underline{B}' + b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1}\underline{P} \qquad (30)$$

We do not seem to have made much progress, because we still have to take an inverse…

However, there is a significant benefit to writing the new matrix in the way that we have written it, and that benefit becomes apparent if we learn a certain matrix relation. This relation is generally referred to as a lemma.

<u>Matrix Inversion Lemma (MIL)</u>: Assume B' is a nonsingular n×n matrix, and let $\underline{c}$ and $\underline{d}$ be n×M matrices with M<n. Then:

$$\left(\underline{B}'+\underline{c}\,\underline{d}^T\right)^{-1}=\underline{B}'^{-1}-\underline{B}'^{-1}\underline{c}\left[\underline{I}^{(M)}+\underline{d}^T\underline{B}'^{-1}\underline{c}\right]^{-1}\underline{d}^T\underline{B}'^{-1}$$

where $\underline{I}^{(M)}$ is the M×M identity matrix.

We neglect the proof but mention that it is proved in [3, p. 100] by simply multiplying the right-hand-side of MIL by the expression inside the brackets of the left-hand-side, and showing that the product is the n×n identity matrix.

We also mention that MIL is *derived* in [4, pp. 138-140].

It may not be very obvious at this point that MIL will help us, since we see 4 different inverses on the right-hand-side of MIL. Let's apply MIL to the inverted term of (30) to see what happens.

Observing that we can define

$$\underline{c}=b_{ij}\underline{e}_{ij}$$
$$\underline{d}^T=\underline{e}_{ij}^T \tag{31}$$

we can apply MIL according to

$$\left(\underline{B}'+b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1} = \underline{B}'^{-1}-\underline{B}'^{-1}b_{ij}\underline{e}_{ij}\left[\underline{I}^{(M)}+\underline{e}_{ij}^{T}\underline{B}'^{-1}b_{ij}\underline{e}_{ij}\right]^{-1}\underline{e}_{ij}^{T}\underline{B}'^{-1} \qquad (32)$$

One of the inverses on the right-hand-side can be addressed right away, however, by identifying the dimensionality of the expression inside the right-hand-side brackets, $[\underline{I}^{(M)}+\underline{d}^{T}\underline{B}'^{-1}\underline{c}]$. Observing from the MIL that M is the number of columns in $\underline{c}$ and $\underline{d}$, and noting from (31) that in our case, $\underline{c}$ and $\underline{d}$ have only M=1 column, we see that what is inside the right-hand-side brackets is a scalar quantity! So that inverse we can take, and accordingly, we can express (32) as:

$$\left(\underline{B}'+b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1} = \underline{B}'^{-1}-\frac{\underline{B}'^{-1}b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}}{1+\underline{e}_{ij}^{T}\underline{B}'^{-1}b_{ij}\underline{e}_{ij}} \qquad (33)$$

Pulling out the scalar multiplier b_{ij} from where it appears in both the numerator and denominator, we have

$$\left(\underline{B}'+b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1} = \underline{B}'^{-1}-\frac{b_{ij}\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}}{1+b_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \qquad (34)$$

Now we can isolate b_{ij} to only one appearance in the expression by dividing top and bottom by it, resulting in:

$$\left(\underline{B}'+b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1} = \underline{B}'^{-1}-\frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}}{\frac{1}{b_{ij}}+\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \qquad (35)$$

What we have just expressed in (35) is the inverted term on the right-hand-side of (30), repeated below for convenience:

$$\underline{\theta}+\Delta\underline{\theta} = \left(\underline{B}'+b_{ij}\underline{e}_{ij}\underline{e}_{ij}^{T}\right)^{-1}\underline{P} \qquad (30)$$

Substituting (35) into (30), we obtain:

$$\underline{\theta} + \Delta\underline{\theta} = \left(\underline{B}'^{-1} - \frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}}{\frac{1}{b_{ij}} + \underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \right)\underline{P} \tag{36}$$

Distributing the injection vector $\underline{P}$ results in

$$\underline{\theta} + \Delta\underline{\theta} = \underline{B}'^{-1}\underline{P} - \frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{P}}{\frac{1}{b_{ij}} + \underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \tag{37}$$

But $\underline{\theta}=\underline{B}'^{-1}\underline{P}$, and therefore we can replace the corresponding expressions in both right-hand-side terms to obtain:

$$\underline{\theta} + \Delta\underline{\theta} = \underline{\theta} - \frac{\underline{B}'^{-1}\underline{e}_{ij}\underline{e}_{ij}^{T}\underline{\theta}}{\frac{1}{b_{ij}} + \underline{e}_{ij}^{T}\underline{B}'^{-1}\underline{e}_{ij}} \tag{38}$$

We can simplify a little more by investigating $\underline{e}_{ij}^{T}\underline{\theta}$ in the numerator. This would be:

$$\underline{e}_{ij}^{T}\underline{\theta} = \begin{bmatrix} 0 & \cdots & \underset{\substack{\uparrow \\ i}}{1} & \cdots & \underset{\substack{\uparrow \\ j}}{-1} & \cdots & 0 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_i \\ \vdots \\ \theta_j \\ \vdots \\ \theta_{n+1} \end{bmatrix} = \theta_i - \theta_j \tag{39}$$

Substituting (39) into (38) results in:

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{\underline{B}'^{-1}\underline{e}_{ij}(\theta_i-\theta_j)}{\frac{1}{b_{ij}}+\underline{e}_{ij}^T\underline{B}'^{-1}\underline{e}_{ij}} \tag{40}$$

Now we have only two inverses left. Interestingly, they both pre-multiply $\underline{e}_{ij}$. That is, we observe that both inverses appear in $\underline{B}'^{-1}\underline{e}_{ij}$, an n×1 vector.

Question: Besides inverting $\underline{B}'^{-1}$, how might we evaluate this term?

Advice: When you don't know how to evaluate something, just name it. Then, if things don't get better right away, you can at least move on with a sort of indicator of where your problem lies.

So let's name this n×1 vector as g^{ij}, i.e.,

$$\underline{g}^{ij}=\underline{B}'^{-1}\underline{e}_{ij} \tag{41}$$

Not sure if that helps much but it does indicate that

$$\underline{B}'\underline{g}^{ij}=\underline{e}_{ij} \tag{42}$$

Equation (42) should stimulate a very good idea within your mind. Since we very well know $\underline{B}'$ and $\underline{e}_{ij}$, we can obtain g^{ij} through LU factorization. Doing so will give us everything we need to evaluate (40), which, when we substitute g^{ij} for $\underline{B}'^{-1}\underline{e}_{ij}$, becomes:

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{(\theta_i-\theta_j)}{\frac{1}{b_{ij}}+\underline{e}_{ij}^T\underline{g}^{ij}}\underline{g}^{ij} \tag{43}$$

One last small change should be made to (43), and that is to recognize that the term in the denominator $\underline{e}_{ij}^T g^{ij}$ can be expressed as

$$\underline{e}_{ij}^{T}\underline{g}^{ij}=\begin{array}{c}\left[\begin{array}{ccccccc}0 & \cdots & 1 & \cdots & -1 & \cdots & 0\end{array}\right]\\ \begin{array}{ccccccc} & & \uparrow_{i} & & \uparrow_{j} & & \end{array}\end{array}\begin{bmatrix}g_{2}^{ij}\\ \vdots\\ g_{i}^{ij}\\ \vdots\\ g_{j}^{ij}\\ \vdots\\ g_{n+1}^{ij}\end{bmatrix}=g_{i}^{ij}-g_{j}^{ij} \tag{44}$$

Therefore, (43) becomes

$$\underline{\theta}+\Delta\underline{\theta}=\underline{\theta}-\frac{(\theta_i-\theta_j)}{\frac{1}{b_{ij}}+(g_i^{ij}-g_j^{ij})}\underline{g}^{ij} \tag{45}$$

Now what is the LODF? Recall the definition of the LODF is

$$d_{\ell,k}=\Delta f_{\ell}\ /\ f_{k0} \tag{16}$$

where we recall that

- k designates the outaged circuit, terminated by buses i and j;
- ℓ designates the circuit for which we want to compute the new flow, terminated by buses m and n.

Inconsistency: Recall the note on p. 22, which indicates an inconsistency in nomenclature with Sec 6.0.

First, let's express the denominator of (16) f_{k0}, which is

$$f_{k0}=-b_{ij}(\theta_i-\theta_j)=-b_{ij}\underline{e}_{ij}^{T}\underline{\theta} \tag{46}$$

Now let's express the numerator of (16) Δf_{ℓ}, which is

$$\Delta f_{\ell}=-b_{mn}(\Delta\theta_m-\Delta\theta_n)=-b_{mn}\underline{e}_{mn}^{T}\Delta\underline{\theta} \tag{47}$$

But note that $\Delta\underline{\theta}$ in (47) can be expressed using the second term of (45), i.e.,

$$\Delta \underline{\theta} = -\frac{(\theta_i - \theta_j)}{\frac{1}{b_{ij}} + (g_i^{ij} - g_j^{ij})} \underline{g}^{ij} \tag{48}$$

Substituting (48) into (47) results in

$$\Delta f_\ell = b_{mn} \underline{e}_{mn}^T \frac{(\theta_i - \theta_j)}{\frac{1}{b_{ij}} + (g_i^{ij} - g_j^{ij})} \underline{g}^{ij} \tag{49}$$

It is helpful at this point to rearrange (49) according to

$$\Delta f_\ell = b_{mn} \frac{b_{ij}(\theta_i - \theta_j)}{1 + b_{ij}(g_i^{ij} - g_j^{ij})} \underline{e}_{mn}^T \underline{g}^{ij} \tag{50}$$

We recognize in (50) that

$$f_{k0} = -b_{ij}(\theta_i - \theta_j) \tag{51}$$

and

$$\underline{e}_{mn}^T \underline{g}^{ij} = g_m^{ij} - g_n^{ij} \tag{52}$$

Substituting (51) and (52) into (50) results in

$$\Delta f_\ell = -b_{mn} \frac{f_{k0}(g_m^{ij} - g_n^{ij})}{1 + b_{ij}(g_i^{ij} - g_j^{ij})} \tag{53}$$

So (53) can be used to obtain the change in flow on circuit ℓ (terminated by buses m and n) due to outage of circuit k (terminated by buses i and j).

To get the LODF, we divide (53) by f_{k0}, resulting in

$$d_{\ell,k} = \frac{\Delta f_\ell}{f_{k0}} = -b_{mn} \frac{(g_m^{ij} - g_n^{ij})}{1 + b_{ij}(g_i^{ij} - g_j^{ij})} \tag{54}$$

The approach, then, to using (54), is to factorize B' into the L and U factors *once*. Then, for each contingency k=1,…,N_C, (per (42)), we use forward and backwards substation to obtain the vector g^{ij}. The LODFs for every branch ℓ (terminated by buses m and n), are then computed from (54).

Example 5:
Consider our 4-bus, 5-branch example problem again. Compute g^{ij} for a line 2-3 outage. Then use it to compute the post-contingency flow on circuit 3-4.

Solution:
Recall (42):

$$\underline{B}'\underline{g}^{ij} = \underline{e}_{ij}$$

where B' is given by:

$$\underline{B}' = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}$$

and $\underline{e}_{23}$ is given by

$$\underline{e}_{23} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow i=2 \\ \leftarrow j=3 \\ \end{array}$$

And so our equation is:

$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} g_2^{23} \\ g_3^{23} \\ g_4^{23} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Performing LU decomposition, we obtain

$$\underline{L}=\begin{bmatrix}20 & 0 & 0\\ -10 & 25 & 0\\ 0 & -10 & 16\end{bmatrix} \qquad \underline{U}=\begin{bmatrix}1 & -0.5 & 0\\ 0 & 1 & -0.4\\ 0 & 0 & 1\end{bmatrix}$$

Notice that the above factors need be computed only one time; they may subsequently be applied to obtain the g-vector for outage of any circuit. In this case, we are interested in outage of the line from bus 2 to bus 3, therefore we write

$$\underline{L}\underline{w}=\underline{e}_{23}\Rightarrow\begin{bmatrix}20 & 0 & 0\\ -10 & 25 & 0\\ 0 & -10 & 16\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}=\begin{bmatrix}0.05\\ -0.02\\ -0.0125\end{bmatrix}$$

$$\underline{U}\,\underline{g^{23}}=\underline{w}\Rightarrow\begin{bmatrix}1 & -0.5 & 0\\ 0 & 1 & -0.4\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}g_2^{23}\\ g_3^{23}\\ g_4^{23}\end{bmatrix}=\begin{bmatrix}0.05\\ -0.02\\ -0.0125\end{bmatrix}\Rightarrow\begin{bmatrix}g_2^{23}\\ g_3^{23}\\ g_4^{23}\end{bmatrix}=\begin{bmatrix}0.0375\\ -0.025\\ -0.0125\end{bmatrix}$$

Then we can compute the LODF for the circuit 3-4 after outage of circuit 2-3:

$$d_{\ell,k}=\frac{\Delta f_\ell}{f_{k0}}=-b_{mn}\frac{(g_m^{ij}-g_n^{ij})}{1+b_{ij}(g_i^{ij}-g_j^{ij})}$$

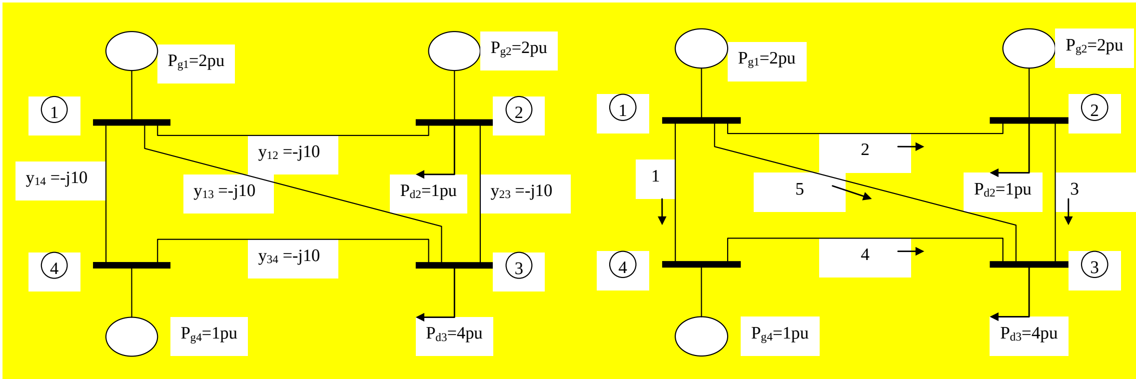
$$=-b_{34}\frac{(g_3^{23}-g_4^{23})}{1+b_{23}(g_2^{23}-g_3^{23})}=10\frac{(-0.025--0.0125)}{1+-10(0.0375--0.025)}$$

$$=10\frac{-0.0125}{0.375}=-0.333$$

If circuit 2-3 has flow of 0.25, then the change in flow on circuit 3-4, following outage of circuit 2-3 becomes

$$\Delta f_\ell=d_{\ell,k}f_{k0}=-0.333*0.25=-0.0833$$

If the pre-contingency flow on circuit 3-4 was 1.25, then
Now let’s check it with the DC power flow.



With all lines in we obtain

$$\underline{P}_B = (\underline{D}\times\underline{A})\left[\underline{B}'\right]^{-1}\underline{P}$$

$$\begin{bmatrix} P_{B1}\\ P_{B2}\\ P_{B3}\\ P_{B4}\\ P_{B5}\end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0\\ 0 & 10 & 0 & 0 & 0\\ 0 & 0 & 10 & 0 & 0\\ 0 & 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 0 & 10\end{bmatrix}\begin{bmatrix} 0 & 0 & -1\\ -1 & 0 & 0\\ 1 & -1 & 0\\ 0 & -1 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix} 20 & -10 & 0\\ -10 & 30 & -10\\ 0 & -10 & 20\end{bmatrix}^{-1}\begin{bmatrix} 1\\ -4\\ 1\end{bmatrix}$$

$$\begin{bmatrix} P_{B1}\\ P_{B2}\\ P_{B3}\\ P_{B4}\\ P_{B5}\end{bmatrix} = \begin{bmatrix} 0.25\\ 0.25\\ 1.25\\ 1.25\\ 1.5\end{bmatrix}$$

Observe that the flow on circuit 3-4 is 1.25 pu.
With circuit 2-3 out, we obtain:

$$
\begin{bmatrix} P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}
$$

$$
\begin{bmatrix} P_{B1} \\ P_{B3} \\ P_{B4} \\ P_{B5} \end{bmatrix} = \begin{bmatrix} 0.6667 \\ -1.0 \\ 1.6667 \\ 2.3333 \end{bmatrix}
$$

Also use W&W's method.

Example 6:

For outage of branch connected to swing.

Do it three ways as in Example 6:

a. Use the above method
b. Use DC flow with and without outage;
c. Use W&W's method.

References:

[1] J. Medina, "Interchange Distribution Calculator (IDC) for Transmission Congestion Management: Implementation and Challenges," presentation at the PSERC IAB Meeting, May 16, 2008, available at http://www.pserc.org/cgi-pserc/getbig/pserconly/iabmeeting/may14-16am/presentati/medina_oati_idc_pserc_iab_may08.pdf.

[2] North American Electric Reliability Council (NERC) Operating Manual, Appendix 9C1, May, 2004, available at www.nerc.com.

[3] A. Debs, "Modern Power Systems Control and Operation," Kluwer, 1988.

[4] A. Monticelli, "State estimation in electric power systems, a generalized approach," Kluwer, 1999.