The Power Flow Equations

1.0 The Admittance Matrix

Current injections at a bus are analogous to power injections. The student may have already been introduced to them in the form of current sources at a node. Current injections may be either positive (into the bus) or negative (out of the bus). Unlike current flowing through a branch (and thus is a branch quantity), a current injection is a nodal quantity. The admittance matrix, a fundamental network analysis tool that we shall use heavily, relates current injections at a bus to the bus voltages. Thus, the admittance matrix relates nodal quantities. We motivate these ideas by introducing a simple example. *We assume that all electrical variables in this document are given in the per-unit system*.

Fig. 1 shows a network represented in a hybrid fashion using one-line diagram representation for the nodes (buses 1-4) and circuit representation for the branches connecting the nodes and the branches to ground. The branches connecting the nodes represent lines. The branches to ground represent any shunt elements at the buses, including the charging capacitance at either end of the line. All branches are denoted with their admittance values y_{ij} for a branch connecting bus i to bus j and y_i for a shunt element at bus i. The current injections at each bus i are denoted by I_i.

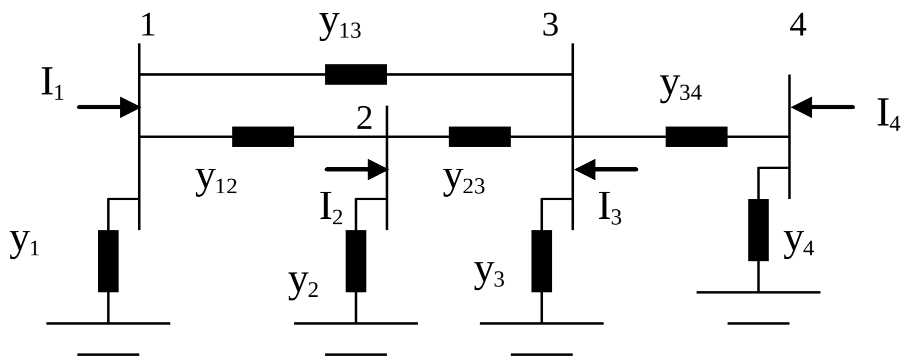

Fig. 1: Network for Motivating Admittance Matrix

Kirchoff's Current Law (KCL) requires that each of the current injections be equal to the sum of the currents flowing out of the bus and into the lines connecting the bus to other buses, or to the ground. Therefore, recalling Ohm's Law, I=V/z=Vy, the current injected into bus 1 may be written as:

$$I_1=(V_1-V_2)y_{12} + (V_1-V_3)y_{13} + V_1y_1 \qquad (1)$$

To be complete, we may also consider that bus 1 is "connected" to bus 4 through an infinite impedance, which implies that the corresponding admittance y_{14} is zero. The advantage to doing this is that it allows us to consider that bus 1 *could be* connected to any bus in the network. Then, we have:

$$I_1=(V_1-V_2)y_{12} + (V_1-V_3)y_{13} + (V_1-V_4)y_{14} + V_1y_1 \qquad (2)$$

Note that the current contribution of the term containing y_{14} is zero since y_{14} is zero. Rearranging eq. 2, we have:

$$I_1= V_1(y_1 + y_{12} + y_{13} + y_{14}) + V_2(-y_{12})+ V_3(-y_{13}) + V_4(-y_{14}) \qquad (3)$$

Similarly, we may develop the current injections at buses 2, 3, and 4 as:

$$I_2= V_1(-y_{21}) + V_2(y_2 + y_{21} + y_{23} + y_{24}) + V_3(-y_{23}) + V_4(-y_{24}) \qquad (4)$$
$$I_3= V_1(-y_{31})+ V_2(-y_{32}) + V_3(y_3 + y_{31} + y_{32} + y_{34}) + V_4(-y_{34})$$
$$I_4= V_1(-y_{41})+ V_2(-y_{42}) + V_3(-y_{34})+ V_4(y_4 + y_{41} + y_{42} + y_{43})$$

where we recognize that the admittance of the circuit from bus k to bus i is the same as the admittance from bus i to bus k, i.e., $y_{ki}=y_{ik}$ From eqs. (3) and (4), we see that the current injections are linear functions of the nodal voltages. Therefore, we may write these equations in a more compact form using matrices according to:

$$
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} y_1 + y_{12} + y_{13} + y_{14} & -y_{12} & -y_{13} & -y_{14} \\ -y_{21} & y_2 + y_{21} + y_{23} + y_{24} & -y_{23} & -y_{24} \\ -y_{31} & -y_{32} & y_3 + y_{31} + y_{32} + y_{34} & -y_{34} \\ -y_{41} & -y_{42} & -y_{43} & y_4 + y_{41} + y_{42} + y_{43} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{5}
$$

The matrix containing the network admittances in eq. (5) is the admittance matrix, also known as the Y-bus, and denoted as:

$$
\underline{Y} = \begin{bmatrix} y_1 + y_{12} + y_{13} + y_{14} & -y_{12} & -y_{13} & -y_{14} \\ -y_{21} & y_2 + y_{21} + y_{23} + y_{24} & -y_{23} & -y_{24} \\ -y_{31} & -y_{32} & y_3 + y_{31} + y_{32} + y_{34} & -y_{34} \\ -y_{41} & -y_{42} & -y_{43} & y_4 + y_{41} + y_{42} + y_{43} \end{bmatrix} \tag{6}
$$

Denoting the element in row i, column j, as Y_{ij}, we rewrite eq. (6) as:

$$
\underline{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \tag{7}
$$

where the terms Y_{ij} are not admittances but rather elements of the admittance matrix. Therefore, eq. (5) becomes:

$$
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \tag{8}
$$

By using eq. (7) and (8), and defining the vectors $\underline{V}$ and $\underline{I}$, we may write eq. (8) in compact form according to:

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \qquad \underline{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \qquad \Rightarrow \qquad \underline{I} = \underline{Y}\underline{V} \quad (9)$$

We make several observations about the admittance matrix given in eqs. (6) and (7). These observations hold true for any linear network of any size.

1. The matrix is symmetric, i.e., $Y_{ij}=Y_{ji}$.
2. A diagonal element Y_{ii} is obtained as the sum of admittances for all branches connected to bus i, including the shunt branch, i.e., $Y_{ii} = y_i + \sum_{k=1,k\neq i}^{N} y_{ik}$, where we emphasize once again that y_{ik} is non-zero only when there exists a physical connection between buses i and k.
3. The off-diagonal elements are the negative of the admittances connecting buses i and j, i.e., $Y_{ij}=-y_{ji}$.

These observations enable us to formulate the admittance matrix very quickly from the network based on visual inspection. The following example will clarify.

Example 1

Consider the network given in Fig. 2, where the numbers indicate admittances.

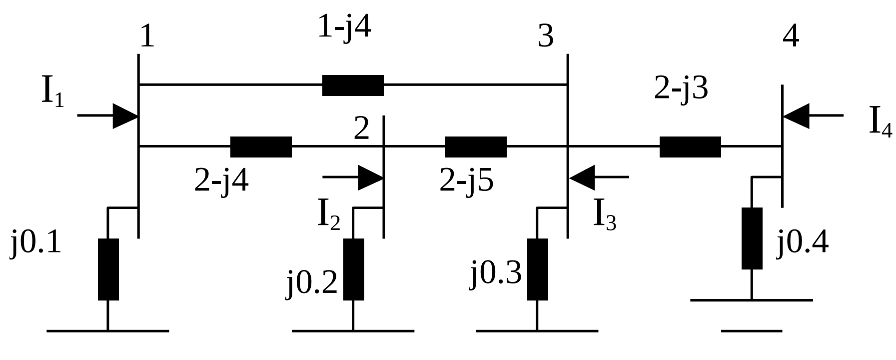

Fig. 2: Circuit for Example 1

The admittance matrix is given by inspection as:

$$\underline{Y}=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}=\begin{bmatrix} 3-j7.9 & -2+j4 & -1+j4 & 0 \\ -2+j4 & 4-j8.8 & -2+j5 & 0 \\ -1+j4 & -2+j5 & 5-j11.7 & -2+j3 \\ 0 & 0 & -2+j3 & 2-j2.6 \end{bmatrix}$$

2.0 The power flow equations

Define the net complex power injection into a bus as $S_k=S_{gk}-S_{dk}$. In this section, we desire to derive an expression for this quantity in terms of network voltages and admittances. We begin by reminding the reader that all quantities are assumed to be in per unit, so we may utilize single-phase power relations. Drawing on the familiar relation for complex power, we may express S_k as:

$$S_k=V_kI_k^* \tag{10}$$

From eq. (8), we see that the current injection into any bus k may be expressed as

$$I_k = \sum_{j=1}^{N} Y_{kj} V_j \tag{11}$$

where, again, we emphasize that the Y_{kj} terms are admittance matrix elements and not admittances. Substitution of eq. (11) into eq. (10) yields:

$$S_k = V_k \left(\sum_{j=1}^{N} Y_{kj} V_j \right)^* = V_k \sum_{j=1}^{N} Y_{kj}^* V_j^* \tag{12}$$

Recall that V_k is a phasor, having magnitude and angle, so that $V_k=|V_k|\angle\theta_k$. Also, Y_{kj}, being a function of admittances, is therefore generally complex, and we define G_{kj} and B_{kj} as the real and imaginary parts of the admittance matrix element Y_{kj}, respectively, so that $Y_{kj}=G_{kj}+jB_{kj}$. Then we may rewrite eq. (12) as

$$\begin{aligned}
S_k &= V_k \sum_{j=1}^{N} Y_{kj}^* V_j^* = |V_k|\angle\theta_k \sum_{j=1}^{N} (G_{kj} + jB_{kj})^* \left(|V_j|\angle\theta_j\right)^* \\
&= |V_k|\angle\theta_k \sum_{j=1}^{N} (G_{kj} - jB_{kj}) \left(|V_j|\angle-\theta_j\right) \\
&= \sum_{j=1}^{N} |V_k|\angle\theta_k \left(|V_j|\angle-\theta_j\right)(G_{kj} - jB_{kj}) \\
&= \sum_{j=1}^{N} \left(|V_k||V_j|\angle(\theta_k - \theta_j)\right)(G_{kj} - jB_{kj})
\end{aligned} \tag{13}$$

Recall, from the Euler relation, that a phasor may be expressed as complex function of sinusoids, i.e., $V=|V|\angle\theta=|V|\{\cos\theta+j\sin\theta\}$. With this, we may rewrite eq. (13) as

$$\begin{aligned} S_k &= \sum_{j=1}^{N} \left(|V_k||V_j| \angle(\theta_k - \theta_j) \right)(G_{kj} - jB_{kj}) \\ &= \sum_{j=1}^{N} |V_k||V_j| \left(\cos(\theta_k - \theta_j) + j\sin(\theta_k - \theta_j) \right)(G_{kj} - jB_{kj}) \end{aligned} \quad (14)$$

If we now perform the algebraic multiplication of the two terms inside the parentheses of eq. (14), and then collect real and imaginary parts, and recall that $S_k=P_k+jQ_k$, we can express eq. (14) as two equations, one for the real part, P_k, and one for the imaginary part, Q_k, according to:

$$\begin{aligned} P_k &= \sum_{j=1}^{N} |V_k||V_j| \left(G_{kj}\cos(\theta_k - \theta_j) + B_{kj}\sin(\theta_k - \theta_j) \right) \\ Q_k &= \sum_{j=1}^{N} |V_k||V_j| \left(G_{kj}\sin(\theta_k - \theta_j) - B_{kj}\cos(\theta_k - \theta_j) \right) \end{aligned} \quad (15)$$

The two equations of (15) are called the power flow equations, and they form the fundamental building block from which we attack the power flow problem.

3.0 Solving the power flow problem

The standard power flow problem is as follows:

Given that for each bus (node) in the network, we know 2 out of the following 4 variables: P_k, Q_k, $|V_k|$, θ_k, so that for each bus, there are two equations available – those of eq. (15) above, and there are two unknown variables. Thus the power flow problem is to solve the power flow equations of (15) for the remaining two variables per bus.

This problem is one where we are required to solve simultaneous nonlinear equations. Because most power systems are very large interconnections, with many buses, the number of power flow equations (and thus the number of unknowns) is very large. For example, a model of the eastern interconnection in the US can have 50,000 buses.

The approach to solving the power flow problem is to use an iterative algorithm. The Newton-Raphson algorithm is the most commonly used algorithm in commercial power flow programs. Starting with a reasonable guess at the solution (where the "solution" is a numerical value of all of the unknown variables), this algorithm checks to see how close the solution is, and then if it is not close enough, updates the solution in a direction that is sure to improve it, and then repeats the check. This process continues until the check is satisfied. Usually, this process requires 5-20 iterations to converge to a satisfactory solution. For large networks, it is computationally intensive.

In this class, we are very interested in optimization methods for finding maximum surplus solutions to the problem of how to dispatch the generation. So far, we have dealt with problems where all generation and load was considered to be at the same bus (node) and were thus able to ignore the network. But in reality, generation and load are located at various buses, and the transportation mechanism for moving electrical energy from supply to consumption is the transmission network. If there are losses or constraints in the transmission network (which there are), these will influence how supply can be allocated, and the most economically desirable solutions may not be feasible.

To account for the network in the economic optimization problems we have posed, we must account for the equations that correspond to the network. These are the power flow equations. This can be done, and problem that results is referred to as the optimal power flow (OPF). However, because the power flow equations are nonlinear, the OPF requires a nonlinear optimization method for its solution. Nonlinear optimization (usually called *nonlinear programming* instead of *linear programming*) is a rich, interesting, and highly applicable area. You can take entire courses on this subject (see, for example, IE 631).

But we do not have time in this class to learn nonlinear programming methods. And fortunately, since we have learned linear programming, we do not need to do so *if we can convert our nonlinear problem into a linear one.*

In our problem, where we desire to maximize social surplus, the objective function may be nonlinear. It is possible to convert a nonlinear objective function into a linear objective function using piecewise linear approximations. This method is very effective in approximating an objective function that is a *separable* function (can be separated into components), since each component is a function of only one variable. In other words, we are able to apply piecewise linear approximation to each individual utility or cost function because each one is a function of only one variable. Fig. 3 illustrates a piecewise linear approximation of a cost function.

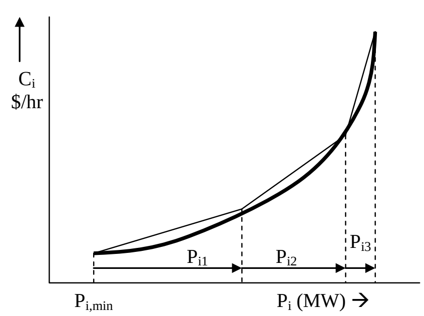

Fig. 3

The power flow equations are functions of many variables, and it is very complex to see how to apply piecewise linear approximations, since a piecewise linear approximation for any one variable will depend on the value of the other variables.

Thus, we seek another method of converting our nonlinear power flow equations into linear equations.

4.0 Approximations to the power flow equations

Let's reconsider the power flow equations:

$$P_k = \sum_{j=1}^{N} |V_k||V_j| \left(G_{kj} \cos(\theta_k - \theta_j) + B_{kj} \sin(\theta_k - \theta_j) \right)$$
$$Q_k = \sum_{j=1}^{N} |V_k||V_j| \left(G_{kj} \sin(\theta_k - \theta_j) - B_{kj} \cos(\theta_k - \theta_j) \right) \qquad (15)$$

We will make use of three practical observations regarding high voltage electric transmission systems.

<u>Observation 1</u>: The resistance of transmission circuits is significantly less than the reactance. Usually, it is the case that the x/r ratio is between 2 and 10. So any given transmission circuit with impedance of z=r-jx will have an admittance of

$$y = \frac{1}{z} = \frac{1}{r + jx} = \frac{1}{r + jx} \bullet \frac{r - jx}{r - jx} = \frac{r - jx}{r^2 + x^2}$$
$$= \frac{r}{r^2 + x^2} - \frac{jx}{r^2 + x^2} = g + jb \qquad (16)$$

From eq. (16), we see that

$$g = \frac{r}{r^2 + x^2} \quad \text{and} \quad b = \frac{-x}{r^2 + x^2} \qquad (17)$$

If r is very small compared to x, then we observe that g will be very small compared to b, and it is reasonable to approximate eqs. (17) as

$$g = 0 \quad \text{and} \quad b = \frac{-1}{x} \tag{18}$$

Now, if g=0, then the real part of all of the Y-bus elements will also be zero, that is, g=0➔G=0.

Applying this conclusion to the power flow equations of eq. (15):

$$P_k = \sum_{j=1}^{N} |V_k||V_j| \left(B_{kj} \sin(\theta_k - \theta_j) \right)$$
$$Q_k = \sum_{j=1}^{N} |V_k||V_j| \left(-B_{kj} \cos(\theta_k - \theta_j) \right) \tag{19}$$

Observation 2: For most typical operating conditions, the difference in angles of the voltage phasors at two buses k and j connected by a circuit, which is θ_k-θ_j for buses k and j, is less than 10-15 degrees. It is extremely rare to ever see such angular separation exceed 30 degrees. Thus, we say that the angular separation across any transmission circuit is "small."

Consider that, in eqs. (19), the angular separation across a transmission circuit, θ_k-θ_j, appears as the argument of the trigonometric functions sine and cosine. What do these functions look like for small angles? We can answer this question by recalling that the sine and cosine functions represent the vertical and horizontal components of a unit (length=1) vector making an angle δ=θ_k-θ_j with the positive x-axis, as illustrated in Fig. 3.

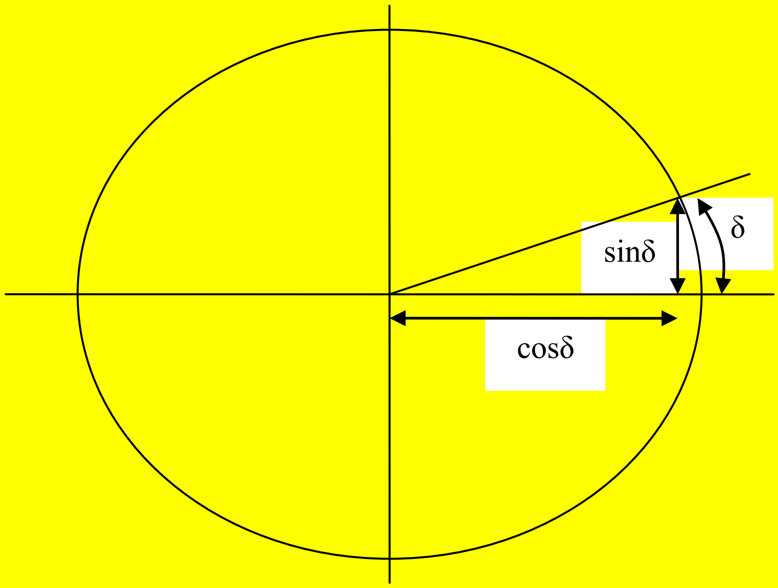

Fig. 3: Trig functions of a small angle

In Fig. 3, it is clear that as the angle $\delta=\theta_k-\theta_j$ gets smaller and smaller, the cosine function approaches 1.0.

One might be tempted to accept the approximation that the sine function goes to zero. This it does, as the angle goes to zero. But an even better approximation is that *the sine of a small angle is the angle itself* (when the angle is given in radians). This can be observed in Fig. 3 from the fact that the vertical line, representing the sine, is almost the same length as the indicated radial distance along the circle, which is the angle (when measured in radians).

Applying these conclusions from observation 2 to eqs. (19):

$$P_k = \sum_{j=1}^{N} |V_k||V_j| \left(B_{kj} (\theta_k - \theta_j) \right)$$

$$Q_k = \sum_{j=1}^{N} |V_k||V_j| \left(- B_{kj} \right) \qquad (20)$$

Note that we have made significant progress at this point, in relation to obtaining linear power flow equations, since we have eliminated the trigonometric terms. However, we still have product terms in the voltage variables, and so we are not done yet. Our next and last observation will take care of these product terms.

Before we do that, however, let's investigate the expressions of eq. (20) a little.

Recall that the quantity B_{kj} is not actually a susceptance but rather an element in the Y-bus matrix.

- If k≠j, then B_{kj}=-b_{kj}, i.e., the Y-bus element in row k column j is the negative of the susceptance of the circuit connecting bus k to bus j.
- If k=j, then $B_{kk} = b_k + \sum_{j=1, j\neq k}^{N} b_{kj}$

Reactive power flow:

The reactive power flow equation of eqs. (20) may be rewritten by pulling out the k=j term from the summation.

$$Q_k = \sum_{j=1}^{N} |V_k||V_j|(-B_{kj}) = -|V_k|^2 B_{kk} - \sum_{\substack{j=1,\\ j\neq k}}^{N} |V_k||V_j|(B_{kj})$$

Then substitute susceptances for the Y-bus elements:

$$Q_k = -|V_k|^2 \left(b_k + \sum_{j=1, j\neq k}^{N} b_{kj} \right) - \sum_{\substack{j=1,\\ j\neq k}}^{N} |V_k||V_j|(-b_{kj})$$

$$= -|V_k|^2 b_k - |V_k|^2 \sum_{j=1, j\neq k}^{N} b_{kj} - \sum_{\substack{j=1,\\ j\neq k}}^{N} |V_k||V_j|(-b_{kj})$$

$$= -|V_k|^2 b_k - \sum_{j=1, j\neq k}^{N} |V_k|^2 b_{kj} + \sum_{\substack{j=1,\\ j\neq k}}^{N} |V_k||V_j|(b_{kj})$$

Now bring all the terms in the two summations under a single summation.

$$Q_k = -|V_k|^2 b_k - \left(\sum_{j=1, j\neq k}^{N} |V_k|^2 b_{kj} - |V_k||V_j|(b_{kj}) \right)$$

Factor out the $|V_k|$ and the $-b_{kj}$ in the summation:

$$Q_k = -|V_k|^2 b_k - \sum_{j=1, j\neq k}^{N} |V_k| b_{kj} \left(|V_k| - |V_j| \right) \qquad (21)$$

Note because we defined the circuit admittance between buses k and j as $y_{kj}=g_{kj}+jb_{kj}$, and because all circuits have series elements that are inductive, the numerical value of b_{kj} is negative. Thus, we can rewrite eq. (21) as

$$Q_k = -|V_k|^2 b_k + \sum_{j=1, j\neq k}^{N} |V_k||b_{kj}| \left(|V_k| - |V_j| \right) \qquad (22)$$

So there are two main terms in eq. (22).

- The first term corresponds to the reactive power supplied (if a capacitor) or consumed (if an inductor) by the shunt susceptance modeled at bus k.
- The second term corresponds to the reactive power flowing on the circuits connected to bus k. Only these circuits will have nonzero b_{kj}. One sees, then, that *each circuit will have per-unit reactive flow in proportion to (a) the bus k voltage magnitude and (b) the difference in per-unit voltages at the circuit's terminating buses*. The direction of flow will be from the higher voltage bus to the lower voltage bus.

Real power flow: Now consider the real power flow equation from eqs. (20), and, as with the reactive power flow equation, let's pull out the j=k term. Thus,

$$P_k = \sum_{j=1}^{N} |V_k||V_j| \left(B_{kj}(\theta_k - \theta_j) \right) = |V_k|^2 \left(B_{kk}(\theta_k - \theta_k) \right) + \sum_{\substack{j=1, \\ j\neq k}}^{N} |V_k||V_j| \left(B_{kj}(\theta_k - \theta_j) \right)$$

Here, we see that the first term is zero, so that:

$$P_k = \sum_{\substack{j=1, \\ j \neq k}}^{N} |V_k||V_j|\left(B_{kj}(\theta_k - \theta_j)\right) \qquad (23)$$

Some comments about this expression:

- There is no "first term" corresponding to shunt elements as there was for the reactive power equation. The reason for this is that, because we assumed that R=0 for the entire network, there can be no shunt resistive element in our model. This actually conforms to reality since we never connect a resistive shunt in the transmission system (this would be equivalent to a giant heater!). The only place where we do actually see an effect which should be modeled as a resistor to ground is in transformers the core loss is so modeled. But the value of this resistance tends to be extremely large, implying the corresponding conductance (G) is extremely small, and it is very reasonable to assume it is zero.
- Therefore the term that we see in eq. (23) represents the real power flow on the circuits connected to bus k. One sees, then, *that each circuit will have per-unit real power flow in proportion to (a) the bus k and j voltage magnitudes and (b) the angular difference across the circuit*. Furthermore, recalling that $B_{kj}=-b_{kj}$, and also that all transmission circuits have series elements that are inductive, the numerical value of b_{kj} is negative, implying that the numerical value of B_{kj} is positive. Therefore, the direction of flow will be from the bus with the larger angle to the bus with the smaller angle.

Observation 3: In the per-unit system, the numerical values of voltage magnitudes $|V_k|$ and $|V_j|$ are very close to 1.0. Typical range under most operating conditions is 0.95 to 1.05. Let's consider the implications of this fact in terms of the expressions for reactive and real power flow eqs. (22) and (23), repeated here for convenience:

$$Q_k = -|V_k|^2 b_k + \sum_{j=1, j\neq k}^{N} |V_k||b_{kj}|\left(|V_k| - |V_j|\right)$$

$$P_k = \sum_{\substack{j=1,\\ j\neq k}}^{N} |V_k||V_j|\left(B_{kj}(\theta_k - \theta_j)\right)$$

Given that $0.95<|V_k|$ and $|V_j|<1.05$, then we incur little error in the above expressions if we assume $|V_k|=|V_j|=1.0$ everywhere that they occur as a multiplying factor. We cannot make this approximation, however, where they occur as a difference, in the reactive power equation, because the difference of two numbers close to 1.0 can range significantly. For example, 1.05-0.95=0.1, but 1.01-1.0=0.01, an order of magnitude difference.

Making this approximation results in:

$$Q_k = -b_k + \sum_{j=1, j\neq k}^{N} |b_{kj}|\left(|V_k| - |V_j|\right) \qquad (24)$$

$$P_k = \sum_{\substack{j=1,\\ j\neq k}}^{N} \left(B_{kj}(\theta_k - \theta_j)\right) \qquad (25)$$

With these equations, we can narrow our statements about power flow.

- Reactive power flow across circuits is determined by the difference in the voltage phasor magnitudes between the terminating buses.
- Real power flow across circuits is determined by the difference in voltage phasor angles between the terminating buses.

Finally, it is interesting to note that the disparity between the maximum reactive power flow and the maximum real power flow across a circuit.

- The reactive power flow equation is proportional to the circuit susceptance and the difference in voltage phasor magnitudes.

The maximum difference in voltage phasor magnitudes will be on the order of 1.05-0.95=0.1.

- The real power flow equation is proportional to the circuit susceptance and the difference in voltage phasor angles. The maximum difference in voltage phasor angles will be, in radians, about 0.52 (which corresponds to 30 degrees).

We see from these last two bullets that real power flow across circuits tends to be significantly larger than reactive power flow, i.e., usually, we will see that

$$P_{kj} >> Q_{kj}$$

This conclusion is consistent with operational experience, which is characterized by an old operator's saying: "Vars don't travel."

5.0 Real vs. Reactive Power Flow

Recall that our original intent was to represent the network in our optimization problem because of our concern that network constraints may limit the ability to most economically dispatch the generation. There are actually several different causes of network constraints, but here, we will limit our interest to the type of constraint that is most common in most networks, and that is circuit overload.

Circuit overload is caused by high current magnitude. When the current magnitude exceeds a given threshold for a particular circuit (called the *rating*), we say that overload has occurred.

In the per-unit system, we recall that

$$S_{kj} = P_{kj} + jQ_{kj} = V_k I_{kj}^*$$

where V_k is the bus k nodal voltage phasor and I_{kj} is the phasor of the current flowing from bus k to bus j. Thus, we have that:

$$I_{kj} = \left(\frac{P_{kj} + jQ_{kj}}{V_k} \right)^*$$

Taking the magnitude (since that is what determines circuit overload), we have:

$$\left| I_{kj} \right| = \frac{\sqrt{P_{kj}^2 + Q_{kj}^2}}{\left| V_k \right|}$$

Given our conclusion on the previous page that generally, $P_{kj} >> Q_{kj}$, we may approximate the above expression according to:

$$\left| I_{kj} \right| \approx \frac{\sqrt{P_{kj}^2}}{\left| V_k \right|} = \frac{\left| P_{kj} \right|}{\left| V_k \right|}$$

and if $|V_k| \approx 1.0$, then we have that

$$\left| I_{kj} \right| \approx \left| P_{kj} \right|$$

Thus, in assessing circuit overload, it is reasonable to look at real power flows only. As a result of this conclusion, we will build into our optimization formulation only the real power flow equations, i.e., eq. (25).

6.0 The DC Power Flow – an example

Let's study the real power flow expression given in eq. (25).

$$P_k = \sum_{\substack{j=1, \\ j \neq k}}^{N} B_{kj} (\theta_k - \theta_j)$$

It is worthwhile to perform a simple example to illustrate use of this expression.

Consider the 4-bus network given in Fig. 4. All 5 lines have the same admittance, and this admittance has no real part indicating we are assuming R=0 for this network. The real power values for each of the three generators and each of the two loads are given. All numerical quantities are given in per-unit.

The problem is to compute the real power flows on all circuits.

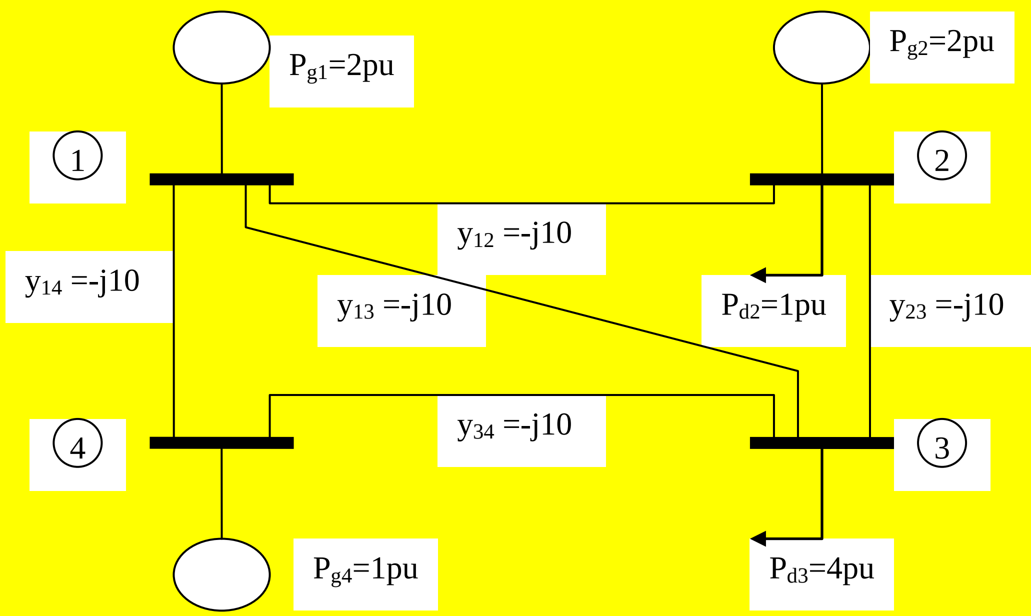

Fig. 4: Four-bus network used in example

We first write down eq. (25) for each bus, beginning with bus 1.

$$P_1 = B_{12}(\theta_1 - \theta_2) + B_{13}(\theta_1 - \theta_3) + B_{14}(\theta_1 - \theta_4)$$

$$= B_{12}\theta_1 - B_{12}\theta_2 + B_{13}\theta_1 - B_{13}\theta_3 + B_{14}\theta_1 - B_{14}\theta_4$$

Collecting terms in the same variables results in:

$$P_1 = \left(B_{12} + B_{13} + B_{14}\right)\theta_1 - B_{12}\theta_2 - B_{13}\theta_3 - B_{14}\theta_4 \qquad (26)$$

Repeating the process for bus 2:

$$P_2 = B_{21}(\theta_2 - \theta_1) + B_{23}(\theta_2 - \theta_3) + B_{24}(\theta_2 - \theta_4)$$
$$= B_{21}\theta_2 - B_{21}\theta_1 + B_{23}\theta_2 - B_{23}\theta_3 + B_{24}\theta_2 - B_{24}\theta_4$$

Again, collecting terms in the same variables results in:

$$P_2 = -B_{21}\theta_1 + (B_{21} + B_{23} + B_{24})\theta_2 - B_{23}\theta_3 - B_{24}\theta_4 \qquad (27)$$

Repeating eqs. (26) and (27), together with the relations for buses 3 and 4, yields:

$$P_1 = (B_{12} + B_{13} + B_{14})\theta_1 - B_{12}\theta_2 - B_{13}\theta_3 - B_{14}\theta_4$$
$$P_2 = -B_{21}\theta_1 + (B_{21} + B_{23} + B_{24})\theta_2 - B_{23}\theta_3 - B_{24}\theta_4$$
$$P_3 = -B_{31}\theta_1 - B_{32}\theta_2 + (B_{31} + B_{32} + B_{34})\theta_3 - B_{34}\theta_4$$
$$P_4 = -B_{41}\theta_1 - B_{42}\theta_2 - B_{43}\theta_3 + (B_{41} + B_{42} + B_{43})\theta_4$$

We can write these equations in matrix form, according to:

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} B_{12}+B_{13}+B_{14} & -B_{12} & -B_{13} & -B_{14} \\ -B_{21} & B_{21}+B_{23}+B_{24} & -B_{23} & -B_{24} \\ -B_{31} & -B_{32} & B_{31}+B_{32}+B_{34} & B_{34} \\ -B_{41} & -B_{42} & -B_{43} & B_{41}+B_{42}+B_{43} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

(28)

Remember, the left-hand-side vector is the injections, which is the generation less the demand.

To get the matrix, it is helpful to first write down the Y-bus:

$$Y = j\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}$$

$$= j\begin{bmatrix} b_1+b_{12}+b_{13}+b_{14} & -b_{12} & -b_{13} & -b_{14} \\ -b_{21} & b_2+b_{21}+b_{23}+b_{24} & -b_{23} & -b_{24} \\ -b_{31} & -b_{32} & b_3+b_{31}+b_{32}+b_{34} & b_{34} \\ -b_{41} & -b_{42} & -b_{43} & b_4+b_{41}+b_{42}+b_{43} \end{bmatrix}$$

$$Y = j\begin{bmatrix} -30 & 10 & 10 & 10 \\ 10 & -20 & 10 & 0 \\ 10 & 10 & -30 & 10 \\ 10 & 0 & 10 & -20 \end{bmatrix}$$

So we readily observe here that, for example, B_{11}=-30, B_{12}=10, B_{13}=10, and B_{14}=10, and it is similar for the other three rows.

So using the Y-bus values, we can express eq. (28) as:

$$\begin{bmatrix} 2 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \qquad (29)$$

(Observe that we omit the j). Then, the angles are given by:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 30 & -10 & -10 & -10 \\ -10 & 20 & -10 & 0 \\ -10 & -10 & 30 & -10 \\ -10 & 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ -4 \\ 1 \end{bmatrix} \qquad (30)$$

However, when we evaluate the above expression by taking the inverse of the indicated matrix, we find it is singular, i.e., it does not have an inverse. The problem here is that we have a dependency in the 4 equations, implying that one of the equations may be obtained from the other three. For example, if we add the bottom three rows and then multiply by -1, we get the top row (in terms of the injection vector, this is just saying that the sum of the generation must equal the demand).

This dependency occurs because all four angles are not independent. What is important are the angular differences. Thus, we are free to choose any one of them as the reference, with a fixed value of 0 degrees. This angle is then no longer a variable (it is 0 degrees), and, referring to eq. (29), the corresponding column in the matrix may be eliminated, since those are the numbers that get multiplied by this 0 degree angle.

To fix this problem, we need to eliminate one of the equations and one of the variables (by setting the variable to zero).

We choose to eliminate the first equation, and set the first variable, θ_1, to zero (which means we are choosing θ_1 as the reference).

This results in:

$$\begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.025 \\ -0.15 \\ -0.025 \end{bmatrix} \qquad (31)$$

The solution on the right-hand-side gives the angles on the bus voltage phasors at buses, 2, 3, and 4.

However, the problem statement requires us to compute the power flows on the lines (this is usually the information needed by operational and planning engineers as they study the power system).

We can get the power flows easily by employing just one term from the summation in eq. (25), which gives the flow across circuit k-j according to:

$$P_{kj} = B_{kj}(\theta_k - \theta_j) \qquad (32)$$

We utilize the Y-bus elements together with the bus angles given by eq. (31) to make these calculations, as follows:

$$P_{12} = B_{12}(\theta_1 - \theta_2) = 10(0 - -0.025) = 0.25$$

$$P_{13} = B_{13}(\theta_1 - \theta_3) = 10(0 - -0.15) = 1.5$$

$$P_{14} = B_{14}(\theta_1 - \theta_4) = 10(0 - -0.025) = 0.25$$
$$P_{23} = B_{23}(\theta_2 - \theta_3) = 10(-0.025 - -0.15) = 1.25$$
$$P_{34} = B_{34}(\theta_3 - \theta_4) = 10(-0.15 - -0.025) = -1.25$$

These computed flows are illustrated in Fig. 5. The power flowing into a bus equals the power flowing out of that bus.

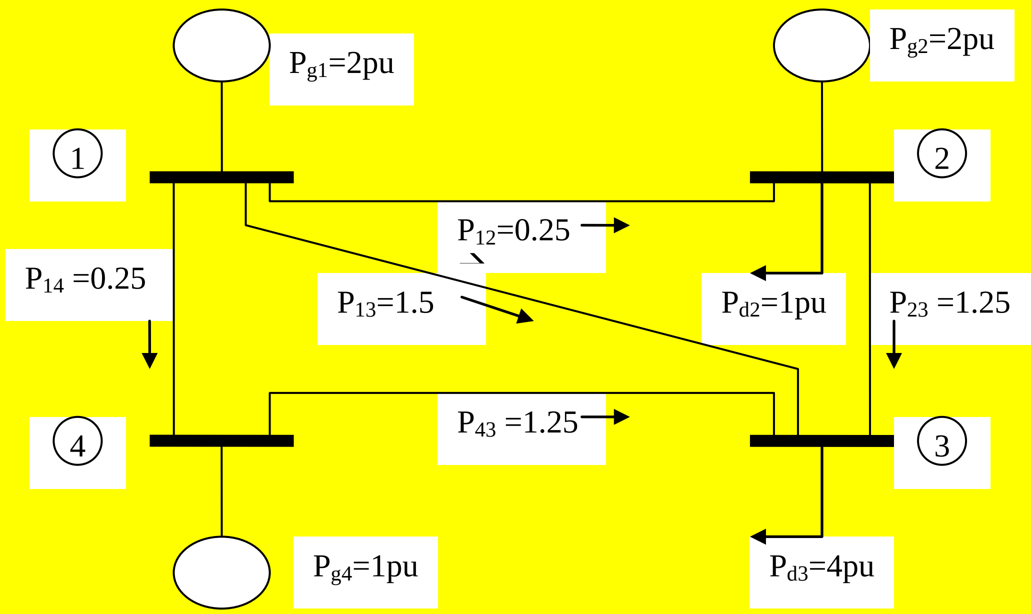

Fig. 5: Computed flows for four-bus network used in example

7.0 The DC Power Flow – Generalization

We desire to generalize the above procedure.

We assume that we are given the network with the following information:

- Total number of buses is N, total number of branches is M.
- Bus number 1 identified as the reference
- Real power injections at all buses except bus 1
- Network topology
- Admittances for all branches.

The DC power flow equations, based on eq. (25) are given in matrix form as

$$\underline{P} = \underline{B}'\underline{\theta} \qquad (33)$$

where

- $\underline{P}$ is the vector of nodal injections for buses 2, …, N
- $\underline{\theta}$ is the vector of nodal phase angles for buses 2,…N
- $\underline{B}$' is the "B-prime" matrix. Generalization of its development requires a few comments.

Development of the B' matrix:

Compare the matrix of eq. (28) with the Y-bus matrix, all repeated here for convenience:

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} B_{12}+B_{13}+B_{14} & -B_{12} & -B_{13} & -B_{14} \\ -B_{21} & B_{21}+B_{23}+B_{24} & -B_{23} & -B_{24} \\ -B_{31} & -B_{32} & B_{31}+B_{32}+B_{34} & B_{34} \\ -B_{41} & -B_{42} & -B_{43} & B_{41}+B_{42}+B_{43} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$Y = j\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}$$

$$= j\begin{bmatrix} b_1+b_{12}+b_{13}+b_{14} & -b_{12} & -b_{13} & -b_{14} \\ -b_{21} & b_2+b_{21}+b_{23}+b_{24} & -b_{23} & -b_{24} \\ -b_{31} & -b_{32} & b_3+b_{31}+b_{32}+b_{34} & b_{34} \\ -b_{41} & -b_{42} & -b_{43} & b_4+b_{41}+b_{42}+b_{43} \end{bmatrix}$$

From the above, we can develop a procedure to obtain the B' matrix from the Y-bus, as follows:

If there are no b_k, then steps 2-3 simplify to "multiply Y-bus by -1

1. Remove the "j" from the Y-bus.
2. Replace diagonal element $\underline{B}'_{kk}$ with the sum of the non-diagonal elements in row k. Alternatively, subtract b_k (shunt term) from B_{kk}, & multiply by -1 (if there is no b_k, then just multiple by -1).
3. Multiply all off-diagonals by -1.
4. Remove row 1 and column 1.

Comparison of the numerical values of the Y-bus with the numerical values of the B' matrix for our example will confirm the above procedure:

$$\underline{Y} = j\begin{bmatrix} -30 & 10 & 10 & 10 \\ 10 & -20 & 10 & 0 \\ 10 & 10 & -30 & 10 \\ 10 & 0 & 10 & -20 \end{bmatrix}$$

$$\underline{B}' = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & -10 \\ 0 & -10 & 20 \end{bmatrix}$$

Another way to remember the B' matrix is to observe that, since its non-diagonal elements are the negative of the Y-bus matrix, the B' non-diagonal elements are susceptances. However, one must be careful to note that the B' matrix element in position row k, column j is the susceptance of the branch connecting buses k+1 and j+1, since the B' matrix does not have a column or row corresponding to bus 1.

Question: Why are shunt terms excluded in the B' matrix? That is, why does excluding them not affect real power calculations?

Although eq. (33) provides the ability to compute the angles, it does not provide the line flows. A systematic method of computing the line flows is:

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} \qquad (34)$$

where:

- $\underline{P}_B$ is the vector of branch flows. It has dimension of M x 1. Branches are ordered arbitrarily, but whatever order is chosen must also be used in $\underline{D}$ and $\underline{A}$.
- $\underline{\theta}$ is (as before) the vector of nodal phase angles for buses 2,…N

- D is an M x M matrix having non-diagonal elements of zeros; the diagonal element in position row k, column k contains the negative of the susceptance of the k^{th} branch.
- A is the M x N-1 *node-arc incidence matrix*. It is also called the adjacency matrix, or the connection matrix. Its development requires a few comments.

Development of the node-arc incidence matrix:

This matrix is well known in any discipline that has reason to structure its problems using a network of nodes and "arcs" (or branches or edges). Any type of transportation engineering is typical of such a discipline.

The node-arc incidence matrix contains a number of rows equal to the number of arcs and a number of columns equal to the number of nodes.

Element (k,j) of A is 1 if the k^{th} branch begins at node j, -1 if the k^{th} branch terminates at node j, and 0 otherwise.

A branch is said to "begin" at node j if the power flowing across branch k is defined positive for a direction *from* node j to the other node.

A branch is said to "terminate" at node j if the power flowing across branch k is defined positive for a direction *to* node j from the other node.

Note that matrix A is of dimension M x N-1, i.e., it has only N-1 columns. This is because we do not form a column with the reference bus, in order to conform to the vector $\underline{\theta}$, which is of dimension (N-1) x 1. This works because the angle being excluded, θ_1, is zero.

We can illustrate development of the node-arc incidence matrix for our example system. Consider numbering the branches as given in Fig. 6. Positive direction of flow is as given by the indicated arrows. The numbers in the circles are bus (node) numbers. The numbers next to each branch are branch numbers.

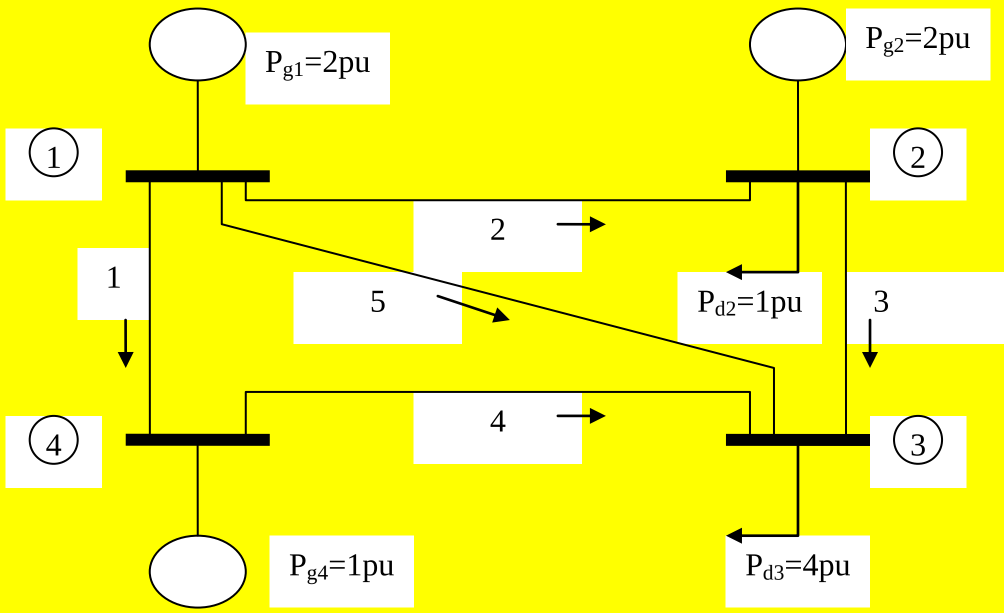

Fig. 6: Branches numbering for development of incidence matrix

Therefore, the node-arc incidence matrix is given as

$$\underline{A} = \overset{\overbrace{nodenumber}}{\begin{matrix} 2 & 3 & 4 \end{matrix}} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \left.\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}\right\} branch\,number$$

The D-matrix is formed by placing the negative of the susceptance of each branch along the diagonal of an M x M matrix, where M=5.

$$\underline{D} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

Combining $\underline{A}$, $\underline{D}$, and $\underline{\theta}$ based on eq. (34) yields

$$\underline{P}_B = (\underline{D} \times \underline{A}) \times \underline{\theta} =$$

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$\begin{bmatrix} P_{B1} \\ P_{B2} \\ P_{B3} \\ P_{B4} \\ P_{B5} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} -\theta_4 \\ -\theta_2 \\ \theta_2 - \theta_3 \\ -\theta_3 + \theta_4 \\ -\theta_3 \end{bmatrix} = \begin{bmatrix} -10\theta_4 \\ -10\theta_2 \\ 10(\theta_2 - \theta_3) \\ 10(-\theta_3 + \theta_4) \\ -10\theta_3 \end{bmatrix}$$

Plugging in the solution for $\underline{\theta}$ that we obtained, which was:

$$\begin{bmatrix}\theta_2\\ \theta_3\\ \theta_4\end{bmatrix}=\begin{bmatrix}-0.025\\ -0.15\\ -0.025\end{bmatrix}$$

We find that the above evaluates to

$$\begin{bmatrix}P_{B1}\\ P_{B2}\\ P_{B3}\\ P_{B4}\\ P_{B5}\end{bmatrix}=\begin{bmatrix}0.25\\ 0.25\\ 1.25\\ 1.25\\ 1.5\end{bmatrix}$$

This solution, obtained systematically, in a way that can be efficiently programmed, agrees with the solution we obtained manually and is displayed in Fig. 5, repeated here for convenience.