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Controllability for a class of discrete-time Hamiltonian systems

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Abstract— **In this paper we study the controllability for a class of discrete-time nonlinear systems which arise from a discretization of a continuous-time integrable Hamiltonian systems. We give necessary and sufficient condition for the global controllability of the discrete-time nonlinear systems. The result in this paper are inspired from Ergodic theory. The basic idea is as follows : for the uncontrolled (drift) system, there exists a ergodic partition, partition of phase space into a subsets. On each of the subset the drift is ergodic i.e., system trajectory will reach any positive measure set within the subset. The aim of the control is only to steer the system from one subset of the partition to another subset.**

I. Introduction

Control of Hamiltonian systems has attracted lot of attention lately because of their applications in the areas like quantum control [1], [2], control of satellite [3], control of mixing [4] and control of power systems. In this paper we study the controllability of a class of discrete-time systems that arise from the discretization of continuous-time integrable Hamiltonian systems. We have combined ergodic theory and control theory ideas to give necessary and sufficient conditions for controllability for a class of discrete-time systems. This particular controllability approach introduced in this paper also gives insight for ways of constructing a control, which is very important from practical point of view.

The approach in this paper is different from the traditional Lie-theoretic approach for proving controllability. In the early 1970s, Brockett, Jurdjevic, Sussmann and others introduced the theory of Lie groups and their associated Lie algebras into the context of nonlinear continuous-time control to express the notions such as controllability, observability. Characterization of controllability for discrete-time systems using Lie-theoretic approach is carried out in [5], [6]. The basic fact that underlines this approach is that one has as analogue for difference equations of the infinitesimal information obtained in the continuous-time case by taking derivatives with respect to time. One uses here derivation with respect to control values. The approach introduced in this paper is inspired from ergodic theory. The key concept is that of the ergodic partition of the drift: partition of the phase space into subsets on which drift is ergodic (for more details on ergodic theory refer [7]). Control is only used to steer the system from one subset of the ergodic partition to another subset. This method of control does not require large control effort because natural dynamics of the system is used very effectively and hence it is advantageous when the control authority is not very big. Controllability result for a two dimensional twist maps using this approach is derived in [8] and for continuous time system in [9]. Methods developed in this paper prove to very powerful when the drift part of the system is integrable, in this case the ergodic partition can be constructed easily. This is particularly true for the case of integrable Hamiltonian systems. For integrable Hamiltonian systems suitable coordinate exist called action-angle coordinates (for more details refer [10]). Control using action-angle coordinate can be related to so called Energy control method introduced by Astrom and Furuta [11]. Action-angle coordinates generalize the idea of the energy of the system for the case where more than one quantity of the system is conserved. Control of quantum systems using action-angle coordinates is studied in [12]. Although the controllability result are proved for the class of discrete-time integrable Hamiltonian systems, the control philosophy is more general and can be applied to a class of discrete-time system for which the drift dynamics is integrable.

This paper is organized as follows. In section 2 we develop some preliminaries for integrable Hamiltonian systems. We derive our discrete-time system by constructing a Poincaré map of continuous time system under the assumption that the control input is small and is held constant over one period of periodic control Hamiltonian. Main result follows in section 3 and conclusion in section 4.

II. Preliminaries

In this section we develop some preliminaries of integrable Hamiltonian systems and show how the discrete-time systems arise from taking the Poincaré map of the continuous-time system. Let M^{2n} be smooth $2n-$ dimensional Poisson manifold with the system of local coordinates $x=(q,p)\in M^{2n}$ and standard Poisson structure [10], [13]. Let $F,G\in C^{\infty}(M^{2n})$ (smooth real valued function defined on M^{2n}), then we define their Poisson bracket as follows

$$\{F,G\}(q,p)=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}-\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right)(p,q)$$

Consider an affine controlled Hamiltonian system (CHS) [13]

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i}(q,p) + \sum_{j=1}^{n} \frac{\partial H_j}{\partial p_i}(p,q,t)u_j$$
$$\dot{p}_i = -\frac{\partial H_0}{\partial q_i}(q,p) - \sum_{j=1}^{n} \frac{\partial H_j}{\partial q_i}(p,q,t)u_j \quad i = 1,..,n \quad (1)$$

where $H_0(p,q) \in C^\infty(M^{2n})$ is a Hamiltonian function of unforced controlled Hamiltonian system. $H_j \in C^\infty(M^{2n}), j = 1,..,n$ are the interaction Hamiltonian being independent functions (in the sense that the corresponding one-form $dH_j \in T^*(M^{2n})$ are linearly independent) and are periodic in t with period T, $u_j, j = 1,..,n$ are control inputs of CHS. The equation (1) provide a convenient mathematical description for various controlled physical and mechanical systems. The q and p components of the phase vector are called "generalized coordinates" and "generalized momenta" correspondingly.

Let the set of independent functions $F_i \in C^\infty(M^{2n}), i = 1,..,k, k \leq n$ and the set of real numbers $f_i, i = 1,..,k$ be given. Then the level set of functions F_i

$$M_f = \{(p,q) \in M^{2n} : F_i(q,p) = f_i, i = 1,..,k\} \qquad (2)$$

is a $(2n-k)$-dimensional submanifold of M^{2n} by virtue of implicit function theorem. It is well known [10] that the condition

$$\{H_0, F_j\} = 0, \quad j = 1,...,k$$

implies that the manifold M_f is invariant set of the unforced controlled Hamiltonian system. We are interested in the case where the unforced controlled Hamiltonian system is integrable. This correspond to the case where $k = n$ (we assume that $F_1 = H_0$). In this case manifold M_f is compact, connected and is diffeomorphic to a $n-$dimensional torus. Moreover there exists a neighborhood $U(M_f) \subset M^{2n}$ such that $U(M_f)$ is diffeomorphic to $B \times T^n$, where $B \subset \mathbf{R}^n$ and $\mathbf{T}^n$ is $n-$torus. There exists canonical coordinates (I, φ) on $B \times T^n$ such that the functions F_i depends only on I. I is called action coordinates and φ is called angle coordinates. In action-angle coordinates the Hamiltonian system is represented by differential equation of the form

$$\dot{I}_i = -\frac{\partial H_0}{\partial \varphi_i} = 0$$
$$\dot{\varphi}_i = -\frac{\partial H_0}{\partial I_i} = \omega_i(I) \ (\text{mod}1) \ \ i = 1,..,n. \qquad (3)$$

We see that I_i is constant along the trajectory of (3). So the action coordinates $\{I_i\}$, $i = 1,..,n$ are the conserved variables of the system. If we assume that the system is non-degenerate i.e., the determinant

$$\det[\frac{\partial \omega}{\partial I}] = \det[\frac{\partial^2 H_0}{\partial I^2}] \neq 0.$$

Then we can make one more coordinate transformation defined as $\omega_i(I) = \bar{I}_i$ for $i = 1,..,n$. In these new coordinate after removing the over-bar notation we get following equation of motion

$$\dot{I}_i = 0$$
$$\dot{\varphi}_i = I_i \ (\text{mod}1) \ \ i = 1,..,n. \qquad (4)$$

Now consider following form of controlled Hamiltonian in action-angle coordinates

$$\mathscr{H} = H_0(I) + \varepsilon \sum_{i=1}^{n} H_i(I, \varphi, t) u_i(t)$$

where $H_0(I)$ is the drift or uncontrolled Hamiltonian and H_i are the control Hamiltonian which are periodic in t with period $T > 0$, $u_i(t) \in [-1, 1]$ are control inputs and $\varepsilon \ll 1$ is a perturbation parameter. We have following equation of motion for the controlled Hamiltonian system.

$$\dot{I} = -\varepsilon \frac{\partial H}{\partial \varphi} u$$
$$\dot{\varphi} = I + \varepsilon \frac{\partial H}{\partial I} u \ (\text{mod}1) \ \ i = 1,..,n \qquad (5)$$

where

$$I = (I_1,..,I_n)^T, \ \varphi = (\varphi_1,..,\varphi_n)^T, \ u = (u_1,..,u_n)^T$$

and the $n \times n$ matrix

$$\frac{\partial H}{\partial I} = \left[\frac{\partial H_j}{\partial I_i}\right], \ \frac{\partial H}{\partial \varphi} = \left[\frac{\partial H_j}{\partial \varphi_i}\right] \ \ i, j = 1,..,n$$

We will derive an approximate form of $2n$-dimensional Poincaré map of the system (5). We will assume that the control inputs $u_i(t)$ are constant over the period T of the control Hamiltonian H_i. Using regular perturbation theory, the solution of (5) are $O(\varepsilon)$ close to the unperturbed solution on the time scale $O(1)$. Hence we have following expansion for the solution of (5).

$$I_\varepsilon(t) = I^0 + \varepsilon I^1(t) + O(\varepsilon^2)$$
$$\varphi_\varepsilon(t) = \varphi^0 + I^0 t + \varepsilon \varphi^1(t) + O(\varepsilon^2) \qquad (6)$$

where $I^1(t)$ and $\varphi^1(t)$ satisfy following first variational equations

$$\dot{I}^1 = -\frac{\partial H(I^0, \varphi^0 + I^0 t, t)}{\partial \varphi} u(t), \ \ \dot{\varphi}^1 = I^1 + \frac{\partial H(I^0, \varphi^0 + I^0 t, t)}{\partial I} u(t)$$

Our aim is to find $2n-$ dimensional Poincaré map that takes variable $I_\varepsilon(0), \varphi_\varepsilon(0)$ to their values after flowing along the solution trajectories of (5) for time T. This map is given by

$$P_\varepsilon : (I_\varepsilon(0), \varphi_\varepsilon(0)) \rightarrow (I_\varepsilon(T), \varphi_\varepsilon(T)) \qquad (7)$$

$$(I^0, \varphi^0) \rightarrow (I^0 + \varepsilon I^1(T), \varphi^0 + I^0 T + \varepsilon \varphi^1(T)) + O(\varepsilon^2) \qquad (8)$$

where we have used (6) and taken following initial condition $I_\varepsilon(0) = I^0$, $\varphi_\varepsilon(0) = \varphi^0$

The expression for $I^1(T)$ and $\varphi^1(T)$ can be obtained by integration under the assumption that the input is held constant over the period of the perturbation. We get

$$I^1(T) = -\int_0^T \frac{\partial H(I^0,\varphi^0+I^0t,t)}{\partial \varphi} dt u(0) = G(I^0,\varphi^0)u(0). \quad (9)$$

Similarly the expression for $\varphi^1(T)$ is

$$\begin{aligned}\varphi^1(T) &= \int_0^T \int_0^t \frac{\partial H(I^0,\varphi^0+I^0\xi,\xi)}{\partial I} u(\xi) d\xi dt \\ &+ \int_0^T \frac{\partial H(I^0,\varphi^0+I^0t,t)}{\partial I} u(t) dt \\ &= F(I^0,\varphi^0)u(0) \quad (10)\end{aligned}$$

Relabelling $(I^0,\varphi^0) = (I,\varphi)$ and $(I(T),\varphi(T)) = (I',\varphi')$ we get following form for the discrete-time control system

$$\mathscr{T}_u \begin{pmatrix} I \\ \varphi \end{pmatrix} = \begin{pmatrix} I' \\ \varphi' \end{pmatrix} = \begin{pmatrix} I+\varepsilon G(I,\varphi)u \\ \varphi+I+\varepsilon F(I,\varphi)u \end{pmatrix} \quad (11)$$

where we have assumed without loss of generality that $T = 1$. For $u = 0$, we write the system as

$$\mathscr{S}(I,\varphi) = (I',\varphi') = (I,\varphi+I) \quad (12)$$

For $u = 0$ case, the system dynamics are easy to understand. The phase space is foliated with $n-$dimensional tori parameterized by I and on each of the tori the motion is either periodic, quasiperiodic or dense orbit depending upon whether I satisfies the resonance condition or not. We say a torus is resonant if the action I parameterizing the torus satisfy following relationship

$$\beta_1 I_1 + ... + \beta_n I_n + n = 0 \text{ for } (\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1} \setminus \{0\}$$

A torus parameterized by I is said to be minimal if orbit from every point on the torus is dense on the torus. It can be shown that [14] on the minimal torus following relation is true

$$\beta_1 I_1 + ... + \beta_n I_n + n \neq 0 \text{ for any } (\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1} \setminus \{0\}$$

III. Main Result

Consider the discrete-time nonlinear system (11), in index notation it can be written as

$$\begin{aligned} I_i' &= I_i + \varepsilon \sum_{j=1}^n u_j g_{ij}(I,\varphi) \\ \varphi_i' &= \varphi_i + I_i + \varepsilon \sum_{j=1}^n u_j f_{ij}(I,\varphi) \pmod 1 \;\; i = 1,..,n(13) \end{aligned}$$

$$\begin{pmatrix} f_{11} & .. & f_{1n} \\ . & .. & . \\ f_{n1} & .. & f_{nn} \end{pmatrix} = F \quad \begin{pmatrix} g_{11} & .. & g_{1n} \\ . & .. & . \\ g_{n1} & .. & g_{nn} \end{pmatrix} = G$$

$I \in B \subset \mathbf{R}^n$, $\varphi \in \mathbf{T}^n$ (n-torus), $\mathscr{A} = B \times \mathbf{T}^n$, $u \in U = [-1,1]^n$, f_{ij} and g_{ij} are functions which are at least $\mathbf{C}^1$ and are periodic in φ with period 1, $\varepsilon \ll 1$ is a perturbation parameter. We denote $I(k) =: I^k$ and $I_i(k) =: I_i^k$ (value of action coordinate on k^{th} iterate) and similarly $\varphi(k) =: \varphi^k, u(k) =: u^k$

We assume the the perturbation g_{ij} are bounded i.e., $|g_{ij}| \leq M$ and satisfies the following regularity condition

For any fixed I let $\mathscr{R}_I = \{\varphi \in \mathbf{T}^n : |\det[G(I,\varphi)]| > \vartheta\}$

$$\text{Then } \mu(\mathscr{R}_I) > \delta > 0 \quad (14)$$

where μ is the lebesgue measure on the torus $\mathbf{T}^n$

Theorem 1: The system (13) is globally controllable under the regularity condition (14) if and only if following controllability conditions are satisfied

1) For all $I = (I_1,..,I_n) \in \mathbf{R}^n \setminus \mathbf{Z}^n$ s.t., $I_1,..,I_n$ satisfies resonance condition i.e., $\sum_{i=1}^n \beta_i I_i + n = 0$ for $(\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1} \setminus \{0\}$
   If $\det[G(\bar{\varphi},I)] = 0$, then there exists an integer $k_1 \in \mathbf{Z}^+$ and a sequence of control input $\{u^0,u^1,..,u^{k_1-1}\}$ such that $T_{u^0,...,u^{k_1-1}}(I,\bar{\varphi}) = (I^{k_1},\varphi^{k_1})$ and I^{k_1} does not satisfy resonance condition i.e., there exists no $(\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1} \setminus \{0\}$ such that $\sum_{i=1}^n \beta_i I_i^{k_1} + n = 0$
2) For $I \in \mathbf{Z}^n$
   a) Both F and G does not vanish simultaneously i.e., $|f_{ij}| + |g_{kl}| \neq 0$ for some $i,j,k,l \in [1,n]$
   b) If $\det[G(\bar{\varphi},I)] = 0$ then there exists an integer $k_2 \in \mathbf{Z}^+$ and a sequence of control input $\{u^0,u^1,..,u^{k_2-1}\}$ such that $T_{u^0,...,u^{k_2-1}}(I,\bar{\varphi}) = (I^{k_2},\varphi^{k_2})$ and I^{k_2} does not satisfy resonance condition i.e., there exists no $(\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1} \setminus \{0\}$ such that $\sum_{i=1}^n \beta_i I_i^{k_2} + n = 0$

Remark 2: Condition 1 ensures that for any initial condition on the torus, which is resonant there exist a sequence of control inputs by which the initial condition is steered to a minimal torus

Condition 2 ensures that there exist no fixed point for the system.

To prove the theorem we need to prove following lemma

Lemma 3: The system (13) satisfying the regularity condition (14) is backward accessible.

Proof:

Consider any point $(I^f,\varphi^f) \in \mathscr{A}$. We have to show that the set of points controllable to (I^f,φ^f) contains an open set $\mathscr{U}$. To prove this we will show that there exists a sequence of control inputs $\{u^k\}$ such that the inverse image of the map under the sequence of control inputs contains an open set. First we will consider the case where I^f does not satisfy the resonant condition i.e., there exists no integers $(\beta_1,..,\beta_n,n) \in \mathbf{Z}^{n+1}$ such that $\beta_1 I_1^f + ... + \beta_n I_n^f + n = 0$. We know the following:

$$\mathscr{S}^{-k}(I',\varphi') = (I',\varphi' - kI')$$

and

$$\mathscr{T}_u^{-1}(I',\varphi') = \{(I,\varphi) : I + \varepsilon Gu - I' = 0;\; \varphi + I + \varepsilon Fu - \varphi' = 0\}$$

Now since I^f does not satisfy the resonance condition we know that the inverse image of (I^f, φ^f) with control inputs zero is dense in $\{I^f\} \times \mathbf{T}^n$ and because of the regularity assumption (14), there exists an integer K_0 such that $|\det[G(I^f, \varphi^f - K_0 I^f)]| > \vartheta$. Now consider the inverse image of (I^f, φ^f) under the following sequence of control input.

$$\mathscr{T}_{u^0}^{-1} \circ \underbrace{\mathscr{S}^{-1} \circ .. \circ \mathscr{S}^{-1}}_{k_0 - 1}(I^f, \varphi^f)$$
$$= \{(I, \varphi) : I + \varepsilon G u^0 - I^* = 0, \varphi + I + \varepsilon F u^0 - \varphi^* = 0\}$$

where $\varphi^* = \varphi^f - (K_0 - 1) I^f$ and $I^* = I^f$. So I and φ satisfies

$$\begin{aligned} \varphi &= \varphi^* - I^* - \varepsilon F(I,\varphi) u^0 + \varepsilon G(I,\varphi) u^0 \\ I &= I^* - \varepsilon G(I,\varphi) u^0 \end{aligned} \tag{15}$$

Since $I^* = I^f$ is nonresonant, we know that there exist an integer K_1 such that $|\det[G(I^*, \varphi^* - K_1 I^*)]| > \vartheta$. Now consider the inverse image of (I, φ) under the following sequence of control inputs.

$$\begin{aligned} \mathscr{T}_{u^1}^{-1} \circ \mathscr{S}^{-(K_1-1)}(I,\varphi) &= \mathscr{T}_{u^1}^{-1}(I^* - \varepsilon G(I,\varphi)u^0,\ \varphi^* - I^* \\ &+ \varepsilon G(I,\varphi)u^0 - \varepsilon F(I,\varphi)u^0 - (K_1 - 1)(I^* - \varepsilon G(I,\varphi)u^0)) \\ &\equiv \mathscr{T}_{u^1}^{-1}(I^1, \varphi^1) \end{aligned} \tag{16}$$

and

$$\mathscr{T}_{u^1}^{-1}(I^1, \varphi^1) = \{ \quad (\bar{I}, \bar{\varphi}) : \bar{I} + \varepsilon G(\bar{I}, \bar{\varphi}) u^1 - I^1 = 0; \quad \bar{\varphi} + \bar{I} + \varepsilon F(\bar{I}, \bar{\varphi}) u^1 - \varphi^1 = 0\}$$

Substituting the value of I^1 and φ^1 from (16), we get following equation to be satisfied by $\bar{I}$ and $\bar{\varphi}$,

$$\begin{aligned} \bar{\varphi} &+ \bar{I} + \varepsilon F(\bar{I}, \bar{\varphi}) u^1 - \varphi^* + I^* - \varepsilon G(I,\varphi) u^0 + \varepsilon F(I,\varphi) u^0 \\ &+ (K_1 - 1)(I^* - \varepsilon G(I,\varphi)u^0) = 0 \\ \bar{I} &+ \varepsilon G(\bar{I}, \bar{\varphi}) u^1 - I^* + \varepsilon G(I,\varphi) u^0 = 0 \end{aligned} \tag{17}$$

So $(\bar{\varphi}, \bar{I})$ satisfying equation (17) with (I, φ) satisfying equation (15) are the set of all points which are mapped to (I^f, φ^f) under the following sequence of control input.

$$\mathscr{T}_{u^1}^{-1} \circ \underbrace{\mathscr{S}^{-1} \circ ... \circ \mathscr{S}^{-1}}_{K_1 - 1} \circ \mathscr{T}_{u^0}^{-1} \circ \underbrace{\mathscr{S}^{-1} \circ ... \circ \mathscr{S}^{-1}}_{K_0 - 1}(I^f, \varphi^f) = (\bar{I}, \bar{\varphi})$$

Now let $\Phi = (\Phi_1, \Phi_2) : \mathbf{R}^{4n} \to \mathbf{R}^n \times \mathbf{R}^n$ be a vector valued function of $(\bar{I}, \bar{\varphi}, u^0, u^1)$ and are defined as follows

$$\begin{aligned} \Phi_1(\bar{I}, \bar{\varphi}, u^0, u^1) &= \bar{\varphi} + \bar{I} + \varepsilon F(\bar{I}, \bar{\varphi}) u^1 - \varphi^* + I^* - \varepsilon G(I,\varphi) u^0 + \\ &\quad \varepsilon F(I,\varphi) u^0 + (K_1 - 1)(I^* - \varepsilon G(\varphi, I) u^0) \\ \Phi_2(\bar{I}, \bar{\varphi}, u^0, u^1) &= \bar{I} + \varepsilon G(\bar{I}, \bar{\varphi}) u^1 - I^* + \varepsilon G(I,\varphi) u^0 \end{aligned}$$

Let $K_0 + K_1 = N$. Then $\Phi(I^f, \varphi^f - N I^f, 0, 0) = 0$ and

$$\det\left[\frac{\partial \Phi}{\partial(\bar{I}, \bar{\varphi})}\right]_{(I^f, \varphi^f - NI^f, 0,0)} = \det \begin{bmatrix} \frac{\partial \Phi_1}{\partial \bar{I}} & \frac{\partial \Phi_1}{\partial \bar{\varphi}} \\ \frac{\partial \Phi_2}{\partial \bar{I}} & \frac{\partial \Phi_2}{\partial \bar{\varphi}} \end{bmatrix}_{(I^f, \varphi^f - NI^f, 0,0)} = 1$$

Hence, by the implicit function theorem, there exists an open neighborhood $\mathscr{O}$ of $(u^0, u^1) = (0,0)$ and unique functions Ψ_1 and Ψ_2 defined on $\mathscr{O}$ and taking values in $\mathbb{R}^n$ such that

$$\Phi_1(\Psi_1(u^0, u^1), \Psi_2(u^0, u^1), u^0, u^1) = 0 \tag{18}$$

$$\Phi_2(\Psi_1(u^0, u^1), \Psi_2(u^0, u^1), u^0, u^1) = 0 \tag{19}$$

for all $(u^0, u^1) \in \mathscr{O}$. Now we have to show that the image of Ψ contains an open set. This is true if $\det[\frac{d\Psi}{du}]|_{(0,0)} \neq 0$. We know that in the neighborhood of $(u^0, u^1) = (0,0)$, we have

$$\left[\frac{d\Phi}{du}\right] = \left[\frac{\partial \Phi}{\partial u}\right] + \left[\frac{\partial \Phi}{\partial \Psi}\right]\left[\frac{d\Psi}{du}\right] = 0$$

$$\det\left[\frac{d\Psi}{du}\right]_{(0,0)} = -\det\left[\frac{\partial \Phi}{\partial \Psi}\right]^{-1}_{(I^f, \varphi^f - NI^f, 0,0)} \det\left[\frac{\partial \Phi}{\partial u}\right]_{(I^f, \varphi^f - NI^f, 0,0)}$$

We know that $\det[\partial \Phi / \partial \Psi]_{(I^f, \varphi^f - NI^f, 0,0)} = 1$. So we need to show that $\det[\partial \Phi / \partial u]_{(I^f, \varphi^f - NI^f, 0,0)} \neq 0$.

$$\det\left[\frac{\partial \Phi}{\partial u}\right]_{(I^f, \varphi^f - NI^f, 0,0)} = \det\begin{pmatrix} \frac{\partial \Phi_1}{\partial u^0} & \frac{\partial \Phi_1}{\partial u^1} \\ \frac{\partial \Phi_2}{\partial u^0} & \frac{\partial \Phi_2}{\partial u^1} \end{pmatrix}_{(I^f, \varphi^f - NI^f, 0,0)}$$

where

$$\begin{aligned} \left[\frac{\partial \Phi_1}{\partial u^0}\right]_{(I^f, \varphi^f - NI^f, 0,0)} &= \varepsilon F(\tilde{I}, \tilde{\varphi}) - \varepsilon K_1 G(\tilde{I}, \tilde{\varphi}) = A \\ \left[\frac{\partial \Phi_2}{\partial u^0}\right]_{(I^f, \varphi^f - NI^f, 0,0)} &= \varepsilon G(\tilde{I}, \tilde{\varphi}) = C \\ \left[\frac{\partial \Phi_1}{\partial u^1}\right]_{(I^f, \varphi^f - NI^f, 0,0)} &= \varepsilon F(\hat{I}, \hat{\varphi}) = B \\ \left[\frac{\partial \Phi_2}{\partial u^1}\right]_{(I^f, \varphi^f - NI^f, 0,0)} &= \varepsilon G(\hat{I}, \hat{\varphi}) = D \end{aligned} \tag{20}$$

and $\tilde{I} = \hat{I} = I^f$, $\tilde{\varphi} = \varphi^f - K_0 I^f$, $\hat{\varphi} = \varphi^f - N I^f$. From matrix analysis we know that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C) \tag{21}$$

if D is invertible. Since $G(\hat{I}, \hat{\varphi})$ is invertible, using (21), we get

$$\begin{aligned} \det\left[\frac{\partial \Phi}{\partial u}\right]_{(\bar{I}, \bar{\varphi}, 0,0)} &= \varepsilon^2 \det[G(\hat{I}, \hat{\varphi})] \det[F(\tilde{I}, \tilde{\varphi}) - K_1 G(\tilde{I}, \tilde{\varphi}) \\ &- G(\tilde{I}, \tilde{\varphi})[G(\hat{I}, \hat{\varphi})]^{-1} F(\hat{I}, \hat{\varphi})] \end{aligned} \tag{22}$$

we know that $\det[G(\hat{I}, \hat{\varphi})] \neq 0$ and $\det[G(\tilde{I}, \tilde{\varphi})] \neq 0$, hence the $\det[\frac{\partial \Phi}{\partial u}]$ can be made nonzero by choosing K_1 sufficiently large. The large choice of K_1 is always possible because we know that points iterated under $\mathscr{S}^{-1}$ land in the set for which $|\det[G]| > 0$ infinitely many times. This proves that the inverse image of (I^f, φ^f) contains an open set, when I^f does not satisfy the resonance condition.

Now we consider the case when I^f satisfies the resonance condition. In this case, we only need to show that there exists a sequence of control inputs $\{u^k\}$ such that inverse image of (I^f, φ^f) under the sequence of control inputs contains a point (I, φ) such that I does not satisfy the resonance condition. Once we have proved this, we can show that inverse image contains an open set by using the previous argument. We know that arbitrary close to I^f there exists $\bar{I}$, which does not satisfy resonance condition. We write

$$I^f = \bar{I} + \delta_I$$

Since $\bar{I}$ is irrational the inverse image of $(\bar{I}, \varphi^f)$ are dense in $\{\bar{I}\} \times \mathbf{T}^n$ and hence there exists integers K_1 and K such that $|\det[G(\bar{I}, \varphi^f - K\bar{I})]| > \vartheta$ and $|\det[G(\bar{I}, \varphi^f - (K+K_1)\bar{I})]| > \vartheta$. Let $N = K_1 + K$ and $\bar{\varphi} = \varphi^f - N\bar{I}$. We claim that there exists control inputs u_1^* and u^* such that

$$\mathscr{S}^{K-1} \circ \mathscr{T}_{u^*} \circ \mathscr{S}^{K_1-1} \circ \mathscr{T}_{u_1^*}(\bar{I}, \bar{\varphi}) = (I^f, \varphi^f).$$

Consider a following map

$$\Gamma_{(\bar{I},\bar{\varphi})}(u_1, u) = \mathscr{S}^{K-1} \circ \mathscr{T}_u \circ \mathscr{S}^{K_1-1} \circ \mathscr{T}_{u_1}(\bar{I}, \bar{\varphi}).$$

Now it can be shown that

$$\det\left[\frac{\partial \Gamma}{\partial(u_1, u)}\right]_{(0,0)} = \varepsilon^2 \det[\bar{G}] \det([\hat{F} + (K-1)\hat{G}] - [\bar{F} + (K_1 + K - 1)\bar{G}][\bar{G}]^{-1}[\hat{G}])$$

and

$$\Gamma_{(\bar{I},\bar{\varphi})}(0,0) = (\bar{I}, \varphi^f)$$

where overbar notation stands for the matrix being evaluated at $(\bar{I}, \bar{\varphi})$ and the overhat for the matrix evaluated at $(\bar{I}, \varphi^f - K\bar{I})$. Since $|\det[\bar{G}]|$ and $|\det[\hat{G}]|$ are greater than ϑ, $|\det[\frac{\partial \Gamma}{\partial(u_1, u)}]|$ can be made nonzero by choosing large value of K_1. This show that Γ is a local diffeomorphism and maps open neighborhood of $(0,0)$ to open neighborhood of $(\bar{I}, \varphi^f)$. The volume of the image set mapped by Γ is directly proportional to the determinant of $\frac{\partial \Gamma}{\partial(u_1, u)}$ and can be made large by choosing large value of K_1. This large choice of K_1 is independent of δ_I because points iterated under $\mathscr{S}$ lands in the set $\mathscr{R}_I$ infinitely many times. So by choosing δ_I sufficiently small and K_1 sufficiently large we can ensure that the image of Γ contains the point (I^f, φ^f). ∎

To prove the theorem we need estimate on $\| G^{-1}(I, \varphi) \|_\infty$ for $\varphi \in \mathscr{R}_I$, $G^{-1} = \frac{1}{\det G}\, adj(G)$

$$\| G^{-1} \|_\infty \leq \frac{1}{\vartheta} \| adj(G) \|_\infty \leq \frac{1}{\vartheta} 2nM^{n-1} = \tilde{M}$$

The control strategy is as follows: Starting from any initial state, the aim is to reach $\mathscr{U}$ (set backward accessible from final state). From given initial state steer the system to a minimal torus (by using controllability conditions of the theorem). On the minimal torus turn OFF the control inputs till the orbit reaches the set $\mathscr{R}_I$. In $\mathscr{R}_I$ the inputs will be turned ON to a proper value to advance in the action space. This process is repeated untill the orbit reaches the set $\mathscr{U}$.

Proof: Let (I^0, φ^0) and (I^f, φ^f) be the initial and final state respectively. By lemma (3) we know that set of points controllable to (I^f, φ^f) contains an open set $\mathscr{U}$. Let $\mathscr{V}$ be a open parallelepiped such that $\mathscr{V} \subset \mathscr{U}$. Let $\pi_I(\mathscr{V}) = \mathscr{V}_I$ and $\pi_\varphi(\mathscr{V}) = \mathscr{V}_\varphi$ where π is the projection map and $\pi_I(I, \varphi) = I$ and $\pi_\varphi(I, \varphi) = \varphi$. We will show that there exist a sequence of control inputs $\{u^0, .., u^N\}$ such that $\pi_I(T_{u^N,...,u^0}(I^0, \varphi^0)) \in \mathscr{V}_I$ and $\pi_\varphi(T_{u^N,...,u^0}(I^0, \varphi^0)) \in \mathscr{V}_\varphi$. We assume that I^0 does not satisfy the resonance condition, because if I^0 satisfy the resonance condition then we know by condition of the theorem that there exist a finite sequence of control inputs $\{u^0, \ldots, u^{k_1-1}\}$ such that $T_{u^0,...,u^{k_1-1}}(I^0, \varphi^0) = (I^{k_1}, \varphi^{k_1})$, where I^{k_1} does not satisfy resonance condition. Hence by relabelling the initial state we can always assume that I^0 does not satisfy the resonance condition. Let $\bar{I} \in \mathscr{V}_I$ be such that $\bar{I}_i - I_i^0$ is rational for all $i = 1, 2.., n$. Let

$\frac{\bar{I}_i - I_i^0}{m} = \lambda_i$ for $i = 1, .., n$ Let $\lambda = (\lambda_1, .., \lambda_n)$ and $\Lambda = \max(|\lambda_i|)$ where $m \in \mathbf{Z}$ be sufficiently large so that $\Lambda \in (-\frac{\varepsilon}{\tilde{M}}, \frac{\varepsilon}{\tilde{M}})$

Since I^0 does not satisfy the resonance condition we know that the orbit starting from (I^0, φ^0) with zero control input is dense in $\{I^0\} \times \mathbf{T}^n$. So by regularity condition (14) there exists an integer $n_1 \in \mathbf{Z}^+$, such that $(I^{n_1}, \varphi^{n_1}) = (I^0, \varphi^{n_1}) \in \mathscr{R}_I$. Once $(I^{n_1}, \varphi^{n_1}) \in \mathscr{R}_I$, turn ON the control inputs u to a value

$$u = G^{-1}\bar{u} \text{ where } \bar{u} = \frac{\lambda}{\varepsilon}$$

So we have

$$I^{n_1+1} = I^{n_1} + \lambda = I^0 + \lambda \tag{23}$$

we need to show that $\| u \|_\infty \leq 1$

$$\| u \|_\infty = \| G^{-1}\bar{u} \|_\infty \leq \| G^{-1} \|_\infty \| \bar{u} \|_\infty \leq \tilde{M} \frac{|\Lambda|}{\varepsilon} \leq 1$$

Since I^0 does not satisfy resonance condition and all the components of $\lambda = (\lambda_1, .., \lambda_n)$ are rational, I^{n_1+1} also does not satisfy the resonance condition. Since I^{n_1+1} does not satisfy the resonance condition the orbit starting from $(I^{n_1+1}, \varphi^{n_1+1})$ with zero control input is dense in $\{I^{n_1+1}\} \times \mathbf{T}^n$. So there exists an integer n_2 such that $(I^{n_1+n_2}, \varphi^{n_1+n_2}) \in \mathscr{R}_I$. With $(I^{n_1+n_2}, \varphi^{n_1+n_2}) \in \mathscr{R}_I$ turn on the control input to a value $u = G^{-1}\frac{\lambda}{\varepsilon}$ so that $I^{n_1+n_2+1} = I^{n_1+n_2} + \lambda = I^0 + 2\lambda$. Repeating the above procedure $m-2$ times more we get

$$I^N = I^0 + m\lambda = \bar{I}$$

where $N = \sum_{i=1}^m n_i + 1$. Since $\bar{I} \in \mathscr{V}_I$ does not satisfy the resonance condition, we turn OFF the control input till the orbit reach the set $\mathscr{V}_\varphi$. With $\bar{I} \in \mathscr{V}_I$ and $\varphi \in \mathscr{V}_\varphi$, $(\bar{I}, \varphi) \in \mathscr{V}$ and hence the system is controllable.

To prove the necessary part we have to show that if the controllability conditions are not satisfied then the system is uncontrollable. Consider an initial condition (I^0, φ^0) which does not satisfy the controllability condition 1 and any final state (I^f, φ^f) such that the torus parameterized by I^f is minimal. We claim that this final state cannot be reached from (I^0, φ^0). Assume that this final state can be reached then there exists a sequence of control inputs $\{u^0, u^1, .., u^{k-1}\}$ such that $\mathscr{T}_{u^0,...,u^{k-1}}(I^0, \varphi^0) = (I^f, \varphi^f)$. This contradict the controllability condition 1 because there exists no sequence of control input by which this initial condition can be steered to the minimal torus.

If controllability condition $2a$ is not satisfied then there exists a fixed point for the system from which the system is not controllable. This is clear because if $I \in \mathbf{Z}^n$ and both F and G vanish simultaneously then for any sequence of control input we have

$$\begin{aligned} I' &= I \\ \varphi' &= \varphi + I \pmod 1 \quad = \varphi \end{aligned}$$

and hence there exists a fixed point. If the controllability condition $2b$ is not satisfied, then again by the same argument as above, the system is uncontrollable. ∎

An easy consequence of the above theorem is the following corollary. Before stating the corollary we give the definition of almost every where controllable.

Definition 4: The system is said to be controllable almost everywhere if for almost every (with respect to Lebesgue measure) given initial state (I^0, φ^0) and almost every final state (I^f, φ^f), there exists a sequence of control inputs $u^0, .., u^k$ such that $T_{u^k,...,u^0}(I^0, \varphi^0) = (I^f, \varphi^f)$

Corollary 5: Under the regularity condition (14), the system (13) is almost everywhere controllable.

Proof: Arbitrary close to any given initial state (I^0, φ^0), there exists (I, φ^0), where I does not satisfy resonance condition. Starting from the state (I, φ^0) we know from the proof of the theorem (1) that we can always reach the set $\mathscr{U}$, which is the backward accessible from the final state (I^f, φ^f). ∎

IV. Conclusion and Future Work

The key ideas introduced in this paper is the ergodic partition of the drift system. By exploiting the ergodic property of the drift system it is possible to control using arbitrary small bounds on the control input. The ideas presented in this paper can be generalized to more general discrete time nonlinear system, where drift part of the system has nice ergodic partition.

V. References

[1] R. Brockett and N. Khaneja, *On the Control of Quantum Ensembles.* System Theory: Modeling, Analysis and Control, Boston: Kluwer, 1999.

[2] D. D'Alessandro, "Small time controllability of systems on compact Lie groups and spin angular momentum," *Journal of Mathematical Physics*, vol. 42, pp. 4488–4496, september 2001.

[3] A. Bloch, P. Krishnaprasad, J. Marsden, and G. Sánchez De Alvarez, "Stabilization of rigid body dynamics by internal and external torque," vol. 28, pp. 745–756, 1992.

[4] D. D'Alessandro, M. Dahleh, and I. Mezic, "Control of mixing in fluid flow: A maximum entropy approach," *IEEE Transactions on Automatic Control*, vol. 44, no. 10, pp. 1852–1863, 1999.

[5] M. Fliess and D. Normand-Cyrot, "A Lie-theoretic appraoch to nonlinear discrete-time controllability via ritt's formal differential groups," *Systems and Control Letters*, vol. 1 (3), pp. 179–183, 1981.

[6] B. Jakubczyk and E. Sontag, "Controllability of nonlinear discrete-time systems: A Lie-algebric appraoch," *SIAM J. Control and Opt.*, vol. 28, pp. 1–33, 1990.

[7] K. Petersen, *Ergodic theory.* Cambridge: Cambridge University Press, 1995.

[8] U. Vaidya and I. Mezić, "Controllability for a class of area preserving twist maps," *Submitted to Physica D.*

[9] I. Mezić, "Controllability of Hamiltonian systems with drift: Action-angle variables and ergodic partition," *Proceedings of CDC 2003.*

[10] V. I. Arnold, *Mathematical methods of classical mechanics.* New York: Springer Verlag, 1978.

[11] K. J. Astrom and K. Furuta, "Swinging up a pendulum by energy control," *Automatica*, vol. 36, pp. 287–295, 2000.

[12] U. Vaidya, D. D'Alessandro, and I. Mezić, "Control of Heisenberg spin system; action control and lie algebraic decomposition," *Proceedings of CDC 2003*, 2003.

[13] H. Neijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems.* New York: Springer Verlag, 1990.

[14] A. Katok and B. Hasselblatt, *Introduction to the theory of dynamical systems.* Cambridge: Cambridge University Press, 1995.