Identification of critical interactions in uncertain network systems with complex dynamics

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Abstract— **In this paper, we propose a novel approach based on tools from ergodic theory of dynamical systems for the identification of critical parameters responsible for the emergence of complex dynamics in network systems. We consider a network system with multiple uncertain parameters and operating in nonequilibrium. The objective is to determine which of these multiple parameters are critical for maintaining the stability of nonequilibrium dynamics. Using combination of tools from linear robust control theory and ergodic theory of dynamical systems, we provide conditions on system parameters to maintain the stability of network systems. The proposed method is applied for the identification of parameters responsible for limit cycle oscillations in biochemical network involved in yeast cell glycolysis and for robust synchronization in network of Kuramoto oscillators with uncertainty in coupling parameters.**

I. Introduction

The problem of analysis and design of robust network systems has received considerable attention in recent literature [1], [2], [3], [4], [5], [6]. The various network systems of interest include biological networks, electric power grid, Internet communication networks, and transportation networks. In biological network, it is of interest to know which of the multiple parameters are responsible for the emergence of robust complex dynamics [7], [8]. In electric power grid, due to uncertainty associated with various load and system parameters it is important to know the relative stability margin with respect to the uncertainty in these systems parameters [9]. In problems involving cyber security of electric power grid, it is of interest to identify the most vulnerable link from where the malicious attack on the grid could be launched [10]. Robust synchronization in network systems at the backdrop of link failure or packet-drop and time delay uncertainty is of interest in Internet communication and sensor networks. The natural dynamics in most of the above discussed network systems of interest is away from equilibrium. For example, in biological networks the nonequilibrium dynamics include limit cycling oscillating solutions and bistable dynamics. The synchronized state of coupled generators in electric power grid is periodic and hence in nonequilibrium.

Because of the lack of systematic methods for the analysis of nonequilibrium dynamics most of the existing approaches for the robustness analysis in network systems focus on equilibrium dynamics [11], [4]. The nonequilibrium network dynamics is not accounted explicitly in the robustness analysis. In this paper, we develop systematic approach for the robust stability analysis of network systems in the presence of stochastic uncertainty while explicitly accounting for the nonequilibrium dynamics of the network. In particular, we provide condition for the stability of network systems with stochastic interaction among network components. Stability condition is expressed in terms of the variance of the stochastic interaction parameter and the input-output property of the network. The problem set-up is general enough to model various uncertain network systems of interest, including biological network with stochastic uncertain parameters and electric power grid with load uncertainty. Furthermore one of the widely studied problem of robust synchronization in a network with identical nonlinear component dynamics interconnected via linear Laplacian and with link failure uncertainty will form a special case of the results developed in this paper. Our proposed framework for robustness analysis is based on combinations of tools from linear robust control theory [12] and ergodic theory of dynamical systems [13]. We provide computable conditions for the network stability expressed in terms of the nominal or mean dynamics of the network and the statistics of the uncertain interactions.

Following are the main contributions of this paper. We propose a systematic framework for robust stability of uncertain network system operating in nonequilibrium. Computable necessary condition is provided to determine critical interaction in network systems responsible for maintaining the stability of nonequilibrium dynamics in the network. Application of the developed framework is demonstrated to the problem of robustness analysis in biological network and synchronization in non-uniform Kuramto oscillator with uncertainty in interactions. The organization of the paper is as follows. In section II, we discuss the problem set-up along with necessary assumption and stability definition. The main results of this paper are presented in section III. Applications of the developed framework is demonstrated in section IV followed by conclusion in section V.

II. Problem set-up and assumptions

The individual components dynamics of the network system is modeled as a single-input single-output discrete time dynamical system as follows:

$$S_k = \begin{cases} x_{n+1}^k &= f_k(x_n^k) + B_k u_n^k \\ y_n^k &= C_k x_n^k, \quad k = 1, \dots, M \end{cases} \tag{1}$$

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where $x_n^k \in X_k \subset R^{N_k}$, $u_n^k \in U_k \subset \mathbb{R}$, and $y_n^k \in Y_k \subset \mathbb{R}$ is the state, input, and output of the k^{th} component sub-system respectively. $f_k : X_k \to X_k$ is assumed to be at least C^r, with $r \geq 1$, function of x. B_k and C_k are column and row vectors of size N_k respectively. We use the notation $X = X_1 \oplus X_2 \cdots \oplus X_M \subset \mathbb{R}^N$ and $Y = Y_1 \oplus Y_2 \cdots \oplus Y_M \subset \mathbb{R}^M$. The interaction among the network components is assumed to be nonlinear uncertain function of output and is modeled as follows:

$$u_n^k = \sum_{\ell=1}^{M} \xi_n^{k\ell} g_{k\ell}(y_n^1, \ldots, y_n^M) \tag{2}$$

where $\xi_n^{k\ell}$ is uncertain and is assumed to be independent identically distributed (i.i.d) random variable with mean $E[\xi_n^{k\ell}] = \mu_{k\ell}$ and second moment $E[\xi_n^{k\ell} - \mu_k] = \sigma_{k\ell}^2$. If particular interaction is certain that will correspond to the case of $\sigma_{k\ell} = 0$ The function $g_{k\ell} : Y \to \mathbb{R}$ is assumed to be C^r, with $r \geq 1$, function of the output for $k, \ell = 1, \ldots, M$. The single-input single-output components dynamics with nonlinear uncertain interaction in Eqs. (1) and (2) can be used to model various uncertain network systems of interest in engineering and natural science. The objective is to determine the most critical interaction responsible for the emergence of complex dynamics in the network. To write the above uncertain network system in compact form, we first define a new random variable $\delta_n^{k\ell} = \xi_n^{k\ell} - \mu_{k\ell}$. So that $E[\delta_n^{k\ell}] = 0$ and $E[(\delta_n^{k\ell})^2] = \sigma_{k\ell}$. With the new random variable, the system Eqs. (1) and (2) can be written in compact form as follows:

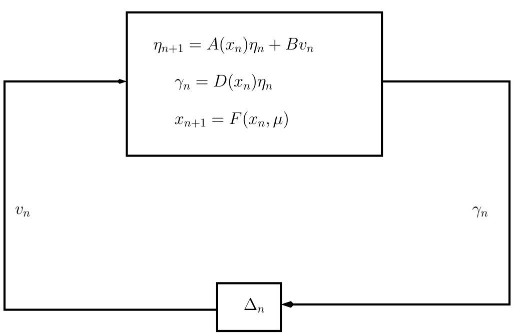

Fig. 1. Schematic of the system as expressed by (4).

$$x_{n+1} = F(x_n, \mu) + \sum_{k=1}^{M}\sum_{\ell=1}^{M} \delta_n^{k\ell} \bar{B}_k g_{k\ell}(y_n^1, \ldots, y_n^M) \tag{3}$$

$x_n = (x_n^1; \ldots; x_n^M)$,

$$F(x_n, \mu) = (F_1(x_n, \mu); \ldots; F_M(x_n, \mu))$$

$$F_k(x_n, \mu) := f_k(x_n^k) + \sum_{\ell=1}^{M} \mu_n^{k\ell} B_k g_{k\ell}(y_n^1, \ldots, y_n^M)$$

and $\bar{B}_k$ is a column vector of size $N = \sum_{\ell=1}^{M} N_\ell$ and is obtained by stacking zero and B_k column vector starting at $\sum_{\ell=1}^{k-1} N_\ell$ location. The network system can be written in feedback control form as follows (refer to Fig. 1 for the schematic):

$$\begin{aligned} x_{n+1} &= F(x_n, \mu) + Bu_n \\ y_n &= G(x_n) \\ u_n &= \Delta_n y_n \end{aligned} \tag{4}$$

where B is a matrix of size $N \times M^2$

$$B = (\underbrace{\bar{B}_1 \ldots, \bar{B}_1}_{M}, \ldots, \underbrace{\bar{B}_M \ldots, \bar{B}_M}_{M})$$

$\Delta_n = diag(\delta_n^{11}, \delta_n^{12}, \ldots, \delta_n^{MM})$ is a diagonal matrix of size M^2, $G(x_n) = (g_{12}(Cx_n), g_{13}(Cx_n), \ldots, g_{M^2}(Cx_n))$, C is a block diagonal matrix of size $M \times N$ and is of the form $C = diag(C_1, C_2, \ldots, C_M)$, and . We now make following assumption on the nominal deterministic system

$$x_{n+1} = F(x_n, \mu) \tag{5}$$

Assumption 1: We assume that the nominal system $x_{n+1} = F(x_n, \mu)$ has a globally stable periodic solution.

Furthermore following assumption is made on the pair $(F, g_{k\ell})$ for the feedback system (4).

Assumption 2: We assume for system described by Eq. (5), the norm of Jacobian matrix is lower bounded i.e. $\frac{\partial F}{\partial x}^T(x_n)\frac{\partial F}{\partial x}(x_n) \geq H > 0$.

Assumption 3: Consider the map $T_k^{N-1}(x) : \mathbb{R}^N \to \mathbb{R}^N$ for $k = 1, \ldots, L$, as

$$T_{k\ell}^{N-1}(x) = (g_{k\ell}(Cx), g_{k\ell}(CF(x, \mu)), \ldots, g_{k\ell}(CF^{N-1}(x, \mu)))$$

where $F^k(x, \mu)$ denotes k times composition of F. We assume that system (4) satisfies following rank condition

$$rank(\frac{\partial T_k^{N-1}(x)}{\partial x}) = N, \quad k, \ell = 1, \ldots, L$$

for almost all, with respect to Lebesgue measure initial condition $x \in X$.

Remark 4: The assumption is equivalent to saying the linearized pair $(\frac{\partial F}{\partial x}, \frac{\partial g_{k\ell}}{\partial x})$ is observable along the dynamics of the nominal system (5).

We make following assumption on system (4).

Assumption 5: We assume that the system (4) has a globally stable periodic solution $\{x_n^*\}$.

The objective is to determine the critical parameter responsible for maintaining the stability of periodic solution for system (4). In particular, we want to determine the parameter with smallest variance responsible for destabilizing the periodic solution of system (4). Since the system (4) is random in nature, we need to make appropriate notion of stochastic stability. Instead of requiring the periodic solution $\{x_n^*\}$ (Assumption 5) to be stable in some stochastic sense, we make slightly stronger assumption on system (4). In particular, we make following assumption.

Assumption 6: We assume that system (4) is incrementally mean square exponentially (MSE) stable in a neighborhood $\mathcal{N}$ of the periodic orbit i.e., there exists positive constants $K < \infty$ and $\beta < 1$ such that

$$E_{\Delta_0^n}\left[\| x_{n+1} - y_{n+1} \|^2\right] \leq \| x_0 - y_0 \|^2 \tag{6}$$

for almost all initial condition $x_0, y_0 \in \mathscr{N}$. Where, $\Delta_0^n = \{\Delta_i | i = 0, 1, \ldots n\}$

Remark 7: The assumption (6) is stronger than requiring stochastic stability of periodic solution as increment MSE stability will imply MSE stability of periodic solution $\{x_n^*\}$. This is because incremental MSE stability implies converge of all trajectories to each other and in particular to the periodic solution as it is one particular trajectory of the system. There are following reasons behind making the stronger assumption of incremental MSE stability as opposed to stochastic stability of periodic orbit. First, the proofs of some of the main results of this paper are considerably simplified with incremental MSE stability. Secondly, one of the main results of this paper is on synchronization of network systems and for synchronization problem the incremental stability is a natural notion of stability. We expect the main results of this paper to hold true under the assumption that the periodic solution is MSE stable, however the proofs will be substantially more complicated. In the rest of the paper, we will refer to Assumption 6 as incremental MSE stability of periodic solution. The objective will be to determine necessary condition in terms of the variance of uncertainty Δ to maintain incremental MSE stability of periodic solution.

III. Main results

The main result section is organized as follows. We first state all the main results of this paper in the form of Theorems 8, 10, and 17. The proof of all the Theorems in differ to section III-C. To state the first main result of this paper

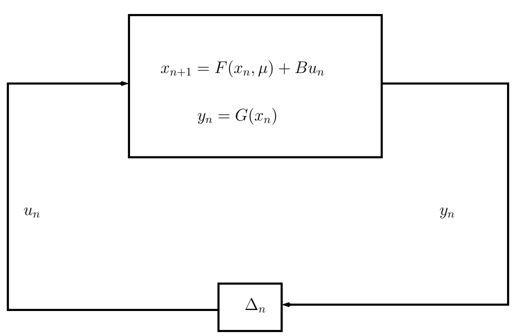

Fig. 2. Schematic of the linearized system as expressed by (7).

on the stability of random dynamical system (4), we define following quantities. We consider the linearization of the system equation (4) along the trajectories of nominal system (5) as follows (refer to Fig. 2 for schematic):

$$\begin{aligned} x_{n+1} &= F(x_n, \mu) \\ \eta_{n+1} &= \frac{\partial F}{\partial x}(x_n, \mu)\eta_n + B v_n, \;\; \gamma_n = \frac{\partial G}{\partial x}(x_n)\eta_n \\ v_n &= \Delta_n \gamma_n \end{aligned} \tag{7}$$

We will use following notation throughout the paper.

$$A(x_n) := \frac{\partial F}{\partial x}(x_n, \mu), \; D_{k\ell}(x_n) := \frac{\partial g_{k\ell}}{\partial x}(Cx_n, \mu)$$
$$D(x_n) := \frac{\partial G}{\partial x}(x_n)$$

We now state the first main result of this paper.

Theorem 8: The necessary condition for the MSE stability for the periodic solution of feedback control system (4) satisfying Assumption 3 is given by

$$\sigma_{k\ell}^2 \bar{B}_k^T P(x_n) \bar{B}_k < 1 \tag{8}$$

$$A^T(x_n) P(x_{n+1}) A(x_n) - P(x_n) = -D_{k\ell}(x_n) D_{k\ell}^T(x_n) \tag{9}$$

for $k, \ell = 1, \ldots, M$ and $P(x)$ is a positive definite matrix with $x_{n+1} = F(x_n, \mu)$.

We postpone the proof of this Theorem to section III-C.

A. Ergodic theory- based computable condition

Theorem 8 provides point-wise condition for the stability of periodic solution for system (4). We now make use of results from ergodic theory to provide computable condition for the stability of (4). The central to this computation is the notion of physical invariant measure which is defined as follows.

Definition 9 (Physical invariant measure): A probability measure $\mu \in \mathscr{M}(X)$ is said to be invariant for the dynamical system $x_{n+1} = F(x_n, \mu)$ if $\mu(F^{-1}(B)) = \mu(B)$ for all sets $B \in \mathscr{B}(X)$, the Borel σ-algebra on X, and where $F^{-1}(B)$ is the inverse image of set B. An invariant probability measure is said to be physical if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \phi(x_k) = \int_X \phi(x) d\mu(x) \tag{10}$$

for all continuous function $\phi : X \to \mathbb{R}$ and positive Lebesgue measure initial condition $x_0 \in X$.

From Assumption 1, it follows that there exits a physical measure supported on the globally stable periodic orbit for system $x_{n+1} = F(x_n, \mu)$. Typically, a periodic orbit in a nonlinear systems will exhibit nonuniform behavior in space and time i.e., different regions of periodic orbit are visited with different frequency. The physical measure supported on the periodic orbit captures this nonuniform behavior. In particular, the physical measure not only provide information about the location of the periodic orbit but also the relative amount of time the system trajectories spend on the different parts of the periodic orbit. It is important to explicitly account for this nonuniform nature of the periodic solution to determine the relative degree of robustness to various parameters variations in the system.

Theorem 10: Let $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_P\}$ be the stable periodic solution for the nominal deterministic system (5) with period P, so that $F^P(\bar{x}_i, \mu) = \bar{x}_i$ for $i = 1, \ldots, P$. Let $\mu(x)$ be the physical invariant measure corresponding to the periodic solution for the nominal deterministic system (5). The necessary condition for the MSE stability of the periodic solution of system (4) satisfying Assumption 3 is given by

$$\log \sigma_{k\ell}^2 + \int_X \log \bar{B}_k^T P(x) \bar{B}_k d\mu(x) < 0 \tag{11}$$

where $P(x)$ is a positive definite matrix and satisfies

$$\mathscr{A}^T(x_i)P(x_i)\mathscr{A}(x_i) - P(x_i) = -\mathscr{R}_{k\ell}(x_i) \quad (12)$$

for $i = 1,\ldots,P$, and $k,\ell = 1,\ldots,M$, and where

$$\mathscr{A}(x_i) = \left[\prod_{r=i}^{P} A(x_r)\right]\left[\prod_{s=1}^{i-1} A(x_s)\right]$$

$$\mathscr{R}(x_i) = D_{k\ell}(x_i)D_{k\ell}^T(x_i) + \sum_{j=1, j\neq i}^{P} \hat{\mathscr{A}}^T(x_i,x_j)D_{k\ell}(x_j)D_{k\ell}^T(x_j)\hat{\mathscr{A}}(x_i,x_j)$$

$$\hat{\mathscr{A}}(x_i,x_j) = \left(\left[\prod_{r=j}^{P} A(x_r)\right]\left[\prod_{s=1}^{min(i,j)-1} A(x_s)\right]\right)$$

We postpone the proof of this Theorem till the end of this section. The results of Theorem 10 can be used to rank order the various interaction parameters for the relative degree of importance in maintaining the stability of periodic solution. In particular, the stability margin $\mathbf{Sm}(k,\ell)$ for interaction parameter $\xi^{k\ell}$ can be defined as

$$\mathbf{Sm}(k,\ell) = -\int_X \log \bar{B}_k^T P(x_i)\bar{B}_k d\mu(x) \quad (13)$$

The larger the value of $\mathbf{Sm}(k,\ell)$, more robust the system behavior is to the variation in the interaction parameter $\xi^{k\ell}$ and hence can tolerate larger variance $\sigma_{k\ell}$. The critical interaction parameter, $\xi^{k^*\ell^*}$, can then be defined as follows:

$$(k^*,\ell^*) = argmin(\mathbf{Sm}(1,2),\ldots,\mathbf{Sm}(M,M)).$$

B. Synchronization in network systems

To study the problem of robust synchronization in network system, we make following assumption on the network component dynamics.

Assumption 11: We assume that all the network components dynamics in Eq. (1) are identical i.e., $f_k = f, B_k = B$, and $C_k = C$ for $k = 1,\ldots,M$.

Furthermore following assumption is made on the nonlinear interaction in Eq. (2).

Assumption 12: We assume that the nonlinear interaction terms vanishes for identical outputs of individual sub-systems i.e., for $x_n^1 = x_n^2 = \ldots = x_n^M = \bar{x}_n$, we have

$$g_{k\ell}(C(\bar{x}_n),\ldots,C(\bar{x}_n)) \equiv 0, \;\; k,\ell = 1,\ldots,M$$

The interaction term, $g_{k\ell}$, in this case represent the generalized nonlinear Laplacian.

Assumption 13: We assume that the individual component system $x_{n+1}^k = f(x_n^k)$ has unique physical invariant measure, $\bar{\mu}$.

Remark 14: The physical invariant measure captures the nonequilibrium dynamics of the individual component system. The nonequilibrium dynamics could range from simple periodic solution to chaotic behavior.

The notion of mean square synchronization is introduced next.

Definition 15: The system described in (4) is mean square exponentially synchronizing if there exists $M_s < \infty$ and $\beta_s < 1$ such that for all x_0,

$$E_{\Delta_0^n} \parallel x_{n+1} - Z_{n+1} \parallel^2 < M_s\beta_s^n \parallel x_0 - Z_0 \parallel^2 \quad (14)$$
$$Z_n = [z_n, z_n, \ldots z_n]^T$$
$$z_{n+1} = f(z_n)$$

for some z_0.

Remark 16: The trajectories of the system described in Eq. (4) synchronizes when $x_n^1 = x_n^2 = \cdots = x_n^M = z_n$(say). This implies $g_{k\ell}(C(z_n),\ldots,C(z_n)) = 0$ and hence $G(x_n) = 0$. This means x_n will evolve as,

$$x_{n+1} = [f(z_n),\ldots f(z_n)]$$

Hence the trajectory x_n eventually converges upon $Z_n = [z_n,\ldots z_n]^T$, and $z_{n+1} = f(z_n)$.

Theorem 17: Consider the uncertain network system (4) satisfying Assumptions 11, 12, and 13. The necessary condition for the synchronization of the network dynamics is given by

$$\log \sigma_{k\ell}^2 + \int_X \log \bar{B}_k^T P(Z)\bar{B}_k d\bar{\mu}(z) < 0 \quad (15)$$

where $Z = [z,z,\ldots,z]^T$ and $P(Z)$ satisfies

$$A^T(Z_n)P(Z_{n+1})A(x_n) - P(Z_n) = -D_{k\ell}^T(Z_n)D_{k\ell}(Z_n)$$

with $Z_n = [z_n,\ldots z_n]^T$ and $z_{n+1} = f(z_n)$.

Remark 18: One of the main difference between the necessary condition for stability provided in Theorem 10 for a general network systems with Theorem 17 is that while condition (11) involves integration with respect to physical measure of the nominal coupled system (i.e., $x_{n+1} = F(x_n,\mu)$), the integration in (15) is with respect to physical measure of individual component system (i.e., $z_{n+1} = f(z_n)$). This is advantageous from the point of view of computation of stability condition for a large size network system as we demonstrate in our application section IV. This computation advantage is possible because of the assumed identical nature of individual component dynamics and also because of the Laplacian nature of the nonlinear interaction term.

C. Proofs of main theorems

Proof of Theorem 8 relies on Lemma 19 and 20, which we prove next.

Lemma 19: The necessary condition for the incremental mean square exponential stability of perodic solution of system (4) (Definition 6 and remark 7) is that the η_n dynamics (Eqn. 7) is mean square exponentially stable i.e. there exists positive constant $M < \infty$ and $\beta < 1$ such that

$$E_{\Delta_0^n}\left[\parallel \eta_{n+1} \parallel^2\right] \leq M\beta^n \parallel \eta_0 \parallel^2$$

Proof: Let us consider the system

$$z_{n+1} = F(z_n,\mu) + B\Delta_n G(z_n) - B\Delta_n G(x_n) \quad (16)$$

Now let us assume two trajectories of starting with initial conditions z_0^1 and z_0^2, evolving as Eqn. 16. Let us define $e_n^z =$

$z_n^2 - z_n^1$.

$$\begin{aligned} e_{n+1}^z &= F(z_n^1 + e_n^z, \mu) - F(z_n^1, \mu) + B\Delta_n G(z_n^1 + e_n^z) - B\Delta_n G(z_n^1) \\ &= \left[\int_0^1 \left(A(z_n^1 + s_n e_n^z) + B\Delta_n D(z_n^1 + s_n e_n^z) \right) ds_n \right] e_n^z \\ &= \left[\prod_{i=0}^{n} \int_0^1 \left(A(z_i^1 + s_i e_i^z) + B\Delta_i D(z_i^1 + s_i e_i^z) \right) ds_i \right] e_0^z \\ &= \mathscr{M}_z^n(z_0^1, e_0^z, \{\Delta_i\}_{i=0}^n) e_0^z \end{aligned}$$

$$E_{\Delta_0^n}\left[{e^z}'_{n+1} e_{n+1}^z \right] = {e^z}'_0 E_{\Delta_0^n}\left[\mathscr{M}_z^{n\prime}(z_0^1, e_z^0, \Delta_0^n) \mathscr{M}_z^n(z_0^1, e_z^0, \Delta_0^n) \right] e^z{}_0$$

Using exponential mean square incremental stability of 4,

$${e^z}'_0 E_{\Delta_0^n}\left[\mathscr{M}_z^{n\prime}(z_0^1, e_z^0, \Delta_0^n) \mathscr{M}_y^n(z_0^1, e_z^0, \Delta_0^n) \right] e^z{}_0 < M\beta^n {e^z}'_0 e_0^z$$

Hence the system described by (16) is exponential mean square incrementally stable. We consider two specific trajectories with initial conditions z_0 and x_0, which evolve according to (16). These two trajectories evolve as,

$$\begin{aligned} z_{n+1} &= F(z_n, \mu) + B\Delta_n G(z_n) - B\Delta_n G(x_n) \\ x_{n+1} &= F(x_n, \mu) \end{aligned}$$

Let $e_n = z_n - x_n$. This gives us,

$$\begin{aligned} e_{n+1} &= F(x_n + e_n, \mu) - F(x_n, \mu) + B\Delta_n G(x_n + e_n) - B\Delta_n G(x_n) \\ &= \left[\prod_{i=0}^{n} \int_0^1 \left(A(x_i + s_i e_i) + B\Delta_i D(x_i + s_i e_i) \right) ds_i \right] e_0 \\ &= \mathscr{M}_z^n(x_0, e_0, \Delta_0^n) e_0^z \end{aligned}$$

Using the above equation we get,

$$E_{\Delta_0^n}\left[e'_{n+1} e_{n+1} \right] = e'_0 E_{\Delta_0^n}\left[\mathscr{M}^{n\prime}(x_0, e_0, \Delta_0^n) \mathscr{M}^n(x_0, e_0, \Delta_0^n) \right] e_0$$

We scale the initial error by $\{\omega_l\}$ and take a sequence such that $\lim_{l\to\infty} \omega_l = 0$. Using Bounded Convergence Theorem and continuity property of $\mathscr{M}^n(x_0, e_0, \Delta_0^n)$,

$$\lim_{l\to\infty} \mathscr{M}^n(x_0, \omega_l e_0, \Delta_0^n) = \mathscr{M}^n(x_0, 0, \Delta_0^n)$$

Where, $A_i = A(x_i)$, and $D_i = D(x_i)$. Now,

$$E_{\Delta_0^n}\left[e'_{n+1} e_{n+1} \right] < K\beta^n e'_0 e_0$$

$$\implies e'_0 E_{\Delta_0^n}\left[\mathscr{M}^{n\prime}(x_0, \omega_l e_0, \Delta_0^l) \mathscr{M}^n(x_0, \omega_l e_0, \Delta_0^l) \right] e_0 < M\beta^n e'_0 e_0$$

Using Fatou's Lemma and taking limit $l \to \infty$ we get,

$$e'_0 E_{\Delta_0^n}\left[\prod_{i=0}^{n} (A_i + B\Delta_i D_i)^T \prod_{i=0}^{n} (A_i + B\Delta_i D_i) \right] e_0 < M\beta^n e'_0 e_0, \forall e_0$$

Now, $E_{\Delta_0^n}\left[\prod_{i=0}^n (A_i + B\Delta_i D_i)^T \prod_{i=0}^n (A_i + B\Delta_i D_i) \right]$ matrix is independent of e_0. Hence it can be also be rewritten as,

$$\eta'_0 E_{\Delta_0^n}\left[\prod_{i=0}^{n} (A_i + B\Delta_i D_i)^T \prod_{i=0}^{n} (A_i + B\Delta_i D_i) \right] \eta_0 < M\beta^n \eta'_0 \eta_0$$

Hence, the proof. ∎

Lemma 20: . The system ,described by (4), is mean square incremental stable only if, for any arbitrary x_0 and n, there exists a symmetric, positive definite $\hat{P}(x_l)$ such that,

$$E_{\Delta_0^n}\left[(A_n + B\Delta_n D_n)^T \hat{P}(x_{n+1}) (A_n + B\Delta_n D_n) \right] < \hat{P}(x_n) \quad (17)$$

where $A_i = A(x_i)$, and $D_i = D(x_i)$, $x_{n+1} = F(x_n, \mu)$, and

$$\gamma_0 \leq \| \hat{P}(x_n) \| \leq \gamma_1 \quad (18)$$

Proof: The system in (4) is mean square incrementally stable. That in turn implies Lemma 19. We construct the function $\hat{P}(x_l)$ as following,

$$\hat{P}(x_n) = \sum_{l=n}^{\infty} E_{\Delta_0^n}\left[\prod_{m=n}^{l} (A_m + B\Delta_m D_m)^T \prod_{m=l}^{n} (A_m + B\Delta_m D_m) \right]$$

From the construction and using the fact $\{\delta_m^{k,\ell}\}$ is a sequence of i.i.d. random variables,

$$\hat{P}(x_n) > A_n^T A_n + \sum_{k,\ell} \sigma_{k\ell}^2 \bar{(}D_{k\ell}^n)^T B_k^T B_k D_{k\ell}^n$$

According to assumption 2, we get $A_n^T A_n \geq H > 0$, which implies (17). Next, we prove the upper bound on norm of $\hat{P}(x_n)$.

$$\eta_0^T \hat{P}(x_n) \eta_0 = \eta_0^T \sum_{l=n}^{\infty} E_{\Delta_0^n} \prod_{m=l}^{n} (A_m + B\Delta_m D_m)^T \prod_{m=l}^{n} (A_m + B\Delta_m D_m) \eta_0$$

$$< M \sum_{l=n}^{\infty} \beta^{l-n} \eta_0^T \ \eta_0 = M \sum_{k=0}^{\infty} \beta^k \eta_0^T \ \eta_0 = \frac{M}{1-\beta} \eta_0^T \ \eta_0$$

We have already shown $\hat{P}(x_n) \geq A_n^T A_n \geq H > 0$. Hence we can choose γ_1, γ_2 accordingly. ∎

Next we continue to prove Theorem 8 using Lemma 20.

Proof: (17) simplifies to,

$$A_n^T \hat{P}(x_{n+1}) A_n - \hat{P}(x_n) + \sum_{k,\ell} \sigma_{k\ell}^2 \bar{(}D_{k\ell}^n)^T B_k^T \hat{P}(x_n) B_k D_{k\ell}^n < 0 \quad (19)$$

This implies,

$$A_n^T \hat{P}(x_{n+1}) A_n - \hat{P}(x_n) + \sigma_{k\ell}^2 \bar{(}D_{k\ell}^n)^T B_k^T \hat{P}(x_n) B_k D_{k\ell}^n < 0$$

Let,

$$Q(x_n) = A_n^T \hat{P}(x_{n+1}) A_n - \hat{P}(x_n) + \alpha_n^{k\ell} (D_{k\ell}^n)^T D_{k\ell}^n$$

and $\alpha_n^{k\ell} > 0$ is chosen such that $\mathscr{R}((D_{k\ell}^n)^T D_{k\ell}^n) \subset \mathscr{N}(Q(x_n))$. This is always possible as,

$$A_n^T \hat{P}(x_{n+1}) A_n - \hat{P}(x_n) < -\sigma_{k\ell}^2 \bar{(}D_{k\ell}^n)^T B_k^T \hat{P}(x_n) B_k D_{k\ell}^n \leq 0$$

The inequality (19) can be rewritten as,

$$Q(x_n) + \left(\sigma_{k\ell}^2 B_k^T \hat{P}(x_{n+1}) B_k - \alpha_n^{k\ell} \right) (D_{k\ell}^n)^T D_{k\ell}^n < 0 \quad (20)$$

From (20) and using the fact $\mathscr{R}((D_{k\ell}^n)^T D_{k\ell}^n) \subset \mathscr{N}(Q(x_n))$ we get,

$$Q(x_n) \leq 0 \text{ and } \sigma_{k\ell}^2 B_k^T \hat{P}(x_{n+1}) B_k < \alpha_n^{k\ell}$$

We have already shown $\| \hat{P}(x_n) \| \geq \gamma_1$. This in turn gives,

$$\alpha_n^{k\ell} > \sigma_{k\ell}^2 B_k^T \hat{P}(x_{n+1}) B_k \geq \sigma_{k\ell}^2 \gamma_1^2 \| B_k \|^2 > 0$$

Using the fact $Q(x_n) \leq 0$ from Eq. (20) we can obtain,

$$A_n^T \hat{P}(x_{n+1}) A_n + \alpha_n^{k\ell} (D_{k\ell}^n)^T D_{k\ell}^n < \hat{P}(x_n)$$

which implies,

$$\alpha_n^{k\ell} (D_{k\ell}^n)^T D_{k\ell}^n < \hat{P}(x_n)$$

Utilizing $\| \hat{P}(x_n) \| \leq \gamma_2$, we get, $\alpha_n^{k\ell} < \frac{\gamma_2^2}{\|D_{k\ell}^n\|^2} < \infty$. Combining upper and lower bounds,

$$0 < \sigma_{k\ell}^2 \gamma_1^2 \| B_k \|^2 < \alpha_n^{k\ell} < \frac{\gamma_2^2}{\| D_{k\ell}^n \|^2} < \infty$$

We define another symmetric, positive definite , matrix function, which is also bounded above and below.

$$\tilde{P}(x_n) = \frac{1}{\alpha_n^{k\ell}} P(x_n)$$

$$A^T(x_n)\tilde{P}(x_{n+1})A(x_n) - \tilde{P}(x_n) \leq -(D_{k\ell}(x_n))^T D_{k\ell}(x_n)$$
$$\sigma^2 B^T \tilde{P}(x_{n+1})B < 1$$

With some abuse of notations,

$$\tilde{P}_n = \sum_{i=n}^{\infty} \left[\left(\prod_{m=n}^{i} A_m \right)^T ((D_{k\ell}^i)^T D_{k\ell}^i - \frac{1}{\alpha_i^{k\ell}} Q_i) \left(\prod_{m=n}^{i} A_m \right) \right]$$

We have already shown, $Q_i \leq 0$. Finally we define, matrix function P by removing terms containing Q_i from above equation.

$$P_n = \sum_{i=n}^{\infty} \left(\prod_{m=n}^{i} A_m \right)^T (D_{k\ell}^n)^T D_{k\ell}^n \left(\prod_{m=n}^{i} A_m \right)$$

It can be observed that $P_n \leq \tilde{P}_n$. This means $\sigma_{k\ell}^2 B_k^T P_{n+1} B_k < 1$. The lower bound of P can be proved by using observability property and also,

$$A_n^T P_{n+1} A_n - P_n = -(D_{k\ell}^n)^T D_{k\ell}^n$$

Hence the proof. ■

Proof: Finally we can proceed to prove Theorem 11. By taking logarithm on both sides of Eq. (9) we get,

$$\log \sigma_{k\ell}^2 + \log \bar{B}_k^T P(x_n) \bar{B}_k < 0, \ \forall n$$

Taking average upto n^{th} step and taking limit $n \to \infty$ we get,

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=0}^{n} \log \sigma_{k\ell}^2 + \frac{1}{n} \sum_{i=0}^{n} \log \bar{B}_k^T P(x_i) \bar{B}_k \right] < 0,$$

Using the definition of physical measure as in Eq. (10),

$$\log \sigma_{k\ell}^2 + \int_X \log \bar{B}_k^T P(x) \bar{B}_k d\mu(x) < 0,$$

The above equation simplifies to Eq. (12) in case of periodic orbit. ■

Next we state two more Lemmas, which are required to prove Theorem 17, which is the necessary condition for mean square synchronization.

Lemma 21: If the system described in (4) is mean square exponentially synchronizing, then there exists positive M_s and β_s such that

$$E_{\{\Delta\}_0^n} \| \eta_{n+1} \|^2 < M_s \beta_s^n \| \eta_0 \|^2$$

where,

$$\begin{aligned} \eta_{n+1} &= A(Z_n)\eta_n + Bv_n, \ \gamma_n = D_{k\ell}(Z_n)\eta_n \\ v_n &= \Delta_n \gamma_n \\ z_{n+1} &= f(z_n), \quad Z_n = [z_n, z_n, \ldots z_n]^T \end{aligned}$$

Lemma 22: The system ,described by (4), is mean square exponentially synchronizing only if, for any arbitrary z_0 and n, there exists a symmetric, positive definite $P_s(Z_l)$ such that,

$$E_{\Delta_0^n} \left[(A_n + B\Delta_n D_n)^T P_s(Z_{n+1})(A_n + B\Delta_n D_n) \right] < P_s(Z_n)$$

and

$$\gamma_0 \leq \| P_s(Z_n) \| \leq \gamma_1 \quad (21)$$

where, $A_i = A(Z_i)$, $D_{k\ell}^i = D_{k\ell}(Z_i)$, and

$$Z_n = [z_n, \ldots z_n]^T, \quad z_{n+1} = f(z_n$$

Finally, the proof of Theorem 17 is completed following the approach outlined in proof of Theorem 8.

Remark 23: The proof of Lemmas 21 and 22 are similar to that of the ones for Lemmas 19 and 20. The only difference is the trajectory for which the P_s matrix is computed is Z_n. It is composed of M copies of z_n, which evolves with f. This reduces the dimention for trajectory computation. This simplification is possible because for synchronization we have the special property,

$$G(Z) = 0, \ \text{where} \ Z = [z, z, \ldots z]^T.$$

The proof of the main theorem on synchronization (i.e., Theorem 17) follows along the lines of proof of Theorem 11. The only difference being the physical measure computation gets reduced to the reduced order system $z_{n+1} = f(z_n)$.

IV. Applications

A. Biological networks

We consider a model of biochemical network involved in yeast cell glycolysis. The model consists of mass balances for the proteins and metabolites, with the mass transfer rates given by fluxes and reaction rate expressions [7]. The biochemical network is described by following nine state equation.

$$\begin{aligned}
\dot{x}_1 &= J_0 - v_1 && (22) \\
\dot{x}_2 &= v_1 - v_2, \quad v_1 = k_1 x_1 x_8 \left[1 + \left(\frac{x_8}{K_i} \right)^n \right]^{-1} \\
\dot{x}_3 &= 2v_2 - v_3 - v_8, \quad v_2 = k_2 x_2 \\
\dot{x}_4 &= v_3 - v_4, \quad v_4 = k_4 x_4 (A - x_8) \\
\dot{x}_5 &= v_4 - v_5, \quad v_5 = k_5 x_5 \\
\dot{x}_6 &= v_5 - v_6 - J, \quad v_6 = k_6 x_6 x_9, \ J = \kappa(x_6 - x_7) \\
\dot{x}_7 &= \phi J - v_9, \quad v_7 = k_7 x_8 \\
\dot{x}_8 &= -2v_1 + v_3 + v_4 - v_7, \quad v_8 = k_8 x_3 x_9 \\
\dot{x}_9 &= v_3 - v_6 - v_8, \quad v_9 = k_9 x_7 \\
v_3 &= \frac{(k_{GAPDH+} k_{PGK+} x_3 N1 (A - x_8) - k_{GAPDH-} k_{PGK-} x_4 x_8 x_9}{k_{GAPDH-} x_9 + k_{PGK+} (A - x_8)}
\end{aligned}$$

where, $N1 = N - x_9$. For the parameter values specified in the Table I, the model exhibit sustained periodic oscillations with time period of T=0.135 s (refer to Fig. 3). We have used the parameter values for simulation purposes as prescribed in [7]. We use Euler method to construct a discrete time system from the continuous time ode (23) with time step for the discretization $\Delta t = 5 \times 10^{-5}$. For the robustness analysis

TABLE I

NOMINAL VALUES OF PARAMETERS

Parameter	Value	Parameter	Value	Parameter	Value
J_0	50	k_1	550	K_i	1.0
k_2	9.8	k_{GAPDH+}	323.8	k_{GAPDH-}	57823.1
k_{PGK+}	76411.1	k_{PGK-}	23.7	k_4	80
k_5	9.7	k_6	2000	k_7	28
k_8	85.7	κ	375	ϕ	0.1
A	4.0	N	1.0	n	4

of limit cycle oscillating solution, we choose seven different parameters namely $k_2, k_5, k_7, k_9, k, k_6, k_8$. The objective is to determine which of these seven parameter is most critical for maintaining the stability of limit cycle oscillating solution. For the purpose of computation, we choose $P = 27$ representative points over one period of the periodic oscillating solution (Fig. 4). The approximation of the physical measure supported on the periodic solution is obtained on these P representative points as shown in Fig. 5. The approximation of physical measure is used to rank order the seven selected parameters with their relative degree of stability margin using formulas (13) and (12). In Fig. 6, we show that plot of stability margin for the seven parameters. From these plots we see that while the parameter k_6 can tolerate the maximum amount of uncertainty, the uncertainty allowed in parameter k_2 is the least.

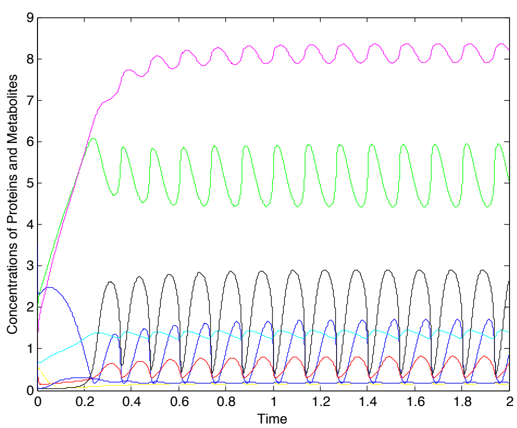

Fig. 3. Limit cycle oscillation in biochemical network model

B. Robust Synchronization of Non Uniform Kuramoto Oscillators

Here we study the problem of robust synchronization in a network of Kuramoto oscillators, which in past has been investigated by many researchers [14], [15].The dynamics of individual oscillators is assumed to be identical and of the form:

$$\dot{\theta}_i = \omega - r\sin\theta_i, \ \ i = 1, \dots, N \tag{23}$$

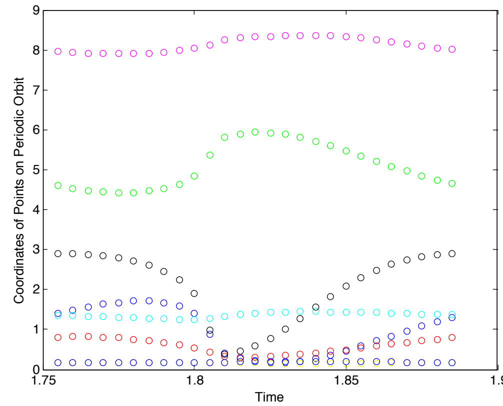

Fig. 4. Representative point over one period of oscillating solution

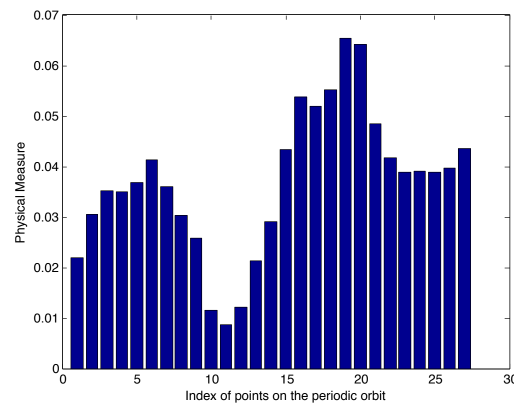

Fig. 5. Approximation of the physical invariant measure with support on periodic orbit

where $\omega = 4\pi$ Hz, $r = 1$, textand $N = 5$. The coupled dynamics is described by following equations

$$\dot{\theta}_i = \omega - r\sin\theta_i + \sum_{j=1}^{N} k_{ij}\sin(\theta_j - \theta_i), \quad i = 1, \dots, N \tag{24}$$

where $k_{ij} = k_{ji}$ are coupling parameters and are assumed to be uncertain. In Fig. 8, we show the nominal values of coupling parameters. Because of the identical nature of individual oscillator dynamics and vanishing property of the coupling terms for $\theta_i = \theta_j$, the synchronized state of the network is characterized by the steady state dynamics of the individual oscillator system. The steady state dynamics of the individual oscillator system is a periodic orbit. In Fig. 7, we show the periodic solution of the individual oscillator. For the purpose of computation $P = 38$ representative points are chosen on the periodic orbit and are shown in Fig. 7. The approximation of the physical measure corresponding to the periodic orbit and its support on the $P = 38$ representative points is shown in Fig. 7. The maximum allowable variance for each of the uncertain link is shown in Fig. 9 and is calculated using the results of Theorem 10 from our main results. Comparing Fig. 8 and Fig. 9, we notice that the link with the smallest nominal value of coupling can tolerate less amount of uncertainty compared to the link with larger

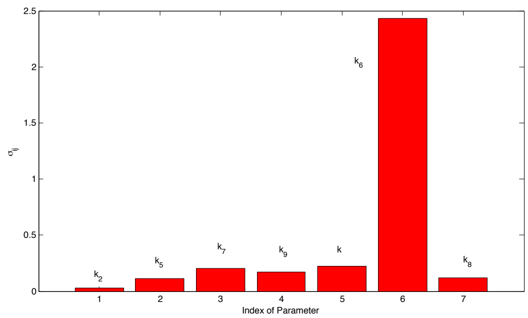

Fig. 6. Maximum allowable variance for different parameters.

nominal value.

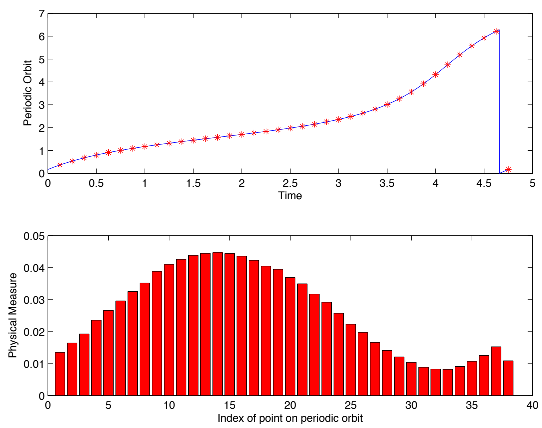

Fig. 7. Periodic orbit and physical measure for an individual oscillator.

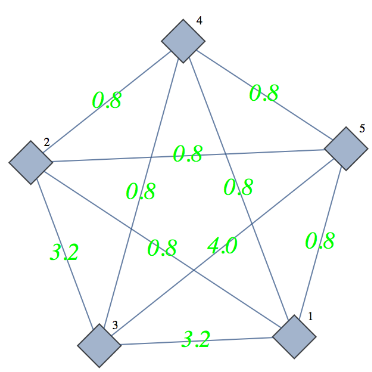

Fig. 8. Mean Value of coupling shown for different links.

V. Conclusions

We developed a systematic approach for the robust stability analysis of nonlinear network systems operating in nonequilibrium. The framework is used for the identification of critical interactions in uncertain network system. The proposed framework is based on combination of tools from linear robust control theory and ergodic theory of dynamical systems. Application of the developed framework is demonstrated on identification of critical parameters responsible for

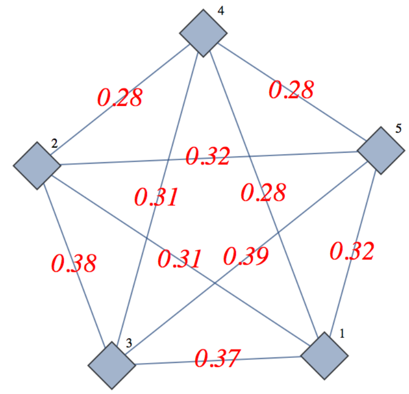

Fig. 9. Allowable σ_{ij} for different links

limit cyclce oscillations in biochemical network involved in yeast cell glycolysis and robust synchronization in netowork of non-uniform Kuramoto oscillators.

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