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Duality in Stability Theory: Lyapunov function and Lyapunov measure

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***Abstract*— In this paper, we introduce Lyapunov measure to verify weak (a.e.) notions of stability for an invariant set of a nonlinear dynamical system. Using certain linear operators from Ergodic theory, we derive Lyapunov results to deduce a.e. stability. In order to highlight the linearity of our approach, we also provide explicit formulas for obtaining Lyapunov function and Lyapunov measure in terms of the resolvent of the two linear operators.**

I. INTRODUCTION

For nonlinear dynamical systems, Lyapunov function based methods play a central role in both stability analysis and control synthesis [Vid02]. Given the complexity of dynamical behavior possible even in low dimensions [ER85], these methods are powerful because they provide an analysis and design approach for *global* stability of an equilibrium solution. However, unlike linear systems, there is no constructive procedure to obtain a Lyapunov function for general nonlinear systems. This lack of constructive procedure for Lyapunov function is an important barrier in nonlinear control theory. For nonlinear ODEs, two ideas have appeared in recent literature towards overcoming this barrier.

In [Ran01], Rantzer introduced a dual to the Lyapunov function, referred to by the author as a **density function**, to define and study weaker notion of stability of an equilibrium solution of nonlinear ODEs. The author shows that the existence of a density function guarantees asymptotic stability in an almost everywhere sense, i.e., with respect to any *set* of initial conditions in the phase space with a positive Lebesgue measure. The second idea involves computation of Lyapunov functions using SOS polynomials. This idea appears in the work of Parrilo [Par00], where the construction of Lyapunov function is cast as a *linear* problem with suitable choice of polynomials (monomials) serving as a basis. In a recent paper by [PPR04], these two ideas have been combined to show that density formulation together with its discretization using SOS methods leads to a convex and linear problem for the joint design of the density function and state feedback controller.

We show that the duality expressed in the paper of [Ran01] and linearity expressed in the paper of [PPR04] is well-understood using the methods of Ergodic theory. Given a dynamical system, one can associate two different *linear* operators known as Koopman and Frobenius-Perron (P-F) operator [LM94], [Man87]. These two operators are adjoint to each other. While the dynamical system describes the evolution of an initial condition, the P-F operator describes the evolution of uncertainty in initial conditions. Under suitable technical conditions, spectrum of these operator on the unit circle provides information about the asymptotic behavior of the system [DJ99], [MB04]. In this paper, spectral analysis of the stochastic operators is used to study the stability properties of the invariant sets of deterministic dynamical systems. In particular, we introduce **Lyapunov measure** as a dual to Lyapunov function. Lyapunov measure is closely related to Rantzer's density function, and like its counterpart it is shown to capture the weaker a.e. notion of stability. Just as invariant measure is a stochastic counterpart of the invariant set, existence of Lyapunov measure is shown to give a stochastic conclusion on the stability of the invariant measure. The key advantage of relating Lyapunov measure to the P-F operator is that the relationship serves to provide explicit formulas of the Lyapunov measure.

For stable linear dynamical systems, the Lyapunov function can be obtained as a positive solution of the so-called Lyapunov equation. The equation is linear and the Lyapunov function is efficiently computed and can even be expressed analytically as an infinite-matrix-series expansion. For the series to converge, there exists a spectral condition on the linear dynamical system ($\rho(A) < 1$). The P-F formulation allows one to generalize these results to the study of stability of invariant and possibly chaotic attractor sets of nonlinear dynamical systems. More importantly, it provides a framework that allows one to carry over the intuition of the linear dynamical systems. For instance, the spectral condition is now expressed in terms of the P-F operator. The Lyapunov measure is shown to be a solution of a linear resolvent operator and admits an infinite-series expansion. The stability result, however, is typically weaker and one can only conclude stability in measure-theoretic (such as a.e.) sense.

The outline of this paper is as follows. In Section II, preliminaries and notation are summarized. In Section III, Lyapunov measure is introduced and related to the stochastic operators and appropriately defined notions of stability of an attractor set. Finally, we summarize some conclusions in Section IV.

II. PRELIMINARIES AND NOTATION

In this paper, *discrete dynamical systems* or mappings of the form

$$x_{n+1} = T(x_n) \tag{1}$$

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are considered. $T : X \to X$ is in general assumed to be only continuous and non-singular with $X \subset \mathbb{R}^n$, a compact set. A mapping T is said to be non-singular with respect to a measure μ if $\mu(T^{-1}B) = 0$ for all $B \in \mathscr{B}(X)$ such that $\mu(B) = 0$. $\mathscr{B}(X)$ denotes the Borel σ-algebra on X, $\mathscr{M}(X)$ the vector space of real valued measures on $\mathscr{B}(X)$. Even though, deterministic dynamics are considered, stochastic approach is employed for their analysis. The stochastic Perron-Frobenius (P-F) operator for a mapping $T : X \to X$ is given by

$$\mathbb{P}[\mu](A) = \mu(T^{-1}(A)), \tag{2}$$

where $\mu \in \mathscr{M}(X)$ and $A \in \mathscr{B}(X)$; cf., [LM94], [DJ00]. The invariant measure are the fixed points of the P-F operator $\mathbb{P}$ that are additionally probability measures. From Ergodic theory, an invariant measure is always known to exist under the assumption that the mapping T is at least continuous and X is compact; cf., [KH95].

For Eq. (1), the operator

$$Uf(x) = f(Tx) \tag{3}$$

is called the Koopman operator for $f \in C^0(X)$. The Koopman operator is a dual to the P-F operator, where the duality is expressed by the following

$$< Uf, \mu >= \int_X Uf(x) d\mu(x) = \int_X f(x) d\mathbb{P}\mu(x) =< f, \mathbb{P}\mu > . \tag{4}$$

A. Physical measure and almost everywhere stability

A set $A \subset X$ is called *T-invariant* if

$$T(A) = A. \tag{5}$$

In this paper, global stability properties of T-invariant sets that are additionally "minimal" in some sense will be investigated. Using Eq. (2), a measure μ is said to be a T-invariant measure if

$$\mu(B) = \mu(T^{-1}(B)) \tag{6}$$

for all $B \in \mathscr{B}(A)$. A T-invariant measure in Eq. (6) is a stochastic counterpart of the T-invariant set in Eq. (5) [ER85], [KH95]. For typical dynamical systems, the set A equals the support of *its* invariant measure μ. To make the correspondence precise, an **attractor** is defined as any set that satisfies the following two properties:

1) Set A is a closed and invariant, i.e., $T(A) = A$,
2) There is a unique **physical invariant measure**, defined below, supported on A.

Definition 1 (Physical measure) *Let A be a T-invariant set. An ergodic measure μ with support on A is called a physical measure if there is an open neighborhood $U \subset X$ with $A \subset \overline{U}$ such that for Lebesgue almost every $x \in U$ and for all continuous functions $\phi : U \to \mathbb{R}$ the equation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \phi(T^k(x)) = \int_A \phi(x)\, d\mu(x) \tag{7}$$

holds.

This definition stresses the statistical behavior of the attractor set. Although rigorous results have been shown only for special cases, it is generally believed that physical measures exists in physical dynamical systems [You02]. This is also supported by numerical evidence with set-oriented methods for approximation of physical measures; cf., [DJ00]. A locally stable equilibrium is the simplest example of an attractor set and the physical measure corresponding to it is the Dirac-delta measure supported on the equilibrium point. The existence of physical measure is related to the notion of a.e. stability. In the following definitions and subsequent sections, we use $U(\varepsilon)$ to denote an ε-neighborhood of an invariant set A, and $A^c := X \setminus A$ to denote the complement set.

Definition 2 (ω-limit set) *A point $y \in X$ is called a ω- limit point for a point $x \in X$ if there exists a sequence of integers $\{n_k\}$ such that $T^{n_k}(x) \to y$ as $k \to \infty$. The set of all ω-limit point for x is denoted by $\omega(x)$ and is called its ω- limit set.*

Definition 3 (Almost everywhere stable) *An invariant set A for the dynamical system $T : X \to X$ is said to be stable almost everywhere (a.e.) with respect to a finite measure $m \in \mathscr{M}(A^c)$ if*

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0 \tag{8}$$

For the special case of a.e. stability of an equilibrium point x_0 with respect to the Lebesgue measure, the definition reduces to

$$m\{x \in X : \lim_{n\to\infty} T^n(x) \neq x_0\} = 0, \tag{9}$$

where m in this case is the Lebesgue measure.

Motivated by the familiar notion of point-wise exponential stability in phase space, we introduce a stronger notion of stability in the measure space. This stronger notion of stability captures a geometric decay rate of convergence.

Definition 4 (Stable a.e. with geometric decay) *The invariant set $A \subset X$ for the dynamical system $T : X \to X$ is said to be stable almost everywhere with geometric decay w.r.t. to a finite measure $m \in \mathscr{M}(A^c)$ if given $\varepsilon > 0$, there exists $K(\varepsilon) < \infty$ and $\beta < 1$ such that*

$$m\{x \in A^c : T^n(x) \in B\} < K\beta^n \quad \forall\, n \geq 0 \tag{10}$$

for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$.

III. MAIN RESULT

In this section, Lyapunov type global stability conditions are presented using the infinite-dimensional P-F operator $\mathbb{P}$ for the mapping $T : X \to X$ in Eq. (1). Recall that an attractor set A is defined to be globally stable w.r.t. to a measure m if

$$\omega(x) \subseteq A, \text{ a.e. } x \in A^c, \tag{11}$$

where *a.e.* is with respect to the measure *m*. Now, consider the restriction of the mapping $T : A^c \to X$ on the complement set. This restriction can be associated with a suitable stochastic operator related to $\mathbb{P}$ that is useful for the stability analysis with respect to the complement set. The following section makes the association precise.

A. Decomposition of Frobenius-Perron operator

Corresponding to the mapping $T : A^c \to X$, the operator

$$\mathbb{P}_1[\mu](A) \doteq \int_{A^c} \delta_{T(x)}(B) d\mu(x) \tag{12}$$

is well-defined for $\mu \in \mathscr{M}(A^c)$ and $B \subset \mathscr{B}(A^c)$. This is because the set A is T-invariant. Denote $\mathbb{P}_0$ as the P-F operator for the restriction $T : A \to A$. It is possible to express the P-F operator $\mathbb{P}$ for $T : X \to X$ in terms of $\mathbb{P}_0$ and $\mathbb{P}_1$. Indeed, consider a splitting of the measure space

$$\mathscr{M}(X) = \mathscr{M}_0 \oplus \mathscr{M}_1, \tag{13}$$

where $\mathscr{M}_0 = \mathscr{M}(A)$ and $\mathscr{M}_1 = \mathscr{M}(A^c)$. Note that $\mathbb{P}_0 : \mathscr{M}_0 \to \mathscr{M}_0$ because $T : A \to A$ and $\mathbb{P}_1 : \mathscr{M}_1 \to \mathscr{M}_1$ by construction. It then follows that with respect to the splitting in Eq. (13), the P-F operator has a lower-diagonal matrix representation given by

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_0 & 0 \\ \times & \mathbb{P}_1 \end{bmatrix}. \tag{14}$$

B. Stability & Lyapunov measure

Using the lower triangular representation of $\mathbb{P}$ in Eq. (14),

$$\mathbb{P}^n = \begin{bmatrix} \mathbb{P}_0^n & 0 \\ \times & \mathbb{P}_1^n \end{bmatrix}. \tag{15}$$

Explicitly, for $B \subset \mathscr{B}(A^c)$,

$$\mathbb{P}_1\mu(B) = \int_{A^c} \chi_B(Tx) d\mu(x) = \mu(T^{-1}(B) \cap A^c) \tag{16}$$

$$\mathbb{P}_1^n\mu(B) = \int_{A^c} \chi_B(T^n x) d\mu(x) = \mu(T^{-n}(B) \cap A^c). \tag{17}$$

These formulas are useful because one can now express the conditions for stability in Definitions 3 and 4 in terms of the asymptotic behavior of the operator $\mathbb{P}_1^n$.

Lemma 5 *Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset X$, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. The following express conditions for a.e. stability w.r.t a finite measure $m \in \mathscr{M}(A^c)$:*

1) *The invariant set A is a.e. stable (definition 3) with respect to a measure m if and only if*

$$\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0 \tag{18}$$

for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$ and every $\varepsilon > 0$.

2) *The invariant set A is a.e. stable with geometric decay (definition 4) with respect to a measure m if and only if for every $\varepsilon > 0$, there exists $K(\varepsilon) < \infty$ and $\beta < 1$ such that*

$$\mathbb{P}_1^n m(B) < K\beta^n \quad \forall\, n \geq 0 \tag{19}$$

and for all sets $B \in \mathscr{B}(X \setminus U(\varepsilon))$.

Proof: For $B \in \mathscr{B}(X \setminus U(\varepsilon))$, denote

$$B_n = \{x \in A^c \text{ and } T^n(x) \in B\}. \tag{20}$$

It is then easy to see that

$$m(B_n) = m(T^{-n}(B) \cap A^c) = \mathbb{P}_1^n m(B), \tag{21}$$

where the last equality follows from Eq. (17). The equivalence for part 2 (Eq. (19)) then follows by applying definition 4. To see part 1, note that

$$\lim_{n\to\infty} \chi_{B_n}(x) = 0 \tag{22}$$

for all x whose ω-limit points lie in A. If A is assumed a.e. stable, the limit in Eq. (22) is a.e. zero and

$$\begin{aligned} 0 &= \int_{A^c} \lim_{n\to\infty} \chi_{B_n}(x) dm(x) = \lim_{n\to\infty} \int_{A^c} \chi_{B_n}(x) dm(x) \\ &= \lim_{n\to\infty} \mathbb{P}_1^n m(B) \end{aligned} \tag{23}$$

by dominated convergence theorem; cf., [Rud87]. Conversely, for $\varepsilon > 0$, consider the set

$$S_n = \{x \in A^c : T^k(x) \in X \setminus U(\varepsilon) \text{ for some } k > n\} \tag{24}$$

and let

$$S = \cap_{n=1}^{\infty} S_n \tag{25}$$

i.e., S is the set of points, some of whose limit points lie in $X \setminus U(\varepsilon)$. Clearly, $x \in S_n$ if and only if $T(x) \in S_{n-1}$. By construction, $x \in S$ if and only if $T(x) \in S$, i.e., $S = T^{-1}(S)$. Furthermore, $S \subset A^c$ and we have,

$$m(S) = m(T^{-1}(S) \cap A^c) = \mathbb{P}_1 m(S). \tag{26}$$

Now, if $\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0$ for all $B \in \mathscr{B}(X \setminus U(\varepsilon))$ and in particular for $B = S$ then Eq. (26) implies that $m(S) = 0$. Since ε here is arbitrary, we have

$$m\{x \in A^c : \omega(x) \not\subseteq A\} = 0, \tag{27}$$

and thus A is a.e. stable in the sense of definition 3.

■

The two conditions in Eq. (18)-(19) represent a certain property, **transience**, of the stochastic operator $\mathbb{P}_1$ w.r.t Lebesgue measure m. In by itself, the two conditions are not very useful to verify stability any more than the definitions themselves. The definition involves iterating the mapping for *all* initial conditions in A^c while the two conditions involve iterating the stochastic operator for *all* Borel set B in A^c. Both are equally complex. However, just as stability can be verified by constructing Lyapunov function for the mapping T, transience can be verified by constructing a **Lyapunov measure** for the operator $\mathbb{P}$.

Definition 6 (Lyapunov measure) *is any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies*

$$\mathbb{P}_1\bar{\mu}(B) < \alpha\bar{\mu}(B), \tag{28}$$

for every set $B \subset \mathscr{B}(A^c)$ with

$$\bar{\mu}(B) > 0. \tag{29}$$

$\alpha \leq 1$ is some positive constant.

This construction and the Lyapunov measure's relationship with the two notions of transience will be a subject of the following three theorems. The first theorem shows that the existence of a Lyapunov measure $\bar{\mu}$ is sufficient for almost everywhere stability with respect to any absolutely continuous measure m.

Theorem 7 *Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset X$, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. Suppose there exists a non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies*

$$\mathbb{P}_1 \bar{\mu}(B) < \bar{\mu}(B) \tag{30}$$

for all $B \subset \mathscr{B}(A^c)$ with $\bar{\mu}(B) > 0$. Then the invariant set A is almost everywhere stable w.r.t. to any absolutely continuous measure $m \prec \bar{\mu}$.

Proof: Consider any set $B \in \mathscr{B}(X \setminus U(\varepsilon))$ with $m(B) > 0$. Using Lemma 5, a.e. stability is equivalent to

$$\lim_{n\to\infty} \mathbb{P}_1^n m(B) = 0. \tag{31}$$

To show Eq. (31), it is first claimed that $\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = 0$. We note that because $m \prec \bar{\mu}$, the claim implies Eq. (31) and thus a.e. stability. In order to prove the claim, we note that $\bar{\mu}(B) > 0$ and consider the sequence of real numbers $\{\mathbb{P}_1^n \bar{\mu}(B)\}$. Using Eq. (30), this is a decreasing sequence of non-negative numbers. As in Lemma 5, let

$$S := \{x \in A^c : \lim_{n_k\to\infty} T^{n_k}(x) \in B\} \tag{32}$$

be the set of points, some of whose ω-limit points lie in B. For $B_n = \{x \in A^c : T^n x \in B\}$, $\chi_{B_n}(x) \to 0$ whenever $x \notin S$. By dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = \lim_{n\to\infty} \int_{A^c} \chi_{B_n}(x) d\bar{\mu}(x) \le \bar{\mu}(S). \tag{33}$$

As in Lemma 5, $T^{-1}(S) = S$ and

$$\mathbb{P}_1 \bar{\mu}(S) = \bar{\mu}(S) \tag{34}$$

and Eq. (30) implies that $\bar{\mu}(S) = 0$. Using Eq. (33),

$$\lim_{n\to\infty} \mathbb{P}_1^n \bar{\mu}(B) = 0, \tag{35}$$

and this verifies the claim and thus proves the theorem. ∎

The following theorem provides a sufficient condition for almost everywhere stability with geometric decay in terms of Lyapunov measure.

Theorem 8 *Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset X$, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. Suppose there exists a non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$ and satisfies*

$$\mathbb{P}_1 \bar{\mu}(B) < \alpha \bar{\mu}(B), \tag{36}$$

for all $B \subset \mathscr{B}(A^c)$ with $\bar{\mu}(B) > 0$, and $\alpha < 1$. Then

1) *A is a.e. stable with respect to any finite measure $m \prec \bar{\mu}$.*
2) *A is a.e. stable with geometric decay with respect to any measure m satisfying $m \le \gamma \bar{\mu}$ for some constant $\gamma > 0$.*

Proof:

1) Since
$$\mathbb{P}_1 \bar{\mu}(B) < \alpha \bar{\mu}(B) < \bar{\mu}(B) \tag{37}$$
the result follows from the proof of Theorem 7.
2) Consider any set $B \in \mathscr{B}(X \setminus U(\varepsilon))$. A simple calculation shows that
$$\mathbb{P}_1^n m(B) \le \gamma \mathbb{P}_1^n \bar{\mu}(B) \le \alpha^n \gamma \bar{\mu}(B) < K \alpha^n, \tag{38}$$
where $K(\varepsilon) = \gamma \bar{\mu}(X \setminus U(\varepsilon))$ is finite. Using Lemma 5, A is stable almost everywhere with geometric decay with respect to the measure m.

∎

The following can be used to construct Lyapunov measure $\bar{\mu}$ as an infinite series involving sub-Markov operator $\mathbb{P}_1$.

Theorem 9 *Consider $T : X \to X$ in Eq. (1) with an invariant set $A \subset X$, $U(\varepsilon)$ is an ε-neighborhood of A, and $A^c = X \setminus A$. Suppose A is stable a.e. with geometric decay w.r.t some finite measure $m \in \mathscr{M}(A^c)$. Then there exists a non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$ which is finite on $\mathscr{B}(X \setminus U(\varepsilon))$, is equivalent to measure m, and satisfies*

$$\mathbb{P}_1 \bar{\mu}(B) < \bar{\mu}(B) \tag{39}$$

for all $B \subset \mathscr{B}(A^c)$ with $\bar{\mu}(B) > 0$. Furthermore, $\bar{\mu}$ may be constructed to dominate measure m i.e., $m(B) \le \bar{\mu}(B)$.

Proof: For any given $\varepsilon > 0$, construct a measure $\bar{\mu}$ as:

$$\bar{\mu}(B) = (I + \mathbb{P}_1 + \mathbb{P}_1^2 + \ldots) m(B) = \sum_{j=0}^{\infty} \mathbb{P}_1^j m(B), \tag{40}$$

where $B \in \mathscr{B}(X \setminus U(\varepsilon))$. For all such sets, the stability definition 4 implies that there exists a $K(\varepsilon) < \infty$ and $\beta < 1$ such that

$$\mathbb{P}_1^n m(B) < K \beta^n. \tag{41}$$

As a result, the infinite-series in Eq. (40) converges, and $\bar{\mu}(B)$ is well-defined, non-negative, and finite. Since, T is assumed non-singular, the individual measures $\mathbb{P}_1^n m$ are absolutely continuous w.r.t. m and thus $\bar{\mu} \prec m$. By construction,

$$m(B) \le \bar{\mu}(B), \tag{42}$$

and thus the two measures are equivalent. Applying $(\mathbb{P}_1 - I)$ to both sides of Eq. (40), we get

$$\mathbb{P}_1 \bar{\mu}(B) - \bar{\mu}(B) = -m(B) < 0 \Longrightarrow \mathbb{P}_1 \bar{\mu}(B) < \bar{\mu}(B) \tag{43}$$

whenever $m(B) > 0$, and equivalently, $\bar{\mu}(B) > 0$. ∎

Remark 10 In the three theorems presented above, A is a.e. stable w.r.t $m \in \mathscr{M}(A^c)$. In general, m can be be any finite measure. Our primary interest is in Lebesgue a.e. stability, and we often take m to be the Lebesgue measure. Another finite measure of interest is

$$m_S(B) = m(B \cap S) \tag{44}$$

where $S \subset A^c$, $B \in \mathscr{B}(X \setminus U(\varepsilon))$, and m is the Lebesgue measure. Using stability definitions, A is a.e. stable w.r.t m_S

iff a.e. initial condition $x \in S$ converges to A. Using the results of this section, m_S can thus be used to characterize and study the domain of attraction of an invariant set A.

Before closing this section, we summarize the salient features of the Lyapunov measure:

1) its existence allows one to verify a.e. asymptotic stability (Theorem 8),
2) for an asymptotically stable system with geometric decay, the infinite-series (see Eq. (40))

$$(I-\mathbb{P}_1)^{-1}m = (I+\mathbb{P}_1+\ldots+\mathbb{P}_1^N+\ldots)m \qquad (45)$$

can be used to construct it.

The series-formulation in fact is very much related to the well-known Lyapunov equation in linear settings.

C. Lyapunov function and Koopman operator

Consider a linear dynamical system

$$x(n+1) = Ax(n), \qquad (46)$$

where $\rho(A) < 1$. With a Lyapunov function candidate $V(x) = x'Px$, the Lyapunov equation is $A'PA - P = -Q$, where Q is positive definite. A positive-definite solution for P is given by

$$P = Q + A'QA + \ldots + A'^n QA^n + \ldots, \qquad (47)$$

where the series converges iff $\rho(A) < 1$. Setting $g(x) = x'Qx$, the infinite-series solution for any $x \in \mathbb{R}^n$ is given by

$$V(x) = \sum_{n=0}^{\infty} g(A^n x) = \sum_{n=0}^{\infty} U^n g(x) = (I-U)^{-1} g(x), \qquad (48)$$

where U is the Koopman operator, the dual to $\mathbb{P}$. The choice of $g(0) = 0$ on the complement set to the attractor $\{0\}$ ensures that the series representation converges. Even though, we have arrived at the series representation in Eq. (48) starting from the linear settings, the series is valid for nonlinear dynamical system or continuous mapping of Eq. (1); U is the Koopman operator for mapping T. If the series converges, one can express the solution in terms of the resolvent operator as in Eq. (48). For a convergent series, it is also easy to check than

$$V(Tx) - V(x) = UV(x) - V(x) = -g(x), \qquad (49)$$

i.e., V is a Lyapunov function for $g(x) > 0$. Note, g need not be quadratic or even a polynomial – any positive C^0 function with $g(0) = 0$ will suffice. Moreover, the description is linear. The following theorem shows that the Lyapunov function can be constructed by using the resolvent of the Koopman operator for a stable system. In particular, we assume that the equilibrium point is globally exponentially stable and prove in essence a converse Lyapunov theorem for stable systems; cf., [Vid02].

Theorem 11 *Consider $T: X \to X$ as in Eq. (1). Suppose $x = 0$ is a fixed-point ($T(0) = 0$), which is globally exponentially stable, i.e.,*

$$\| T^n(x) \| \leq K\alpha^n \| x \| \quad \forall x \in X \qquad (50)$$

where $\alpha < 1$, $K > 1$, and $\|\cdot\|$ is the Euclidean norm in X. Then there exists a non-negative function $V: X \to \mathbb{R}^+$ satisfying

$$\begin{aligned} a\| x \|^p &\leq V(x) \leq b \| x \|^p, \\ V(Tx) &\leq c \cdot V(x), \end{aligned} \qquad (51)$$

where a, b, c, p are positive constants; $c < 1$. Also, V can be expressed as

$$V(x) = (I-U)^{-1} f(x), \qquad (52)$$

where $f(x) = \| x \|^p$ and U is the koopman operator corresponding to the dynamical system T.

Proof: Let $f(x) = \| x \|^p$ with $p \geq 1$ and set

$$V_N(x) = \sum_{n=0}^{N} f(T^n x) = \sum_{n=0}^{N} U^n f(x). \qquad (53)$$

Now,

$$\|V_N(x)\| \leq \sum_{n=0}^{\infty} \| T^n x \|^p \leq K^p \sum_{n=0}^{\infty} \alpha^{np} \| x \|^p \leq \frac{K^p}{1-\alpha^p} \|x\|^p \qquad (54)$$

satisfies a uniform bound because of globally exponentially stability (Eq. (50)) and because X is compact. As a result, $V(x) = \lim_{N\to\infty} V_N(x)$ is well-defined and can be expressed as an infinite-series,

$$V(x) = \lim_{N\to\infty} \sum_{n=0}^{N} U^n f(x) = (I-U)^{-1} f(x). \qquad (55)$$

By Eqs. (53) and (54),

$$\| x \|^p \leq V(x) \leq \frac{K^p}{1-\alpha^p} \| x \|^p = b \| x \|^p, \qquad (56)$$

where $b > 1$. Finally, because $T: X \to X$, $V(Tx) = UV(x)$, Eq. (55) gives

$$(U-I)V(x) = -f(x) = -\| x \|^p \leq \frac{-1}{b} \cdot V(x) \qquad (57)$$

Set $c = (1 - \frac{1}{b})$. Clearly, $c < 1$ and using Eq. (57),

$$V(Tx) \leq c \cdot V(x). \qquad (58)$$

■

The series formulation in Eq. (45) using the P-F operator on the complement set A^c is a dual to the series expansion using the Koopman operator in Eq. (55). The Lyapunov *measure* description thus is a dual to the Lyapunov *function* description. The measure-theoretic description provides a *set-wise* counterpart to the *point-wise* description with Lyapunov function. One of the advantage is that weaker notions of stability, such as a.e stability are possible with measure-theoretic description.

IV. Conclusions

In nonlinear control, Lyapunov functions have primarily been used for verifying stability and stabilization, using control, of an equilibrium solution. An equilibrium is only one of the many recurrent behavior that are possible in nonlinear dynamical systems. A stable periodic orbit is a simple example of non-equilibrium behavior but stranger attractors (for e.g., Lorentz attractor) arise even in low-dimensions. In higher dimensions such as distributed fluid systems, non-equilibrium behavior is the norm. This paper is based on the premise that measure-theoretic stochastic approaches are important to the study of non-equilibrium behavior in dynamical systems. Indeed, stochastic dynamic methods have come to be viewed as increasingly relevant for the study of global recurrent behavior such as attractor sets in dynamical systems. Lyapunov measures, introduced in this paper, are a stochastic counterpart to the notion of transience and thus useful for verifying (weak forms of) stability of the recurrent attractor sets.

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