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Connection between almost everywhere stability of an ODE and advection PDE

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Abstract— **A result on the necessary and sufficient conditions for almost everywhere stability of an invariant set in continuous-time dynamical systems is presented. It is shown that the existence of a *Lyapunov density* is equivalent to the *almost everywhere stability* of an invariant set. Furthermore, such a density can be obtained as the positive solution of a linear partial differential equation analogous to the positive solution of Lyapunov equation for stable linear systems.**

I. Introduction

In this paper, a linear transfer operator framework is used to study the almost everywhere stability problem for continuous time dynamical systems. The notion of almost everywhere stability and density function verifying this notion of stability were introduced for the first time by Rantzer [1]. In [2] [3] [4] the Lyapunov measure is introduced for verifying almost everywhere stability of an invariant set for a discrete time dynamical system. The density function and the Lyapunov measure were shown to be dual to the Lyapunov function and capture the weaker notion of almost everywhere stability. One of the remarkable properties enjoyed by the density function and the Lyapunov measure, which is not true for the Lyapunov function, is convexity in terms of controller design [5] [6]. This makes the density function and the Lyapunov measure a very useful design tool for nonlinear control systems.

In this paper, we continue the investigation of the extended notion of stability. In particular, we study a new notion of stability, introduced in [4] for discrete time dynamical systems, called *almost everywhere uniform stability*. We give necessary and sufficient conditions for almost everywhere uniform stability of an invariant set for a continuous time dynamical system. Results on necessary and sufficient conditions for almost everywhere stability of an equilibrium point using the density function already exist in [7] [8]. However, these results were proven under the assumption that the equilibrium point is *locally stable*. In this paper we do not assume local stability of the invariant set. Instead we assume a weaker form of stability of an invariant set which we call almost everywhere uniform stability. The notion of almost everywhere uniform stability and necessary and sufficient conditions verifying this stability using the Lyapunov measure were proved in [4] for discrete time dynamical systems. So the result in this paper can be viewed as the continuous time counterpart of the discrete time result presented in [4]. We show that almost everywhere uniform stability of an invariant set, according to Definition (2), is equivalent to the existence of a positive solution, i.e. *Lyapunov density*, to the advection equation with a positive right hand side. We believe that this connection between stability and the solution of a partial differential equation will be an important step towards computing the Lyapunov density and opening up a new approach towards verifying stability.

The paper is organized as follows. In Section (II), we briefly mention the preliminary concepts required for this paper. We introduce the new notion of almost everywhere uniform stability in Section (III). The main result is stated and proven in Section (IV). We show an important link between a.e. uniform stability and the spectrum of a certain operator in Section (V). Numerical simulations for three different systems are shown in Section (VI). Conclusion follows in section (VII)

II. Preliminaries

In this paper we are interested in the global stability property of the invariant set for the following ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

where $f : X \to X$ is assumed to be smooth and X is a compact subset of $\mathbb{R}^n$. We use the notation $\phi_t(x)$ to denote the solution or *flow* of (1) at time t, having started from the initial condition x. Equation (1) can be used to study the evolution of a single trajectory. The evolution of ensembles of trajectories or the densities on the phase space can be studied using a linear operator called the Perron-Frobenius (P-F) operator $\mathbb{P}_t : L^1(X) \to L^1(X)$ defined as follows

$$\int_A \mathbb{P}_t\rho(x)dx = \int_{\phi_{-t}(A)} \rho(x)dx = \int_A \rho(\phi_{-t}(x))|\frac{\partial\phi_{-t}(x)}{\partial x}|dx \tag{2}$$

for every set $A \subset X$. Hence the following identity is true

$$\mathbb{P}_t\rho(x) = \rho(\phi_{-t}(x))|\frac{\partial\phi_{-t}(x)}{\partial x}|, \tag{3}$$

where $|\frac{\partial\phi_{-t}(x)}{\partial x}|$ is the determinant of the Jacobian of the flow ϕ_{-t}.

Furthermore, the Perron-Frobenius operator introduced above is the semigroup corresponding to the operator $\mathbb{A}\rho = -\bigtriangledown\cdot(\rho f)$. In other words, $\mathbb{P}_t = e^{\mathbb{A}t}$ describes the evolution of densities ρ via the *advection equation*

$$\frac{\partial\rho}{\partial t} = -\bigtriangledown\cdot(\rho f) =: \mathbb{A}\rho. \tag{4}$$

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For more details on the Perron-Frobenius operator and its infinitesimal generator see ([9]). If $(X,\mathscr{B},\mu)$ is a measure space and $\mathbb{P}_t$ is the Perron-Frobenius operator corresponding to the dynamical system (1), then the $\mathbb{P}_t$ satisfies the following properties [9].

1) $\mathbb{P}_t(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\mathbb{P}_t f_1+\alpha_2\mathbb{P}_t f_2$ for all $f_1,f_2\in L^2$ and $\alpha_1,\alpha_2\in\mathbb{R}$.
2) $\mathbb{P}_t f\geq 0$ if $f\geq 0$
3) $\int_X \mathbb{P}_t f(x)\mu(dx)=\int_X f(x)\mu(dx)$

Roughly speaking, the Perron-Frobenius operator and the advection equation can be thought of as describing the evolution of the density of a fluid as it moves under the influence of the vector-field (1). Properties 2) and 3) above can then be thought of as the fluid always having positive density and the total mass of the fluid remaining constant as it moves under (1).

III. Almost Everywhere Stability

We are interested in the global *almost everywhere stability* of the invariant set Λ. Let us first describe what we mean by an invariant set.

Definition 1 (Invariant Set): A closed set $\Lambda\subset X$ is said to be an invariant set for (1) if for every $x\in\Lambda$, $\phi_t(x)\in\Lambda$ for all time $t\in\mathbb{R}$, i.e., $\phi_t(\Lambda)=\Lambda$.

To define almost everywhere stability of an invariant set Λ, we let

$$A_t:=\{x\in\Lambda^c:\phi_t(x)\in A\}$$

for any set $A\subset X\setminus B_\delta$, where $B_\delta\supset\Lambda$, is the δ neighborhood of the set Λ for some fixed $\delta>0$.

Definition 2 (Almost Everywhere Uniformly Stability): The invariant set $\Lambda\subset B_\delta$ for the differential equation (1) is said to be almost everywhere uniformly stable with respect to measure m if for any given $\varepsilon>0$, there exists a $T(\varepsilon,\delta)>0$ such that

$$\int_T^\infty m(A_t)dt<\varepsilon \tag{5}$$

for every set $A\subset X\setminus B_\delta$.

Remark 3: The measure m in the definition of almost everywhere uniform stability will be assumed to be the Lebesgue measure or any measure that is absolutely continuous with respect to Lebesgue measure.

The definition says that the measure (w.r.t. m) of set of points that stay outside the δ neighborhood of the invariant set can be made arbitrarily small with increasing time. Since the global stability is with respect to the set of points which are outside the δ neighborhood of the invariant set, this motivates us to look at the restriction of the Perron-Frobenius semigroup to the space $L^1(X\setminus B_\delta)$. Hence we define the new semigroup corresponding to the restriction of the flow $\phi_t:X\setminus B_\delta\to X$ as follows

$$\mathbb{P}_t^1\rho(x):=\chi_{X\setminus B_\delta}(x)\rho(\phi_{-t}(x))|\frac{\partial\phi_{-t}(x)}{\partial x}|, \tag{6}$$

where $\rho(x)$ is supported on the set $X\setminus B_\delta$ and $\chi_{X\setminus B_\delta}(x)$ is the indicator function of the set $X\setminus B_\delta$. So we have

$$\mathbb{P}_t^1=\Sigma\mathbb{P}_t:L^1(X\setminus B_\delta)\to L^1(X\setminus B_\delta),$$

where $\Sigma:L^1(X)\to L^1(X\setminus B_\delta)$ is the projection operator, and

$$\Sigma\rho(x)=\chi_{X\setminus B_\delta}(x)\rho(x).$$

Let $\mathbb{A}^1$ be the infinitesimal generator corresponding to the semigroup of the restriction $\mathbb{P}_t^1$. The relation between the infinitesimal generator corresponding to $\mathbb{P}_t^1$ and that of $\mathbb{P}_t$ is established in the following Lemma.

Lemma 4: Let $\mathbb{A}^1$ and $\mathbb{A}$ be the infinitesimal generators corresponding to the semigroups $\mathbb{P}_t^1$ and $\mathbb{P}_t$ respectively and $\Sigma:L^1(X)\to L^1(X\setminus B_\delta)$ be the projection operator, then we have

$$\mathbb{A}^1=\Sigma\mathbb{A}\quad\text{on}\quad L^1(X\setminus B_\delta).$$

Proof: For $\rho\in L^1(X\setminus B_\delta)$, we have

$$\Sigma\rho=\rho.$$

Now, we have the following identity:

$$\mathbb{P}_t^1\rho(x)-\rho(x)=\Sigma\mathbb{P}_t\rho(x)-\Sigma\rho(x).$$

Therefore we have,

$$\lim_{t\to 0}\frac{\mathbb{P}_t^1\rho(x)-\rho(x)}{t}=\Sigma\lim_{t\to 0}\frac{\mathbb{P}_t(\rho(x))-\rho(x)}{t}=\Sigma\mathbb{A}\rho.$$

■

Since by assumption the measure m is either the Lebesgue measure or absolutely continuous with respect to the Lebesgue measure, we let ρ_0 be the density of the measure m with support on the set $X\setminus B_\delta$, i.e.,

$$m(A)=\int_A\rho_0(x)dx$$

for $A\subset X\setminus B_\delta$. The following Lemma establishes the connection between the almost everywhere uniform stability of the invariant set and the asymptotic property of the restricted semigroup $\mathbb{P}_t^1$.

Lemma 5: The invariant set $\Lambda\subset X$ for the system of differential equations (1) is almost everywhere uniformly stable with respect to measure m if and only if for every $\varepsilon>0$ there exits a $T(\delta,\varepsilon)$ such that

$$\int_A\int_T^\infty\mathbb{P}_t^1\rho_0(x)dtdx<\varepsilon \tag{7}$$

for any set $A\subset X\setminus B_\delta$.

Proof: Almost everywhere uniform stability with respect to measure m implies that for any $\varepsilon>0$ there exists an $T(\delta,\varepsilon)>0$ such that

$$\int_T^\infty m(A_t)dt<\varepsilon$$

For $A\subset X\setminus B_\delta$, we have

$$A_t=\phi_{-t}(A)\cap\Lambda^c.$$

$$\begin{aligned}\int_T^\infty m(A_t)dt&=\int_T^\infty\int_{A_t}\rho_0(x)dxdt\\&=\int_T^\infty\int_A\rho_0(\phi_{-t}(x))|\frac{\partial\phi_{-t}(x)}{\partial x}|dxdt\\&=\int_T^\infty\int_A\chi_{X\setminus B_\delta}(x)\rho_0(\phi_{-t}(x))|\frac{\partial\phi_{-t}(x)}{\partial x}|dxdt\\&=\int_T^\infty\int_A\mathbb{P}_t^1\rho_0(x)dxdt<\varepsilon.\end{aligned} \tag{8}$$

where we have use the fact that $A \subset X \setminus B_\delta$. ■

IV. MAIN RESULT

In this section, we prove the main result of this paper giving a necessary and sufficient condition for almost everywhere uniform stability of the invariant set Λ. Before proving the main theorem, we prove the following lemma, which is used in the proof of the main theorem.

Lemma 6: Let $\phi_t(x)$ denote the solution of Equation (1). Then we have the following identity

$$\det \frac{d\phi_t(x)}{dx} = e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}. \tag{9}$$

Proof: We first note that the following integral identity is true by the assumption that $\phi_t(x)$ is the solution of Equation (1)

$$\phi_t(x) = \phi_0(x) + \int_0^t f(\phi_s(x))ds, \tag{10}$$

and $\phi_0(x) = x$. Next we differentiate Equation (10) with respect to x to get the following

$$\frac{d\phi_t(x)}{dx} = I + \int_0^t \frac{df}{dx}(\phi_s(x))\frac{d\phi_s(x)}{dx}ds.$$

Let us denote $M(t) = \frac{d\phi_t(x)}{dx}$. We note that $M(0) = I$. Furthermore we have the following

$$M'(t) = \frac{df}{dx}(\phi_t(x))\frac{d\phi_t(x)}{dx} = \frac{df}{dx}(\phi_t(x))M(t) = A(t)M(t),$$

where we have denoted $A(t) = \frac{df}{dx}(\phi_t(x))$. Hence we have shown that $M(t)$ is a solution to the following differential equation in time

$$M'(t) = A(t)M(t); \quad M(0) = I.$$

By Abel's formula, we have the following

$$\begin{aligned}\det M(t) &= \det M(0)\; e^{\int_0^t \text{trace(A(s))}ds}\\ &= e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}.\end{aligned}$$

■

Now we state and prove the main result of this paper.

Theorem 7: The invariant set $\Lambda \subset X$ for the system of differential equation (1) is almost everywhere uniformly stable with respect to measure m, with density ρ_0, if and only if the following steady-state partial differential equation

$$\mathbb{A}^1\rho(x) = -\rho_0(x) \tag{11}$$

admits a positive solution $\rho(x) > 0$ and $\rho(x)$ is integrable on $X \setminus B_\delta$.

Proof: To prove the sufficiency of (11) we construct a solution $\rho(x)$ as follows

$$\rho(x) = \int_0^\infty \mathbb{P}_t^1\rho_0(x)dt. \tag{12}$$

The definition of a.e. uniform stability w.r.t. ρ_0 given by (7) guarantees the convergence of (12) for a.e. $x \in X \setminus B_\delta$. Also, by the property of the $\mathbb{P}_t$, we have that $\rho(x) > 0$. It remains to verify that (15) defines a solution for (11). To see this, we apply the operator $\mathbb{A}^1$ to (15). We get the following

$$\mathbb{A}^1\rho(x) = \mathbb{A}^1\int_0^\infty \mathbb{P}_t^1\rho_0(x)dt = \int_0^\infty \mathbb{A}^1\mathbb{P}_t^1\rho_0(x)dt,$$

where we have used the closedness of the operator $\mathbb{A}^1$ (guaranteed by the Hille-Yosida semigroup generation theorem) to obtain the last equality. We now have the following:

$$\int_0^\infty \mathbb{A}^1\mathbb{P}_t^1\rho_0(x)dt = \int_0^\infty \frac{d}{dt}\mathbb{P}_t^1\rho_0(x)dt = \lim_{t\to\infty}\mathbb{P}_t^1\rho_0(x) - \lim_{t\to 0}\mathbb{P}_t^1\rho_0(x) = -\rho_0(x),$$

where we have used $\lim_{t\to\infty}\mathbb{P}_t^1\rho_0(x) = 0$ (implied by the definition of a.e. uniform stability) and the semigroup property of $\mathbb{P}_t^1$ to obtain the first equality above and $\lim_{t\to 0}\mathbb{P}_t^1\rho_0(x) = \rho_0(x)$.

To prove the necessity of (11), we first find a representation formula for the following PDE

$$-\mathbb{A}\rho = \nabla\cdot(f\rho) = \rho_0. \tag{13}$$

Equation (13) can be rewritten as

$$\sum_{i=1}^n f_i(x)\rho_{x_i} + \rho(x)(\nabla\cdot f) = \rho_0(x). \tag{14}$$

Since this is a first order PDE, we use the method of characteristics to obtain a solution formula. The characteristic curves are given by the solution of the following ODE

$$\dot{x}(t) = f(x), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{15}$$

Let $\phi_t(x)$ denote the solution of (15). Then (14) can be rewritten as

$$\frac{d}{dt}\rho(\phi_t(x)) + \rho(\phi_t(x))(\nabla\cdot f) = \rho_0(\phi_t(x)), \tag{16}$$

which is a first order ODE in the t variable. The solution of (16) is obtained by multiplying (16) by the integrating factor

$$e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}.$$

We obtain the following

$$\frac{d}{dt}\left(\rho(\phi_t(x))e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}\right) = e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}\rho_0(\phi_t(x)). \tag{17}$$

Hence we obtain the following solution formula for (13) along the characteristic curves given by the solution of (15):

$$\begin{aligned}\rho(\phi_t(x)) &= e^{-\int_0^t \nabla\cdot f(\phi_s(x))ds}\rho(\phi_0(x)) \\ &+ e^{-\int_0^t \nabla\cdot f(\phi_s(x))ds}\int_0^t e^{\int_0^s \nabla\cdot f(x(\tau))d\tau}\rho_0(\phi_s(x))ds.\end{aligned} \tag{18}$$

From Lemma (6) we have

$$\det \frac{d(\phi_t(x))}{dx} = e^{\int_0^t \nabla\cdot f(\phi_s(x))ds}.$$

Using the above equation and rearranging Equation (18), we have the following

$$|\frac{d(\phi_t(x))}{dx}|\rho(\phi_t(x)) = \rho(\phi_0(x)) + \int_0^t |\frac{d(\phi_s(x))}{dx}|\rho_0(\phi_s(x))ds$$

$$\Rightarrow \quad \mathbb{P}_{-t}\rho(x) = \rho(\phi_0(x)) + \int_0^t \mathbb{P}_{-s}\rho_0(x)dt$$

$$\Rightarrow \quad \rho(x) = \mathbb{P}_t\rho(x) + \int_0^t \mathbb{P}_{t-s}\rho_0(x)dt. \qquad (19)$$

Next, we note that $\phi_0(x) = x$ and integrate Equation (19) in space with respect to $A \subset X \setminus B_\delta$ to obtain the following

$$\int_A \rho(x)dx = \int_A \mathbb{P}_t\rho(x) + \int_0^t \int_A \mathbb{P}_{t-s}\rho_0(x)$$

$$\Rightarrow \quad \int_A \rho(x)dx = \int_A \mathbb{P}_t^1\rho(x) + \int_0^t \int_A \mathbb{P}_{t-s}^1\rho_0(x)dxds$$

$$\Rightarrow \quad \int_A \rho(x)dx = \int_A \mathbb{P}_t^1\rho(x) + \int_0^t \int_A \mathbb{P}_\tau^1\rho_0(x)d\tau$$

Since $\rho(x), \rho_0(x)$ are both finite positive densities on $X \setminus B_\delta$, we have the following

$$\int_0^t \int_A \mathbb{P}_\tau^1\rho_0(x)d\tau < \qquad (20)$$

$$\int_A \mathbb{P}_t^1\rho(x) + \int_0^t \int_A \mathbb{P}_\tau^1\rho_0(x)d\tau = \int_A \rho(x)dx < \infty, \ \forall\ t > 0,$$

which is equivalent to Definition (2). ■

Remark 8: The solution of Equation (11) has to be understood in the weak sense i.e., the derivatives that appear in Equation (11) are weak derivatives since the regularity of the initial density $L^1(X \setminus B_\delta)$ is maintained by the solution.

For the case when m is the Lebesgue measure, we have $\rho_0(x) = \chi_{X\setminus B_\delta}$. If the equilibrium point is stable with respect to the Lebesgue measure then it is easy to prove the following theorem.

Theorem 9: Assume that the invariant set $\Lambda \subset X$ for the system of differential equation (1) is almost everywhere uniformly stable with respect to Lebesgue measure, then for any given positive $\rho_0 \in L^1(X \setminus B_\delta)$, the following steady state partial differential equation

$$\mathbb{A}^1\rho = -\rho_0(x) \qquad (21)$$

admits a positive solution ρ which is integrable on $X \setminus B_\delta$.

Proof: Fix $\varepsilon > 0$. We have that any function $\rho_0(x) \in L^1(X \setminus B_\delta)$ is a strong limit (in L^1 norm) of a sequence of simple functions $\{\psi_N(x)\}_{N=0}^\infty$. Also, we can choose $\{\psi_N(x)\}$ to be an increasing sequence satisfying $0 \le \psi_N(x) \le \rho_0(x)$. We denote the sequence as follows:

$$\psi_N(x) = \sum_{i=1}^N \lambda_i \chi_{A_i}(x),$$

where $A_i \subset X \setminus B_\delta$. We have

$$0 \le \int_{X\setminus B_\delta} \sum_i \lambda_i \chi_{A_i} dx \le M \implies 0 \le \sum_i \lambda_i \le \frac{M}{m(X \setminus B_\delta)} = C. \qquad (22)$$

Now

$$\int_T^\infty \int_A \mathbb{P}_t^1\rho_0(x)dxdt = \int_T^\infty \int_A \mathbb{P}_t^1\psi_N(x)dxdt + \int_T^\infty \int_A \mathbb{P}_t^1(\rho_0(x) - \psi_N(x))dxdt. \qquad (23)$$

First we estimate the first term using Equation (22):

$$\int_T^\infty \int_A \mathbb{P}_t^1\psi_N(x)dxdt \le \sum_{i=1}^N \lambda_i \int_T^\infty \int_A \mathbb{P}_t^1\chi_{A_i}dxdt$$

$$\sum_{i=1}^N \lambda_i \int_T^\infty \int_A \mathbb{P}_t^1\chi_{A_i}dxdt \le \sum_{i=1}^N \lambda_i \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{A_i}dxdt$$

Since $A_i \subset X \setminus B_\delta$ we have $\chi_{A_i}(x) \le \chi_{X\setminus B_\delta}(x)$. Furthermore, it is easy to see that property 2 holds true for $\mathbb{P}_t^1$ also. Hence we have $\mathbb{P}_t^1\chi_{A_i} \le \mathbb{P}_t^1\chi_{X\setminus B_\delta}$. and the following:

$$\sum_{i=1}^N \lambda_i \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{A_i}dxdt \le \sum_{i=1}^N \lambda_i \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{X\setminus B_\delta}dxdt$$

$$\le \left(\sum_{i=1}^N \lambda_i\right) \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{X\setminus B_\delta}dxdt$$

$$\le \frac{M}{m(X \setminus B_\delta)} \int_T^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{X\setminus B_\delta}dxdt.$$

Since we have a.e uniform stability w.r.t Lebesgue measure there exists $T = T_0$ such that $\int_{T_0}^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1\chi_{X\setminus B_\delta}dxdt < \frac{\varepsilon}{2C}$, where $C = \frac{M}{m(X\setminus B_\delta)}$. Hence we have

$$\int_{T_0}^\infty \int_A \mathbb{P}_t^1\psi_N(x)dxdt \le \frac{\varepsilon}{2}. \qquad (24)$$

Next we look at the second term. We note that $\mathbb{P}_t^1(\rho_0(x) - \psi_N(x)) \ge 0\ \forall N \in \mathbb{N}$, by property 2 applied to $\mathbb{P}_t^1$. Hence we have the following:

$$\int_{T_0}^\infty \int_A \mathbb{P}_t^1(\rho_0(x) - \psi_N(x))dxdt$$

$$\le \int_{T_0}^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1(\rho_0(x) - \psi_N(x))dxdt$$

Since $\rho_0(x)$ and $\psi_N(x)$ are both supported on $X \setminus B_\delta$, we have the following:

$$\int_{X\setminus B_\delta} \mathbb{P}_t^1(\rho_0(x) - \psi_N(x))dx = ||\mathbb{P}_t^1(\rho_0(x) - \psi_N(x))||_{L^1(X\setminus B_\delta)}.$$

Using the continuity of $\mathbb{P}_t^1$ on $L^1(X \setminus B_\delta)$ for fixed $t > 0$, there exists an N large enough such that

$$||\mathbb{P}_t^1(\rho_0(x) - \psi_N(x))||_{L^1(X\setminus B_\delta)} \le \varepsilon e^{-t}$$

Hence we have,

$$\int_{T_0}^\infty \int_{X\setminus B_\delta} \mathbb{P}_t^1(\rho_0(x) - \psi_N(x))dxdt \le \int_{T_0}^\infty \varepsilon e^{-t}dt = \frac{\varepsilon}{2}e^{-T_0} \le \frac{\varepsilon}{2}.$$

Hence we have,

$$\int_{T_0}^\infty \int_A \mathbb{P}_t(\rho_0(x) - \psi_N(x))dxdt \le \frac{\varepsilon}{2}. \qquad (25)$$

From Equations (24) and (25) we have

$$\int_{T_0}^{\infty}\int_A \mathbb{P}_t\rho_0(x)dxdt \le \varepsilon \ \forall A \in X\setminus B_\delta. \quad (26)$$

■

Corollary 10: Assume that the invariant set $\Lambda \subset X$ for the system of differential equation (1) is almost everywhere uniformly stable with respect to Lebesgue measure. Then

$$\int_0^{\infty} \| \mathbb{P}_t^1\rho_0(x) \| \, dt < \infty \quad (27)$$

for all $\rho_0(x) \in L^1(X \setminus B_\delta)$, where $\| \mathbb{P}_t^1\rho_0(x) \| = \int_{X\setminus B_\delta} |\mathbb{P}_t^1\rho_0(x)|dx$

Proof: From theorem (9), we know that for any positive $\rho_0 \in L^1(X\setminus B_\delta)$ and any given $\varepsilon > 0$ there exists a time T_0 such that

$$\int_{T_0}^{\infty} \| \mathbb{P}_t^1\rho_0(x) \| \le \varepsilon$$

and hence

$$\int_0^{\infty} \| \mathbb{P}_t^1\rho_0(x) \| < \infty \quad (28)$$

Any $\rho_0(x) \in L^1(X\setminus B_\delta)$ can be written as

$$\rho_0(x) = \rho_0^+(x) - \rho_0^-(x)$$

where both $\rho_0^+(x) > 0$ and $\rho_0^-(x) > 0$. Hence we have

$$\int_0^{\infty} \| \mathbb{P}_t^1\rho_0(x) \| \, dt \le \int_0^{\infty} \| \mathbb{P}_t^1\rho_0^+(x) \| \, dt + \int_0^{\infty} \| \mathbb{P}_t^1\rho_0^-(x) \| \, dt < \infty \quad (29)$$

■

V. Spectrum of $\mathbb{A}^1$

In this section, we show that almost everywhere uniform stability implies that the spectrum of the operator $\mathbb{A}^1$ has to necessarily be a subset of the left half plane. We recall the following definitions and theorems from Semigroup theory [10].

Theorem 11: A strongly continuous semigroup $(T(t))t \ge 0$ on a Banach space $L^p(X)$ is uniformly exponentially stable if and only if for one/all $p \in [1,\infty)$ one has

$$\int_0^{\infty} \| T(t)\rho(x) \|^p \, dt < \infty$$

for all $\rho(x) \in L^p(X)$

Proof: Refer [10], page 300. ■

From Theorem (11) and Corollary (10), we have the following Theorem:

Theorem 12: The invariant set $\Lambda \subset X$ for the system of differential equation (1) is almost everywhere uniformly stable with respect to Lebesgue measure if and only if the semigroup $\mathbb{P}_t^1$ is uniformly exponentially stable.

Next, we relate the spectrum of the semigroup $\mathbb{P}_t^1$ to that of its generator $\mathbb{A}^1$, towards this we use following definition of growth bound and spectral bound.

Definition 13 (Growth bound): For a strongly continuous semigroup $(T(t))_{t\ge 0}$, we call

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \exists M_\omega \ge 1 s.t. \| T(t) \| \le M_\omega e^{\omega t} \forall t \ge 0\}$$

its growth bound.

Definition 14 (Spectral bound): Let $\mathbb{A}$ be a closed operator. Then

$$s(\mathbb{A}) := \sup\{Re\lambda : \lambda \in \sigma(\mathbb{A})\}$$

is called the spectral bound of A.

We have the following proposition relating the spectral bound of the generator $\mathbb{A}$ to the growth bound of the generated semigroup $(T(t))_{t\ge 0}$.

Proposition 15: For the spectral bound $s(\mathbb{A})$ of a generator $\mathbb{A}$ and for the growth bound ω_0 of the generated semigroup $(T(t))_{t\ge 0}$, one has

$$-\infty \le s(\mathbb{A}) \le \omega_0$$

Proof: Refer [10], page 251. ■

Proposition 16: A strongly continuous semigroup $T(t)_{t\ge 0}$ is uniformly exponentially stable if and only if $\omega_0 < 0$, where ω_0 is the growth bound of semigroup.

Proof: Refer [10] ■

From Propositions (15),(16) and Theorem (12), we have the following theorem about the spectrum of the operator $\mathbb{A}^1$.

Theorem 17: Assume that the invariant set $\Lambda \subset X$ for the system of differential equation (1) is almost everywhere uniformly stable with respect to Lebesgue measure then

$$s(\mathbb{A}^1) < 0. \quad (30)$$

VI. Numerical Simulation

In this section, we show numerical results for the computation of density ρ for the following three systems:

$$\dot{x} = Ax \text{ where } A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad (31)$$

$$\dot{x} = \begin{cases} y \\ -y - \sin(x) \end{cases}, \quad (32)$$

$$\dot{x} = \begin{cases} y \\ 0.5(1-x)^2 y - x \end{cases}. \quad (33)$$

The equilibrium point $(0,0)$ is well known to be exponentially stable for (31) and (32). The Van der Pol oscillator (33) is also known to have a stable limit cycle. For each case, we show the characteristic curves on a region outside a neighborhood of the invariant set. The initial density ρ_0 is chosen to be the characteristic function of this region. The characteristic curves were calculated backwards in time first and the density was computed using a discretized version of Equation (19) to compute the density function. The results are shown below. Since the density function assumes large values near the invariant set, we choose to plot the natural logarithm of the density in order to show the growth of the density better.

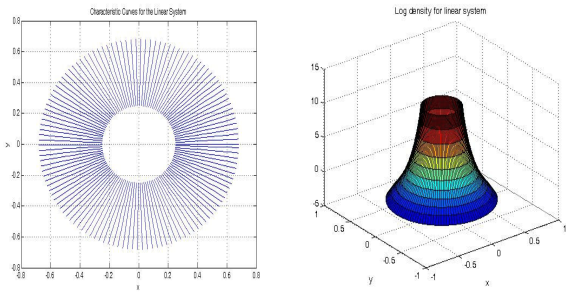

Fig. 1. Characteristic curves and $\ln(\rho)$ for the linear system (31) with $(0,0)$ as the invariant set

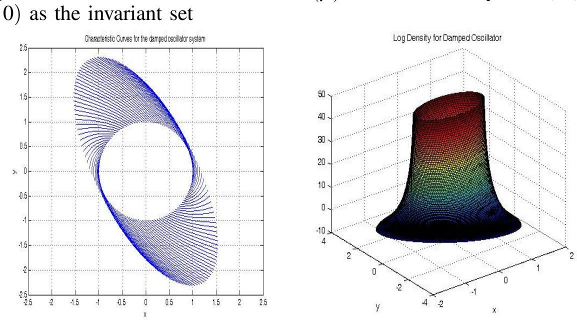

Fig. 2. Characteristic curves and $\ln(\rho)$ for the damped oscillator (31) with $(0,0)$ as the invariant set

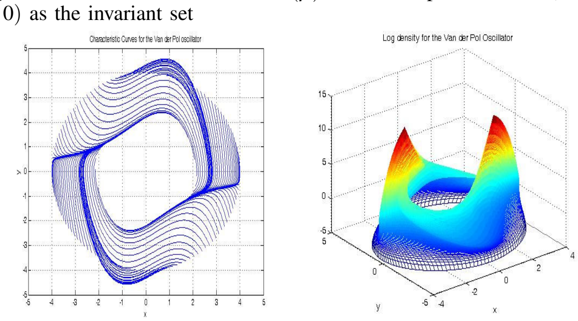

Fig. 3. Characteristic curves and $\ln(\rho)$ for the Van der Pol system (31) with stable limit cycle as the invariant set

VII. Conclusion

We have proved result on the necessary and sufficient condition for almost everywhere uniform stability of an invariant set in continuous time dynamical system. Lyapunov density verifying the almost everywhere uniform stability of an invariant set is shown to be obtained as a positive solution of steady state linear partial differential equation. Use of this result for an efficient computation of the Lyapunov density and its application to controller design is currently under investigation.

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